

## Chapter 8. Converter Transfer Functions

### 8.1. Review of Bode plots

- 8.1.1. Single pole response
- 8.1.2. Single zero response
- 8.1.3. Right half-plane zero
- 8.1.4. Frequency inversion
- 8.1.5. Combinations
- 8.1.6. Double pole response: resonance
- 8.1.7. The low-Q approximation
- 8.1.8. Approximate roots of an arbitrary-degree polynomial

### 8.2. Analysis of converter transfer functions

- 8.2.1. Example: transfer functions of the buck-boost converter
- 8.2.2. Transfer functions of some basic CCM converters
- 8.2.3. Physical origins of the right half-plane zero in converters

---

---

---

---

---

---

---

---

---

---

## Converter Transfer Functions

### 8.3. Graphical construction of converter transfer functions

- 8.3.1. Series impedances: addition of asymptotes
- 8.3.2. Parallel impedances: inverse addition of asymptotes
- 8.3.3. Another example
- 8.3.4. Voltage divider transfer functions: division of asymptotes

### 8.4. Measurement of ac transfer functions and impedances

### 8.5. Summary of key points

---

---

---

---

---

---

---

---

---

---

## The Engineering Design Process

1. *Specifications and other design goals* are defined.
2. *A circuit is proposed.* This is a creative process that draws on the physical insight and experience of the engineer.
3. *The circuit is modeled.* The converter power stage is modeled as described in Chapter 7. Components and other portions of the system are modeled as appropriate, often with vendor-supplied data.
4. *Design-oriented analysis* of the circuit is performed. This involves development of equations that allow element values to be chosen such that specifications and design goals are met. In addition, it may be necessary for the engineer to gain additional understanding and physical insight into the circuit behavior, so that the design can be improved by adding elements to the circuit or by changing circuit connections.
5. *Model verification.* Predictions of the model are compared to a laboratory prototype, under nominal operating conditions. The model is refined as necessary, so that the model predictions agree with laboratory measurements.

---

---

---

---

---

---

---

---

---

---

## Design Process

6. *Worst-case analysis* (or other reliability and production yield analysis) of the circuit is performed. This involves quantitative evaluation of the model performance, to judge whether specifications are met under all conditions. Computer simulation is well-suited to this task.
7. *Iteration*. The above steps are repeated to improve the design until the worst-case behavior meets specifications, or until the reliability and production yield are acceptably high.

This Chapter: steps 4, 5, and 6

---

---

---

---

---

---

---

---

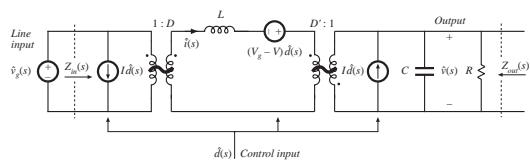
---

---

---

---

## Buck-boost converter model From Chapter 7



$$G_{vd}(s) = \left. \frac{\hat{v}(s)}{\hat{v}_g(s)} \right|_{\hat{d}(s)=0}$$

$$G_{vd}(s) = \left. \frac{\hat{v}(s)}{\hat{d}(s)} \right|_{\hat{v}_g(s)=0}$$

---

---

---

---

---

---

---

---

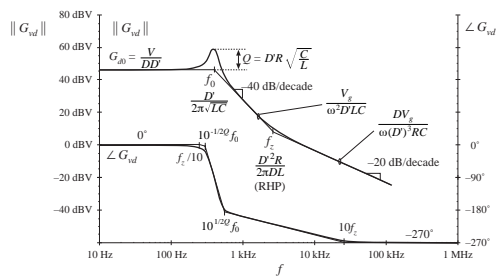
---

---

---

---

## Bode plot of control-to-output transfer function with analytical expressions for important features




---

---

---

---

---

---

---

---

---

---

---

---

## Design-oriented analysis

How to approach a real (and hence, complicated) system

Problems:

- Complicated derivations
- Long equations
- Algebra mistakes

Design objectives:

- Obtain physical insight which leads engineer to synthesis of a good design
- Obtain simple equations that can be inverted, so that element values can be chosen to obtain desired behavior. Equations that cannot be inverted are useless for design!

*Design-oriented analysis* is a structured approach to analysis, which attempts to avoid the above problems

---

---

---

---

---

---

---

---

---

---

## Some elements of design-oriented analysis, discussed in this chapter

- Writing transfer functions in normalized form, to directly expose salient features
- Obtaining simple analytical expressions for asymptotes, corner frequencies, and other salient features, allows element values to be selected such that a given desired behavior is obtained
- Use of inverted poles and zeroes, to refer transfer function gains to the most important asymptote
- Analytical approximation of roots of high-order polynomials
- Graphical construction of Bode plots of transfer functions and polynomials, to
  - avoid algebra mistakes
  - approximate transfer functions
  - obtain insight into origins of salient features

---

---

---

---

---

---

---

---

---

---

## 8.1. Review of Bode plots

**Decibels**

$$|G|_{\text{dB}} = 20 \log_{10}(|G|)$$

Decibels of quantities having units (impedance example):  
normalize before taking log

$$|Z|_{\text{dB}} = 20 \log_{10}\left(\frac{|Z|}{R_{\text{base}}}\right)$$

Table 8.1. Expressing magnitudes in decibels

Actual magnitude	Magnitude in dB
1/2	-6dB
1	0 dB
2	6 dB
5 = 10/2	20 dB - 6 dB = 14 dB
10	20dB
1000 = 10 <sup>3</sup>	3 · 20dB = 60 dB

5Ω is equivalent to 14dB with respect to a base impedance of  $R_{\text{base}} = 1\Omega$ , also known as 14dBΩ.

60dBμA is a current 60dB greater than a base current of 1μA, or 1mA.

---

---

---

---

---

---

---

---

---

---

## Bode plot of $f^n$

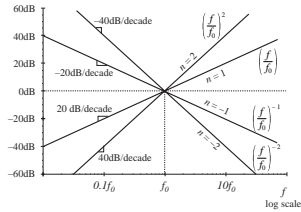
Bode plots are effectively log-log plots, which cause functions which vary as  $f^n$  to become linear plots. Given:

$$|G| = \left(\frac{f}{f_0}\right)^n$$

Magnitude in dB is

$$|G|_{dB} = 20 \log_{10} \left(\frac{f}{f_0}\right)^n = 20n \log_{10} \left(\frac{f}{f_0}\right)$$

- Slope is 20n dB/decade
- Magnitude is 1, or 0dB, at frequency  $f = f_0$




---

---

---

---

---

---

---

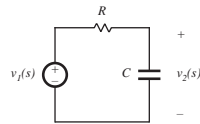
---

---

---

## 8.1.1. Single pole response

Simple R-C example



Transfer function is

$$G(s) = \frac{v_o(s)}{v_i(s)} = \frac{1}{sC + R}$$

Express as rational fraction:

$$G(s) = \frac{1}{1 + sRC}$$

This coincides with the normalized form

$$G(s) = \frac{1}{1 + \frac{s}{\omega_0}}$$

with  $\omega_0 = \frac{1}{RC}$

---

---

---

---

---

---

---

---

---

---

## $G(j\omega)$ and $\|G(j\omega)\|$

Let  $s = j\omega$ :

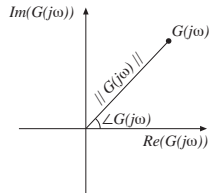
$$G(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}} = \frac{1 - j\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

Magnitude is

$$\begin{aligned} |G(j\omega)| &= \sqrt{[\text{Re}(G(j\omega))]^2 + [\text{Im}(G(j\omega))]^2} \\ &= \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}} \end{aligned}$$

Magnitude in dB:

$$|G(j\omega)|_{dB} = -20 \log_{10} \left( \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \right) \text{ dB}$$




---

---

---

---

---

---

---

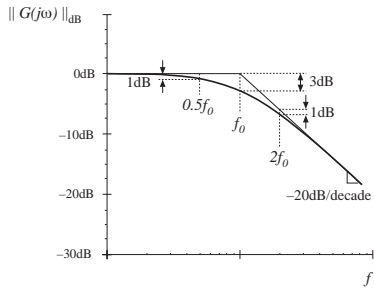
---

---

---



### Summary: magnitude




---

---

---

---

---

---

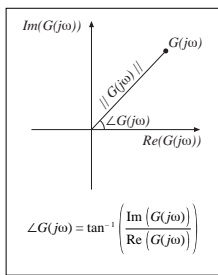
---

---

---

---

### Phase of $G(j\omega)$



$$G(j\omega) = \frac{1}{1 + j \frac{\omega}{\omega_0}} = \frac{1 - j \frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

$$\angle G(j\omega) = -\tan^{-1} \left( \frac{\omega}{\omega_0} \right)$$

$$\angle G(j\omega) = \tan^{-1} \left( \frac{\text{Im}(G(j\omega))}{\text{Re}(G(j\omega))} \right)$$

---

---

---

---

---

---

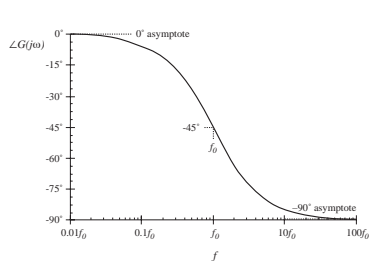
---

---

---

---

### Phase of $G(j\omega)$



$$\angle G(j\omega) = -\tan^{-1} \left( \frac{\omega}{\omega_0} \right)$$

$\omega$	$\angle G(j\omega)$
0	0°
$\omega_0$	-45°
$\infty$	-90°

---

---

---

---

---

---

---

---

---

---

## Phase asymptotes

Low frequency:  $0^\circ$

High frequency:  $-90^\circ$

Low- and high-frequency asymptotes do not intersect

Hence, need a midfrequency asymptote

Try a midfrequency asymptote having slope identical to actual slope at the corner frequency  $f_0$ . One can show that the asymptotes then intersect at the break frequencies

$$\begin{aligned} f_c &= f_0 e^{-\pi/2} \approx f_0 / 4.81 \\ f_h &= f_0 e^{\pi/2} \approx 4.81 f_0 \end{aligned}$$

---

---

---

---

---

---

---

---

---

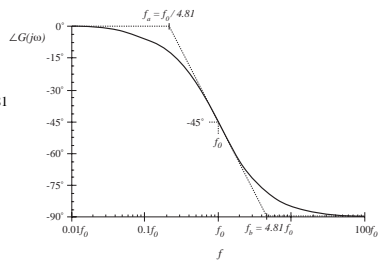
---

---

---

## Phase asymptotes

$$\begin{aligned} f_c &= f_0 e^{-\pi/2} \approx f_0 / 4.81 \\ f_h &= f_0 e^{\pi/2} \approx 4.81 f_0 \end{aligned}$$




---

---

---

---

---

---

---

---

---

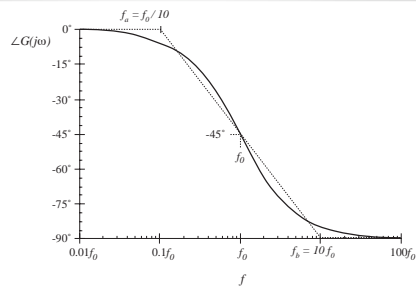
---

---

---

## Phase asymptotes: a simpler choice

$$\begin{aligned} f_c &= f_0 / 10 \\ f_h &= 10 f_0 \end{aligned}$$




---

---

---

---

---

---

---

---

---

---

---

---



### 8.1.3. Right half-plane zero

Normalized form:

$$G(s) = \left(1 - \frac{s}{\omega_0}\right)$$

Magnitude:

$$|G(j\omega)| = \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

— same as conventional (left half-plane) zero. Hence, magnitude asymptotes are identical to those of LHP zero.

Phase:

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

— same as real pole.

The RHP zero exhibits the magnitude asymptotes of the LHP zero, and the phase asymptotes of the pole

---

---

---

---

---

---

---

---

---

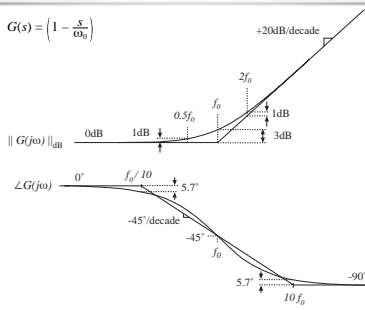
---

---

---

### Summary: Bode plot, RHP zero

$$G(s) = \left(1 - \frac{s}{\omega_0}\right)$$




---

---

---

---

---

---

---

---

---

---

---

---

### 8.1.4. Frequency inversion

Reversal of frequency axis. A useful form when describing mid- or high-frequency flat asymptotes. Normalized form, inverted pole:

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)}$$

An algebraically equivalent form:

$$G(s) = \frac{\left(\frac{s}{\omega_0}\right)}{\left(1 + \frac{s}{\omega_0}\right)}$$

The inverted-pole format emphasizes the high-frequency gain.

---

---

---

---

---

---

---

---

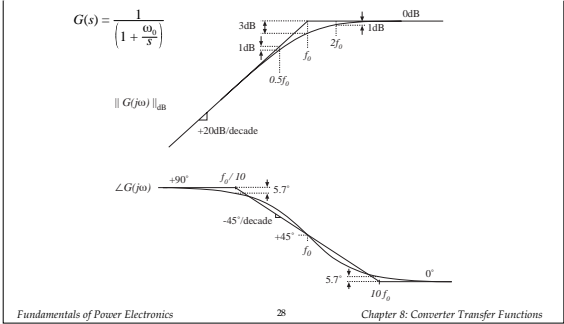
---

---

---

---

### Asymptotes, inverted pole




---

---

---

---

---

---

---

---

---

---

### Inverted zero

Normalized form, inverted zero:

$$G(s) = \left(1 + \frac{\omega_0}{s}\right)$$

An algebraically equivalent form:

$$G(s) = \frac{\left(1 + \frac{s}{\omega_0}\right)}{\left(\frac{s}{\omega_0}\right)}$$

Again, the inverted-zero format emphasizes the high-frequency gain.

---

---

---

---

---

---

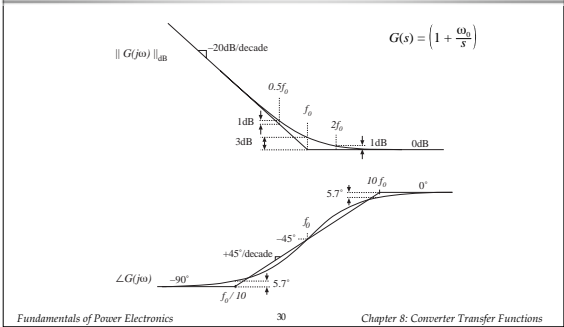
---

---

---

---

### Asymptotes, inverted zero




---

---

---

---

---

---

---

---

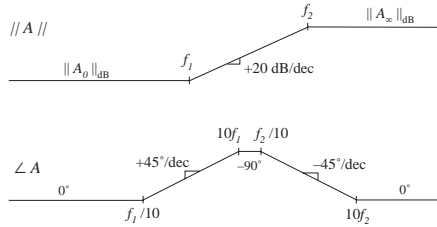
---

---



### Example 2

Determine the transfer function  $A(s)$  corresponding to the following asymptotes:




---

---

---

---

---

---

---

---

---

---

### Example 2, continued

One solution:

$$A(s) = A_0 \frac{\left(1 + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_2}\right)}$$

Analytical expressions for asymptotes:

For  $f < f_1$   $\left| A_0 \frac{\left(1 + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_2}\right)} \right|_{s=j\omega} = A_0 \frac{1}{1} = A_0$

For  $f_1 < f < f_2$   $\left| A_0 \frac{\left(\frac{s}{\omega_1} + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_2}\right)} \right|_{s=j\omega} = A_0 \frac{\frac{s}{\omega_1}}{1} \Big|_{s=j\omega} = A_0 \frac{\omega}{\omega_1} = A_0 \frac{f}{f_1}$

---

---

---

---

---

---

---

---

---

---

### Example 2, continued

For  $f > f_2$   $\left| A_0 \frac{\left(\frac{s}{\omega_1} + \frac{s}{\omega_1}\right)}{\left(\frac{s}{\omega_2} + \frac{s}{\omega_2}\right)} \right|_{s=j\omega} = A_0 \frac{\left[\frac{s}{\omega_1}\right]_{s=j\omega}}{\left[\frac{s}{\omega_2}\right]_{s=j\omega}} = A_0 \frac{\omega_2}{\omega_1} = A_0 \frac{f_2}{f_1}$

So the high-frequency asymptote is

$$A_\infty = A_0 \frac{f_2}{f_1}$$

Another way to express  $A(s)$ : use inverted poles and zeroes, and express  $A(s)$  directly in terms of  $A_\infty$

$$A(s) = A_\infty \frac{\left(1 + \frac{\omega_1}{s}\right)}{\left(1 + \frac{\omega_2}{s}\right)}$$

---

---

---

---

---

---

---

---

---

---

### 8.1.6 Quadratic pole response: resonance

Example

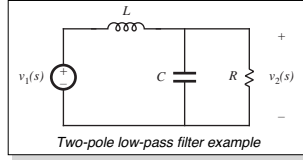
$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2LC}$$

Second-order denominator, of the form

$$G(s) = \frac{1}{1 + a_1s + a_2s^2}$$

with  $a_1 = L/R$  and  $a_2 = LC$

How should we construct the Bode diagram?




---

---

---

---

---

---

---

---

---

---

---

---

### Approach 1: factor denominator

$$G(s) = \frac{1}{1 + a_1s + a_2s^2}$$

We might factor the denominator using the quadratic formula, then construct Bode diagram as the combination of two real poles:

$$G(s) = \frac{1}{\left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s}{s_2}\right)} \quad \text{with} \quad s_1 = -\frac{a_1}{2a_2} \left[1 - \sqrt{1 - \frac{4a_2}{a_1^2}}\right]$$

$$s_2 = -\frac{a_1}{2a_2} \left[1 + \sqrt{1 - \frac{4a_2}{a_1^2}}\right]$$

- If  $4a_2 \leq a_1^2$ , then the roots  $s_1$  and  $s_2$  are real. We can construct Bode diagram as the combination of two real poles.
- If  $4a_2 > a_1^2$ , then the roots are complex. In Section 8.1.1, the assumption was made that  $\omega_0$  is real; hence, the results of that section cannot be applied and we need to do some additional work.

---

---

---

---

---

---

---

---

---

---

---

---

### Approach 2: Define a standard normalized form for the quadratic case

$$G(s) = \frac{1}{1 + 2\zeta\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \quad \text{or} \quad G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

- When the coefficients of  $s$  are real and positive, then the parameters  $\zeta$ ,  $\omega_0$ , and  $Q$  are also real and positive
- The parameters  $\zeta$ ,  $\omega_0$ , and  $Q$  are found by equating the coefficients of  $s$
- The parameter  $\omega_0$  is the angular corner frequency, and we can define  $f_0 = \omega_0/2\pi$
- The parameter  $\zeta$  is called the *damping factor*.  $\zeta$  controls the shape of the exact curve in the vicinity of  $f = f_0$ . The roots are complex when  $\zeta < 1$ .
- In the alternative form, the parameter  $Q$  is called the *quality factor*.  $Q$  also controls the shape of the exact curve in the vicinity of  $f = f_0$ . The roots are complex when  $Q > 0.5$ .

---

---

---

---

---

---

---

---

---

---

---

---

## The Q-factor

In a second-order system,  $\zeta$  and  $Q$  are related according to

$$Q = \frac{1}{2\zeta}$$

$Q$  is a measure of the dissipation in the system. A more general definition of  $Q$ , for sinusoidal excitation of a passive element or system is

$$Q = 2\pi \frac{\text{(peak stored energy)}}{\text{(energy dissipated per cycle)}}$$

For a second-order passive system, the two equations above are equivalent. We will see that  $Q$  has a simple interpretation in the Bode diagrams of second-order transfer functions.

---

---

---

---

---

---

---

---

---

---

## Analytical expressions for $f_0$ and $Q$

Two-pole low-pass filter  
example: we found that

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2LC}$$

Equate coefficients of like powers of  $s$  with the standard form

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Result:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$Q = R\sqrt{\frac{C}{L}}$$

---

---

---

---

---

---

---

---

---

---

## Magnitude asymptotes, quadratic form

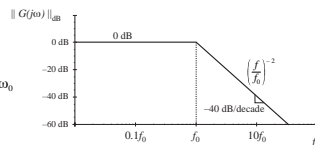
In the form  $G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$

let  $s = j\omega$  and find magnitude:  $|G(j\omega)| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$

Asymptotes are

$$|G| \rightarrow 1 \quad \text{for } \omega \ll \omega_0$$

$$|G| \rightarrow \left(\frac{f}{f_0}\right)^{-2} \quad \text{for } \omega \gg \omega_0$$




---

---

---

---

---

---

---

---

---

---

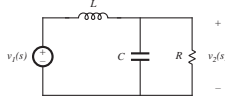


## An example

A problem with this procedure is the complexity of the quadratic formula used to find the corner frequencies.

R-L-C network example:

$$G(s) = \frac{v_o(s)}{v_i(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$



Use quadratic formula to factor denominator. Corner frequencies are:

$$\omega_1, \omega_2 = \frac{L/R \pm \sqrt{(L/R)^2 - 4LC}}{2LC}$$

---

---

---

---

---

---

---

---

---

---

## Factoring the denominator

$$\omega_1, \omega_2 = \frac{L/R \pm \sqrt{(L/R)^2 - 4LC}}{2LC}$$

This complicated expression yields little insight into how the corner frequencies  $\omega_1$  and  $\omega_2$  depend on  $R$ ,  $L$ , and  $C$ .

When the corner frequencies are well separated in value, it can be shown that they are given by the much simpler (approximate) expressions

$$\omega_1 = \frac{R}{L}, \quad \omega_2 = \frac{1}{RC}$$

$\omega_1$  is then independent of  $C$ , and  $\omega_2$  is independent of  $L$ .

These simpler expressions can be derived via the Low- $Q$  Approximation.

---

---

---

---

---

---

---

---

---

---

## Derivation of the Low- $Q$ Approximation

Given

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Use quadratic formula to express corner frequencies  $\omega_1$  and  $\omega_2$  in terms of  $Q$  and  $\omega_0$  as:

$$\omega_1 = \frac{\omega_0}{Q} \frac{1 - \sqrt{1 - 4Q^2}}{2} \quad \omega_2 = \frac{\omega_0}{Q} \frac{1 + \sqrt{1 - 4Q^2}}{2}$$

---

---

---

---

---

---

---

---

---

---

### Corner frequency $\omega_2$

$$\omega_2 = \frac{\omega_0}{Q} \frac{1 + \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

$$\omega_2 = \frac{\omega_0}{Q} F(Q)$$

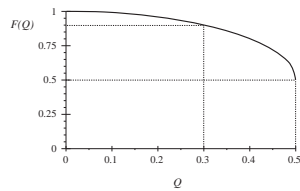
where

$$F(Q) = \frac{1}{2} (1 + \sqrt{1 - 4Q^2})$$

For small  $Q$ ,  $F(Q)$  tends to 1.

We then obtain

$$\omega_2 \approx \frac{\omega_0}{Q} \quad \text{for } Q \ll \frac{1}{2}$$



For  $Q < 0.3$ , the approximation  $F(Q) = 1$  is within 10% of the exact value.

---

---

---

---

---

---

---

---

---

---

### Corner frequency $\omega_1$

$$\omega_1 = \frac{\omega_0}{Q} \frac{1 - \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

$$\omega_1 = \frac{Q \omega_0}{F(Q)}$$

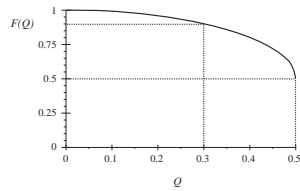
where

$$F(Q) = \frac{1}{2} (1 + \sqrt{1 - 4Q^2})$$

For small  $Q$ ,  $F(Q)$  tends to 1.

We then obtain

$$\omega_1 \approx Q \omega_0 \quad \text{for } Q \ll \frac{1}{2}$$



For  $Q < 0.3$ , the approximation  $F(Q) = 1$  is within 10% of the exact value.

---

---

---

---

---

---

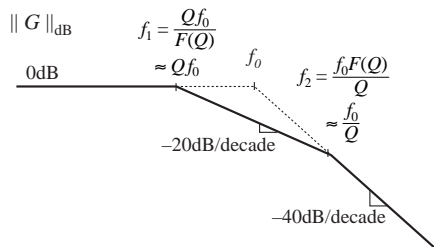
---

---

---

---

### The Low- $Q$ Approximation




---

---

---

---

---

---

---

---

---

---

## R-L-C Example

For the previous example:

$$G(s) = \frac{v_o(s)}{v_i(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC} \quad f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$Q = R\sqrt{\frac{C}{L}}$$

Use of the Low- $Q$  Approximation leads to

$$\omega_1 = Q \omega_0 = R\sqrt{\frac{C}{L}} \frac{1}{\sqrt{LC}} = \frac{R}{L}$$

$$\omega_2 = \frac{\omega_0}{Q} = \frac{1}{\sqrt{LC}} \frac{1}{R\sqrt{\frac{C}{L}}} = \frac{1}{RC}$$

---

---

---

---

---

---

---

---

---

---

---

---

## 8.1.8. Approximate Roots of an Arbitrary-Degree Polynomial

Generalize the low- $Q$  approximation to obtain approximate factorization of the  $n^{\text{th}}$ -order polynomial

$$P(s) = 1 + a_1 s + a_2 s^2 + \dots + a_n s^n$$

It is desired to factor this polynomial in the form

$$P(s) = (1 + \tau_1 s)(1 + \tau_2 s) \dots (1 + \tau_n s)$$

When the roots are real and well separated in value, then approximate analytical expressions for the time constants  $\tau_1, \tau_2, \dots, \tau_n$  can be found, that typically are simple functions of the circuit element values.

**Objective:** find a general method for deriving such expressions. Include the case of complex root pairs.

---

---

---

---

---

---

---

---

---

---

---

---

## Derivation of method

Multiply out factored form of polynomial, then equate to original form (equate like powers of  $s$ ):

$$a_1 = \tau_1 + \tau_2 + \dots + \tau_n$$

$$a_2 = \tau_1(\tau_2 + \dots + \tau_n) + \tau_2(\tau_3 + \dots + \tau_n) + \dots$$

$$a_3 = \tau_1\tau_2(\tau_3 + \dots + \tau_n) + \tau_2\tau_3(\tau_4 + \dots + \tau_n) + \dots$$

$$\vdots$$

$$a_n = \tau_1\tau_2\tau_3 \dots \tau_n$$

- Exact system of equations relating roots to original coefficients
- Exact general solution is hopeless
- Under what conditions can solution for time constants be easily approximated?

---

---

---

---

---

---

---

---

---

---

---

---







## Example

Approximate factorization of a third-order denominator

The filter transfer function from the previous slide is

$$G(s) = \frac{i_g(s)}{i_s(s)} = \frac{1 + s \frac{L_1 + L_2}{R}}{1 + s \frac{L_1 + L_2}{R} + s^2 L_1 C + s^3 \frac{L_1 L_2 C}{R}}$$

— contains a third-order denominator, with the following coefficients:

$$a_1 = \frac{L_1 + L_2}{R}$$

$$a_2 = L_1 C$$

$$a_3 = \frac{L_1 L_2 C}{R}$$

---

---

---

---

---

---

---

---

---

---

## Real roots case

Factorization as three real roots:

$$\left(1 + s \frac{L_1 + L_2}{R}\right) \left(1 + sRC \frac{L_1}{L_1 + L_2}\right) \left(1 + s \frac{L_2}{R}\right)$$

This approximate analytical factorization is justified provided

$$\frac{L_1 + L_2}{R} \gg RC \frac{L_1}{L_1 + L_2} \gg \frac{L_2}{R}$$

Note that these inequalities cannot be satisfied unless  $L_1 \gg L_2$ . The above inequalities can then be further simplified to

$$\frac{L_1}{R} \gg RC \gg \frac{L_2}{R}$$

And the factored polynomial reduces to

$$\left(1 + s \frac{L_1}{R}\right) \left(1 + sRC\right) \left(1 + s \frac{L_2}{R}\right)$$

• Illustrates in a simple way how the roots depend on the element values

---

---

---

---

---

---

---

---

---

---

## When the second inequality is violated

$$\frac{L_1 + L_2}{R} \gg RC \frac{L_1}{L_1 + L_2} \not\gg \frac{L_2}{R}$$

↑  
not satisfied

Then leave the second and third roots in quadratic form:

$$P(s) = \left(1 + a_1 s\right) \left(1 + \frac{a_2}{a_1} s + \frac{a_3}{a_1} s^2\right)$$

which is

$$\left(1 + s \frac{L_1 + L_2}{R}\right) \left(1 + sRC \frac{L_1}{L_1 + L_2} + s^2 L_1 L_2 C\right)$$

---

---

---

---

---

---

---

---

---

---

### Validity of the approximation

This is valid provided

$$\frac{L_1 + L_2}{R} \gg RC \quad \frac{L_1}{L_1 + L_2} \gg \frac{L_1 L_2}{L_1 + L_2} RC \quad (\text{use } a_0 = 1)$$

These inequalities are equivalent to

$$L_1 \gg L_2, \quad \text{and} \quad \frac{L_1}{R} \gg RC$$

It is no longer required that  $RC \gg L_2/R$

The polynomial can therefore be written in the simplified form

$$\left(1 + s \frac{L_1}{R}\right) \left(1 + sRC + s^2 L_2 C\right)$$

---

---

---

---

---

---

---

---

---

---

### When the first inequality is violated

$$\frac{L_1 + L_2}{R} \not\gg RC \quad \frac{L_1}{L_1 + L_2} \gg \frac{L_2}{R}$$

↑  
not satisfied

Then leave the first and second roots in quadratic form:

$$P(s) = \left(1 + a_1 s + a_2 s^2\right) \left(1 + \frac{a_3}{a_2} s\right)$$

which is

$$\left(1 + s \frac{L_1 + L_2}{R} + s^2 L_1 C\right) \left(1 + s \frac{L_2}{R}\right)$$

---

---

---

---

---

---

---

---

---

---

### Validity of the approximation

This is valid provided

$$\frac{L_1 RC}{L_2} \gg \frac{L_1 + L_2}{R} \gg \frac{L_2}{R}$$

These inequalities are equivalent to

$$L_1 \gg L_2, \quad \text{and} \quad RC \gg \frac{L_2}{R}$$

It is no longer required that  $L_1/R \gg RC$

The polynomial can therefore be written in the simplified form

$$\left(1 + s \frac{L_1}{R} + s^2 L_1 C\right) \left(1 + s \frac{L_2}{R}\right)$$

---

---

---

---

---

---

---

---

---

---

## 8.2. Analysis of converter transfer functions

- 8.2.1. Example: transfer functions of the buck-boost converter
- 8.2.2. Transfer functions of some basic CCM converters
- 8.2.3. Physical origins of the right half-plane zero in converters

---

---

---

---

---

---

---

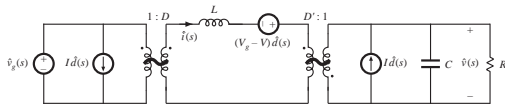
---

---

---

### 8.2.1. Example: transfer functions of the buck-boost converter

Small-signal ac model of the buck-boost converter, derived in Chapter 7:




---

---

---

---

---

---

---

---

---

---

### Definition of transfer functions

The converter contains two inputs,  $\hat{d}(s)$  and  $\hat{v}_g(s)$  and one output,  $\hat{v}(s)$ . Hence, the ac output voltage variations can be expressed as the superposition of terms arising from the two inputs:

$$\hat{v}(s) = G_{vd}(s) \hat{d}(s) + G_{vg}(s) \hat{v}_g(s)$$

The control-to-output and line-to-output transfer functions can be defined as

$$G_{vd}(s) = \left. \frac{\hat{v}(s)}{\hat{d}(s)} \right|_{\hat{v}_g(s)=0} \quad \text{and} \quad G_{vg}(s) = \left. \frac{\hat{v}(s)}{\hat{v}_g(s)} \right|_{\hat{d}(s)=0}$$

---

---

---

---

---

---

---

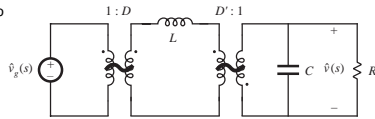
---

---

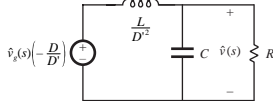
---

## Derivation of line-to-output transfer function $G_{vD}(s)$

Set  $\hat{d}$  sources to zero:



Push elements through transformers to output side:




---

---

---

---

---

---

---

---

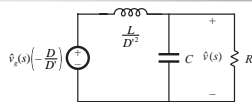
---

---

## Derivation of transfer functions

Use voltage divider formula to solve for transfer function:

$$G_{vD}(s) = \left. \frac{\hat{v}(s)}{\hat{v}_g(s)} \right|_{\hat{d}(s)=0} = -\frac{D}{D^2} \frac{\left( R \parallel \frac{1}{sC} \right)}{\frac{L}{D^2} + \left( R \parallel \frac{1}{sC} \right)}$$



Expand parallel combination and express as a rational fraction:

$$\begin{aligned} G_{vD}(s) &= \left( -\frac{D}{D^2} \right) \frac{\left( \frac{R}{1+sRC} \right)}{\frac{L}{D^2} + \left( \frac{R}{1+sRC} \right)} \\ &= \left( -\frac{D}{D^2} \right) \frac{R}{R + \frac{sL}{D^2} + \frac{s^2RLC}{D^2}} \end{aligned}$$

We aren't done yet! Need to write in normalized form, where the coefficient of  $s^0$  is 1, and then identify salient features

---

---

---

---

---

---

---

---

---

---

## Derivation of transfer functions

Divide numerator and denominator by  $R$ . Result: the line-to-output transfer function is

$$G_{vD}(s) = \left. \frac{\hat{v}(s)}{\hat{v}_g(s)} \right|_{\hat{d}(s)=0} = \left( -\frac{D}{D^2} \right) \frac{1}{1 + s \frac{L}{D^2 R} + s^2 \frac{LC}{D^2}}$$

which is of the following standard form:

$$G_{vD}(s) = G_{vD0} \frac{1}{1 + \frac{s}{Q\omega_0} + \left( \frac{s}{\omega_0} \right)^2}$$

---

---

---

---

---

---

---

---

---

---

### Salient features of the line-to-output transfer function

Equate standard form to derived transfer function, to determine expressions for the salient features:

$$G_{vo} = -\frac{D}{D'}^2$$

$$\frac{1}{\omega_0^2} = \frac{LC}{D'^2} \quad \omega_0 = \frac{D'}{\sqrt{LC}}$$

$$\frac{1}{Q\omega_0} = \frac{L}{D'^2 R} \quad Q = D' R \sqrt{\frac{C}{L}}$$

---

---

---

---

---

---

---

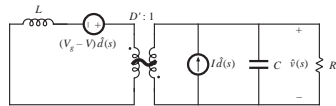
---

---

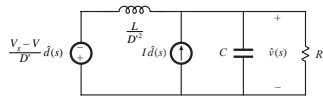
---

### Derivation of control-to-output transfer function $G_{vd}(s)$

In small-signal model, set  $\hat{v}_g$  source to zero:



Push all elements to output side of transformer:



There are two  $\hat{d}$  sources. One way to solve the model is to use superposition, expressing the output  $\hat{v}$  as a sum of terms arising from the two sources.

---

---

---

---

---

---

---

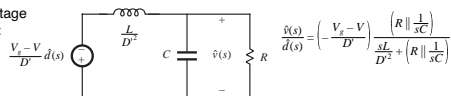
---

---

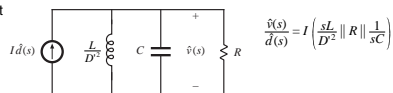
---

### Superposition

With the voltage source only:



With the current source alone:



Total: 
$$G_{vd}(s) = \left( -\frac{V_g - V}{D'} \right) \frac{\left( R \parallel \frac{1}{sC} \right)}{\frac{sL}{D'^2} + \left( R \parallel \frac{1}{sC} \right)} + I \left( \frac{sL}{D'^2} \parallel R \parallel \frac{1}{sC} \right)$$

---

---

---

---

---

---

---

---

---

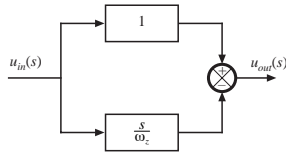
---





### 8.2.3. Physical origins of the right half-plane zero

$$G(s) = \left(1 - \frac{s}{\omega_0}\right)$$



- phase reversal at high frequency
- transient response: output initially tends in wrong direction

---

---

---

---

---

---

---

---

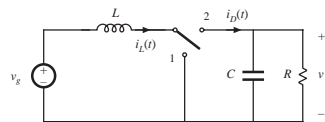
---

---

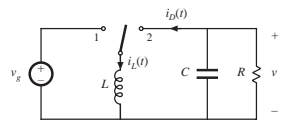
### Two converters whose CCM control-to-output transfer functions exhibit RHP zeroes

$$\langle i_b \rangle_{r_s} = d' \langle i_i \rangle_{r_s}$$

Boost



Buck-boost




---

---

---

---

---

---

---

---

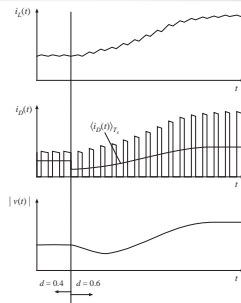
---

---

### Waveforms, step increase in duty cycle

$$\langle i_b \rangle_{r_s} = d' \langle i_i \rangle_{r_s}$$

- Increasing  $d(t)$  causes the average diode current to initially decrease
- As inductor current increases to its new equilibrium value, average diode current eventually increases




---

---

---

---

---

---

---

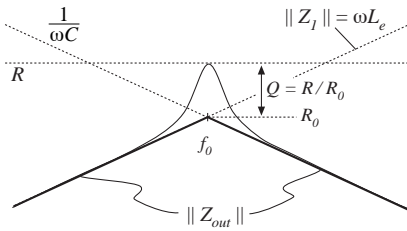
---

---

---



### Graphical construction of output impedance




---

---

---

---

---

---

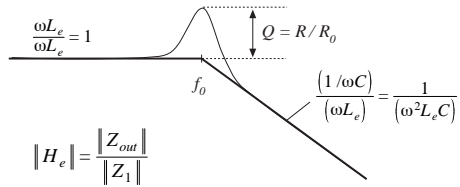
---

---

---

---

### Graphical construction of filter effective transfer function




---

---

---

---

---

---

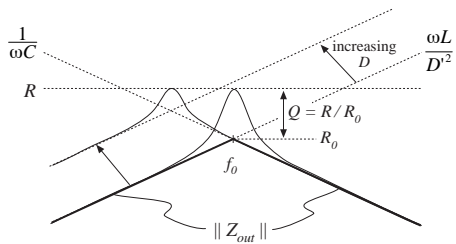
---

---

---

---

### Boost and buck-boost converters: $L_e = L / D'^2$




---

---

---

---

---

---

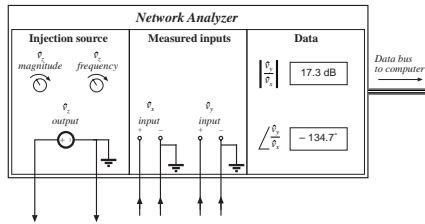
---

---

---

---

## 8.4. Measurement of ac transfer functions and impedances




---

---

---

---

---

---

---

---

---

---

## Swept sinusoidal measurements

- Injection source produces sinusoid  $\hat{v}_1$  of controllable amplitude and frequency
- Signal inputs  $\hat{v}_2$  and  $\hat{v}_1$  perform function of narrowband tracking voltmeter:
  - Component of input at injection source frequency is measured
  - Narrowband function is essential: switching harmonics and other noise components are removed
- Network analyzer measures

$$\left| \frac{\hat{v}_2}{\hat{v}_1} \right| \quad \text{and} \quad \angle \frac{\hat{v}_2}{\hat{v}_1}$$

---

---

---

---

---

---

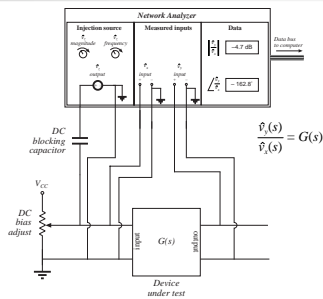
---

---

---

---

## Measurement of an ac transfer function



- Potentiometer establishes correct quiescent operating point
- Injection sinusoid coupled to device input via dc blocking capacitor
- Actual device input and output voltages are measured as  $\hat{v}_1$  and  $\hat{v}_2$
- Dynamics of blocking capacitor are irrelevant

---

---

---

---

---

---

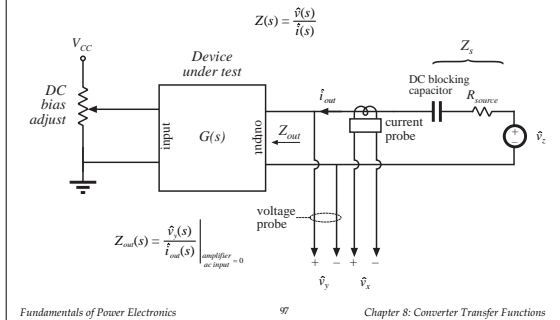
---

---

---

---

## Measurement of an output impedance




---

---

---

---

---

---

---

---

---

---

## Measurement of output impedance

- Treat output impedance as transfer function from output current to output voltage:

$$Z(s) = \frac{\hat{v}(s)}{\hat{i}(s)} \quad Z_{meas}(s) = \frac{\hat{v}_x(s)}{\hat{i}_{out}(s)} \Big|_{\text{amplifier ac input} = 0}$$

- Potentiometer at device input port establishes correct quiescent operating point
- Current probe produces voltage proportional to current; this voltage is connected to network analyzer channel  $\hat{v}_x$
- Network analyzer result must be multiplied by appropriate factor, to account for scale factors of current and voltage probes

---

---

---

---

---

---

---

---

---

---

## Measurement of small impedances

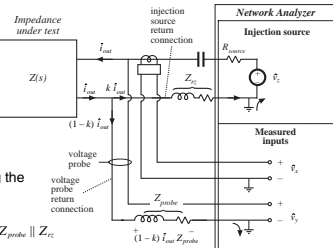
Grounding problems cause measurement to fail:

Injection current can return to analyzer via two paths. Injection current which returns via voltage probe ground induces voltage drop in voltage probe, corrupting the measurement. Network analyzer measures

$$Z + (1-k)Z_{probe} = Z + Z_{probe} \parallel Z_{rc}$$

For an accurate measurement, require

$$|Z| \gg |(Z_{probe} \parallel Z_{rc})|$$




---

---

---

---

---

---

---

---

---

---



### Summary of key points

8. Salient features of the transfer functions of the buck, boost, and buck-boost converters are tabulated in section 8.2.2. The line-to-output transfer functions of these converters contain two poles. Their control-to-output transfer functions contain two poles, and may additionally contain a right half-plane zero.
9. Approximate magnitude asymptotes of impedances and transfer functions can be easily derived by graphical construction. This approach is a useful supplement to conventional analysis, because it yields physical insight into the circuit behavior, and because it exposes suitable approximations. Several examples, including the impedances of basic series and parallel resonant circuits and the transfer function  $H_v(s)$  of the boost and buck-boost converters, are worked in section 8.3.
10. Measurement of transfer functions and impedances using a network analyzer is discussed in section 8.4. Careful attention to ground connections is important when measuring small impedances.

---

---

---

---

---

---

---

---