# Chapter 9. Controller Design





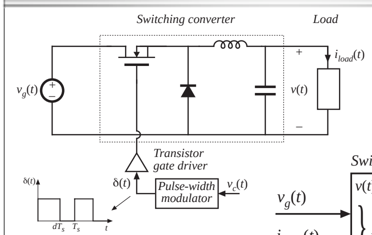
# Controller design



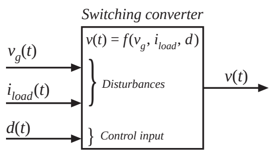


# Controller design

# 9.1. Introduction




Output voltage of a switching converter depends on duty cycle $d$, input voltage $v_g$, and load current $i_{load}$.

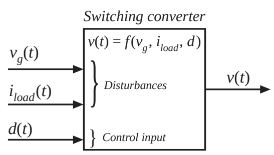





# The dc regulator application

Objective: maintain constant output voltage $v(t) = V$, in spite of disturbances in $v_g(t)$ and $i_{load}(t)$.

Typical variation in $v_g(t)$: 100Hz or 120Hz ripple, produced by rectifier circuit.




Load current variations: a significant step-change in load current, such as from 50% to 100% of rated value, may be applied.

A typical output voltage regulation specification: $5\text{V} \pm 0.1\text{V}$.

Circuit elements are constructed to some specified tolerance. In high volume manufacturing of converters, all output voltages must meet specifications.

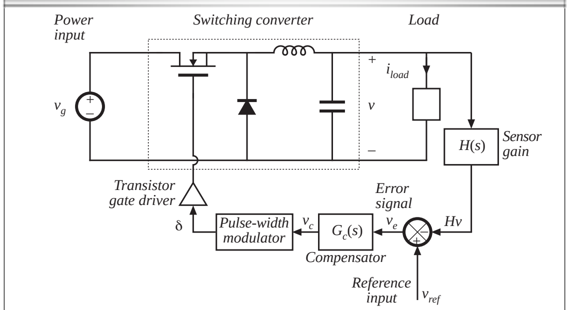


# The dc regulator application

So we cannot expect to set the duty cycle to a single value, and obtain a given constant output voltage under all conditions.

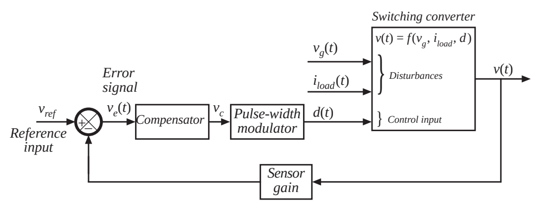
Negative feedback: build a circuit that automatically adjusts the duty cycle as necessary, to obtain the specified output voltage with high accuracy, regardless of disturbances or component tolerances.

## Negative feedback:
a switching regulator system





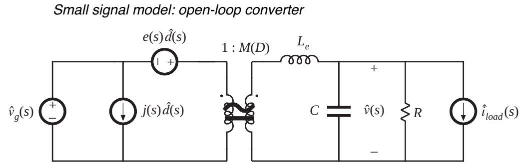

## Negative feedback






## 9.2. Effect of negative feedback on the network transfer functions

*Small signal model: open-loop converter*




Output voltage can be expressed as

$$\hat{v}(s) = G_{vd}(s)\,\hat{d}(s) + G_{vg}(s)\,\hat{v}_g(s) - Z_{out}(s)\,\hat{i}_{load}(s)$$

where

$$G_{vd}(s) = \left.\frac{\hat{v}(s)}{\hat{d}(s)}\right|_{\substack{\hat{v}_g = 0 \\ \hat{i}_{load} = 0}} \qquad G_{vg}(s) = \left.\frac{\hat{v}(s)}{\hat{v}_g(s)}\right|_{\substack{\hat{d} = 0 \\ \hat{i}_{load} = 0}} \qquad Z_{out}(s) = -\left.\frac{\hat{v}(s)}{\hat{i}_{load}(s)}\right|_{\substack{\hat{d} = 0 \\ \hat{v}_g = 0}}$$

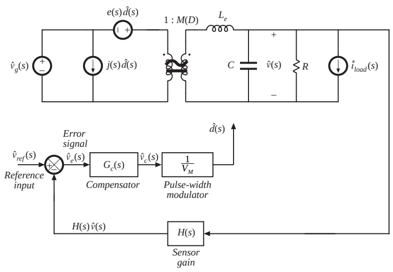
## Voltage regulator system small-signal model

- *Use small-signal converter model*
- *Perturb and linearize remainder of feedback loop:*

$v_{ref}(t) = V_{ref} + \hat{v}_{ref}(t)$

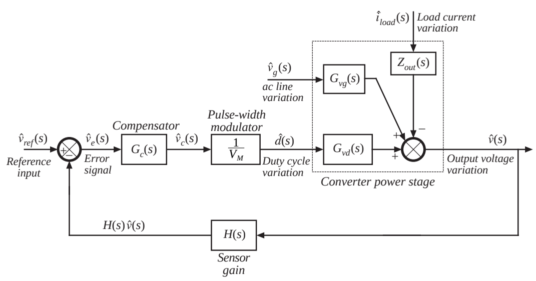
$v_e(t) = V_e + \hat{v}_e(t)$

*etc.*






## Regulator system small-signal block diagram






## Solution of block diagram

Manipulate block diagram to solve for $\hat{v}(s)$. Result is

$$\hat{v} = \hat{v}_{ref} \frac{G_c G_{vd} / V_M}{1 + H G_c G_{vd} / V_M} + \hat{v}_g \frac{G_{vg}}{1 + H G_c G_{vd} / V_M} - \hat{i}_{load} \frac{Z_{out}}{1 + H G_c G_{vd} / V_M}$$

which is of the form

$$\hat{v} = \hat{v}_{ref} \frac{1}{H} \frac{T}{1+T} + \hat{v}_g \frac{G_{vg}}{1+T} - \hat{i}_{load} \frac{Z_{out}}{1+T}$$

*with* $T(s) = H(s)\, G_c(s)\, G_{vd}(s) / V_M$ = "*loop gain*"

Loop gain $T(s)$ = products of the gains around the negative feedback loop.

## 9.2.1. Feedback reduces the transfer functions from disturbances to the output

Original (open-loop) line-to-output transfer function:

$$G_{vg}(s) = \left. \frac{\hat{v}(s)}{\hat{v}_g(s)} \right|_{\substack{\hat{d} = 0 \\ \hat{i}_{load} = 0}}$$

With addition of negative feedback, the line-to-output transfer function becomes:

$$\left. \frac{\hat{v}(s)}{\hat{v}_g(s)} \right|_{\substack{\hat{v}_{ref} = 0 \\ \hat{i}_{load} = 0}} = \frac{G_{vg}(s)}{1 + T(s)}$$

Feedback reduces the line-to-output transfer function by a factor of

$$\frac{1}{1 + T(s)}$$

If $T(s)$ is large in magnitude, then the line-to-output transfer function becomes small.



## Closed-loop output impedance

Original (open-loop) output impedance:

$$Z_{out}(s) = - \left. \frac{\hat{v}(s)}{\hat{i}_{load}(s)} \right|_{\substack{\hat{d} = 0 \\ \hat{v}_g = 0}}$$

With addition of negative feedback, the output impedance becomes:

$$\left. \frac{\hat{v}(s)}{-\hat{i}_{load}(s)} \right|_{\substack{\hat{v}_{ref} = 0 \\ \hat{v}_g = 0}} = \frac{Z_{out}(s)}{1 + T(s)}$$

Feedback reduces the output impedance by a factor of

$$\frac{1}{1 + T(s)}$$

If $T(s)$ is large in magnitude, then the output impedance is greatly reduced in magnitude.



## 9.2.2. Feedback causes the transfer function from the reference input to the output to be insensitive to variations in the gains in the forward path of the loop

Closed-loop transfer function from $\hat{v}_{ref}$ to $\hat{v}(s)$ is:

$$\left. \frac{\hat{v}(s)}{\hat{v}_{ref}(s)} \right|_{\substack{\hat{v}_g = 0 \\ \hat{i}_{load} = 0}} = \frac{1}{H(s)} \frac{T(s)}{1 + T(s)}$$

If the loop gain is large in magnitude, i.e., $\| T \| >> 1$, then $(1+T) \approx T$ and $T/(1+T) \approx T/T = 1$. The transfer function then becomes

$$\frac{\hat{v}(s)}{\hat{v}_{ref}(s)} \approx \frac{1}{H(s)}$$

which is independent of the gains in the forward path of the loop.

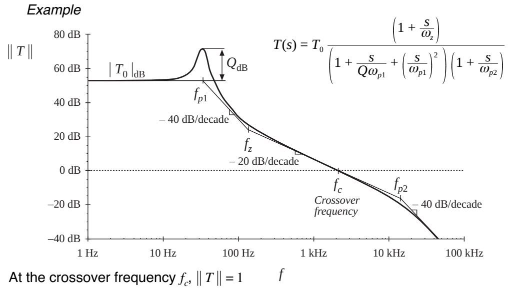
This result applies equally well to dc values:

$$\frac{V}{V_{ref}} = \frac{1}{H(0)} \frac{T(0)}{1 + T(0)} \approx \frac{1}{H(0)}$$

## 9.3. Construction of the important quantities $1/(1+T)$ and $T/(1+T)$

*Example*

$$T(s) = T_0 \frac{\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{Q\omega_{p1}} + \left(\frac{s}{\omega_{p1}}\right)^2\right)\left(1 + \frac{s}{\omega_{p2}}\right)}$$




At the crossover frequency $f_c$, $\| T \| = 1$



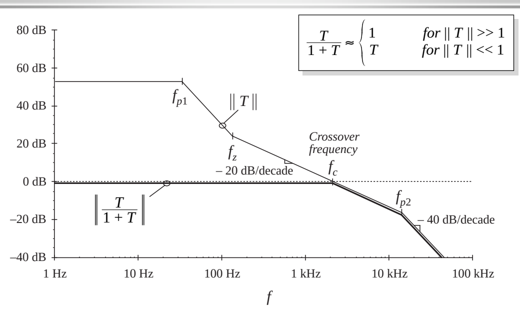
## Approximating $1/(1+T)$ and $T/(1+T)$

$$\frac{T}{1+T} \approx \begin{cases} 1 & \text{for } \| T \| >> 1 \\ T & \text{for } \| T \| << 1 \end{cases}$$

$$\frac{1}{1+T(s)} \approx \begin{cases} \frac{1}{T(s)} & \text{for } \| T \| >> 1 \\ 1 & \text{for } \| T \| << 1 \end{cases}$$



## Example: construction of $T/(1+T)$



## Example: analytical expressions for approximate reference to output transfer function

At frequencies sufficiently less that the crossover frequency, the loop gain $T(s)$ has large magnitude. The transfer function from the reference to the output becomes

$$\frac{\hat{v}(s)}{\hat{v}_{ref}(s)} = \frac{1}{H(s)}\frac{T(s)}{1+T(s)} \approx \frac{1}{H(s)}$$

This is the desired behavior: the output follows the reference according to the ideal gain $1/H(s)$. The feedback loop works well at frequencies where the loop gain $T(s)$ has large magnitude.

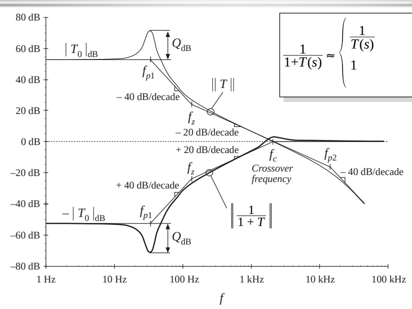
At frequencies above the crossover frequency, $\| T \| < 1$. The quantity $T/(1+T)$ then has magnitude approximately equal to 1, and we obtain

$$\frac{\hat{v}(s)}{\hat{v}_{ref}(s)} = \frac{1}{H(s)}\frac{T(s)}{1+T(s)} \approx \frac{T(s)}{H(s)} = \frac{G_c(s)G_{vd}(s)}{V_M}$$

This coincides with the open-loop transfer function from the reference to the output. At frequencies where $\| T \| < 1$, the loop has essentially no effect on the transfer function from the reference to the output.



## Same example: construction of 1/(1+T)




$$\frac{1}{1+T(s)} \approx \begin{cases} \frac{1}{T(s)} & \text{for } \| T \| >> 1 \\ 1 & \text{for } \| T \| << 1 \end{cases}$$



## Interpretation: how the loop rejects disturbances

Below the crossover frequency: $f < f_c$ and $\| T \| > 1$

- Then $1/(1+T) \approx 1/T$, and disturbances are reduced in magnitude by $1/\| T \|$

Above the crossover frequency: $f > f_c$ and $\| T \| < 1$

- Then $1/(1+T) \approx 1$, and the feedback loop has essentially no effect on disturbances

$$\frac{1}{1+T(s)} \approx \begin{cases} \frac{1}{T(s)} & \text{for } \| T \| >> 1 \\ 1 & \text{for } \| T \| << 1 \end{cases}$$

## Terminology: open-loop vs. closed-loop

Original transfer functions, before introduction of feedback ("open-loop transfer functions"):

$$G_{vd}(s) \qquad G_{vg}(s) \qquad Z_{out}(s)$$

Upon introduction of feedback, these transfer functions become ("closed-loop transfer functions"):

$$\frac{1}{H(s)}\frac{T(s)}{1+T(s)} \qquad \frac{G_{vg}(s)}{1+T(s)} \qquad \frac{Z_{out}(s)}{1+T(s)}$$

The loop gain:

$$T(s)$$



## 9.4. Stability

Even though the original open-loop system is stable, the closed-loop transfer functions can be unstable and contain right half-plane poles. Even when the closed-loop system is stable, the transient response can exhibit undesirable ringing and overshoot, due to the high $Q$ -factor of the closed-loop poles in the vicinity of the crossover frequency.

When feedback destabilizes the system, the denominator $(1+T(s))$ terms in the closed-loop transfer functions contain roots in the right half-plane (i.e., with positive real parts). If $T(s)$ is a rational fraction of the form $N(s) / D(s)$, where $N(s)$ and $D(s)$ are polynomials, then we can write

$$\frac{T(s)}{1+T(s)} = \frac{\frac{N(s)}{D(s)}}{1+\frac{N(s)}{D(s)}} = \frac{N(s)}{N(s)+D(s)}$$

$$\frac{1}{1+T(s)} = \frac{1}{1+\frac{N(s)}{D(s)}} = \frac{D(s)}{N(s)+D(s)}$$

- Could evaluate stability by evaluating $N(s) + D(s)$, then factoring to evaluate roots. This is a lot of work, and is not very illuminating.



## Determination of stability directly from $T(s)$

- Nyquist stability theorem: general result.
- A special case of the Nyquist stability theorem: the phase margin test

  Allows determination of closed-loop stability (i.e., whether $1/(1+T(s))$ contains RHP poles) directly from the magnitude and phase of $T(s)$.

  A good design tool: yields insight into how $T(s)$ should be shaped, to obtain good performance in transfer functions containing $1/(1+T(s))$ terms.

## 9.4.1. The phase margin test

A test on $T(s)$, to determine whether $1/(1+T(s))$ contains RHP poles.

The crossover frequency $f_c$ is defined as the frequency where

$$\| T(j2\pi f_c) \| = 1 \Rightarrow 0\text{dB}$$

The phase margin $\varphi_m$ is determined from the phase of $T(s)$ at $f_c$ , as follows:

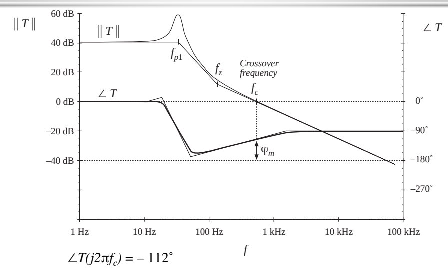
$$\varphi_m = 180^\circ + \angle T(j2\pi f_c)$$

If there is exactly one crossover frequency, and if $T(s)$ contains no RHP poles, then

the quantities $T(s)/(1+T(s))$ and $1/(1+T(s))$ contain no RHP poles whenever the phase margin $\varphi_m$ is positive.



## Example: a loop gain leading to a stable closed-loop system

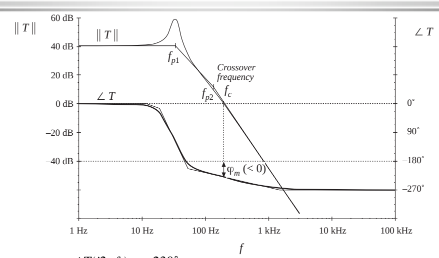



$\angle T(j2\pi f_c) = -\ 112^\circ$

$\varphi_m = 180^\circ - 112^\circ = +\ 68^\circ$



## Example: a loop gain leading to an unstable closed-loop system




$\angle T(j2\pi f_c) = -\ 230^\circ$

$\varphi_m = 180^\circ - 230^\circ = -\ 50^\circ$

## 9.4.2. The relation between phase margin and closed-loop damping factor

How much phase margin is required?

A small positive phase margin leads to a stable closed-loop system having complex poles near the crossover frequency with high $Q$. The transient response exhibits overshoot and ringing.

Increasing the phase margin reduces the $Q$. Obtaining real poles, with no overshoot and ringing, requires a large phase margin.

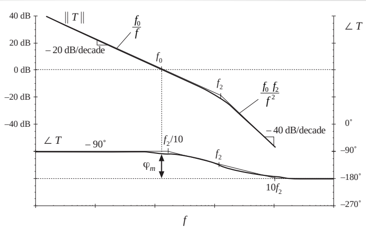
The relation between phase margin and closed-loop $Q$ is quantified in this section.



## A simple second-order system

Consider the case where $T(s)$ can be well-approximated in the vicinity of the crossover frequency as

$$T(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{s}{\omega_2}\right)}$$






## Closed-loop response

If

$$T(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

Then

$$\frac{T(s)}{1 + T(s)} = \frac{1}{1 + \frac{1}{T(s)}} = \frac{1}{1 + \frac{s}{\omega_0} + \frac{s^2}{\omega_0 \omega_2}}$$

or,

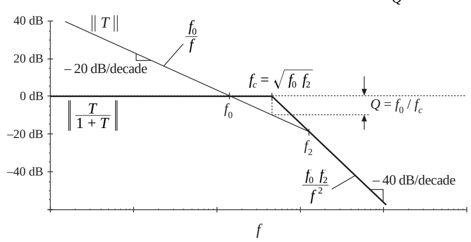
$$\frac{T(s)}{1 + T(s)} = \frac{1}{1 + \frac{s}{Q\omega_c} + \left(\frac{s}{\omega_c}\right)^2}$$

where

$$\omega_c = \sqrt{\omega_0 \omega_2} = 2\pi f_c \qquad Q = \frac{\omega_0}{\omega_c} = \sqrt{\frac{\omega_0}{\omega_2}}$$

# Low-$Q$ case

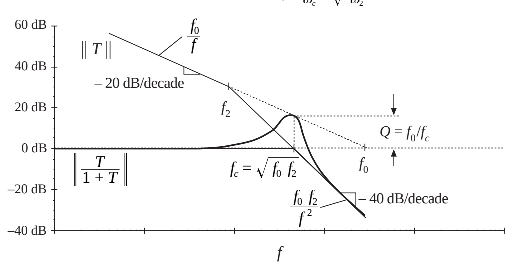
$Q = \frac{\omega_0}{\omega_c} = \sqrt{\frac{\omega_0}{\omega_2}}$ low-$Q$ approximation: $Q\,\omega_c = \omega_0$ $\frac{\omega_c}{Q} = \omega_2$






# High-$Q$ case

$\omega_c = \sqrt{\omega_0\omega_2} = 2\pi f_c$ $Q = \frac{\omega_0}{\omega_c} = \sqrt{\frac{\omega_0}{\omega_2}}$





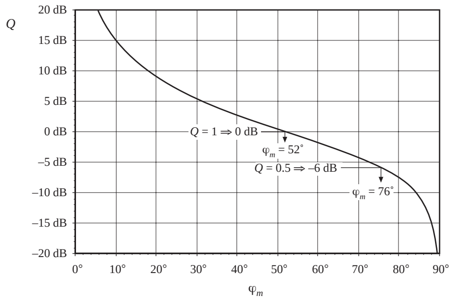

# $Q$ vs. $\varphi_m$

Solve for exact crossover frequency, evaluate phase margin, express as function of $\varphi_m$. Result is:

$$Q = \frac{\sqrt{\cos \varphi_m}}{\sin \varphi_m}$$

$$\varphi_m = \tan^{-1}\sqrt{\frac{1 + \sqrt{1 + 4Q^4}}{2Q^4}}$$

# $Q$ vs. $\varphi_m$






# 9.4.3. Transient response vs. damping factor

Unit-step response of second-order system $T(s)/(1+T(s))$

$$\hat{v}(t) = 1 + \frac{2Q\, e^{-\omega_c t/2Q}}{\sqrt{4Q^2-1}} \sin\left[\frac{\sqrt{4Q^2-1}}{2Q}\,\omega_c t + \tan^{-1}\left(\sqrt{4Q^2-1}\right)\right] \qquad Q > 0.5$$

$$\hat{v}(t) = 1 - \frac{\omega_2}{\omega_2-\omega_1} e^{-\omega_1 t} - \frac{\omega_1}{\omega_1-\omega_2} e^{-\omega_2 t} \qquad Q < 0.5$$

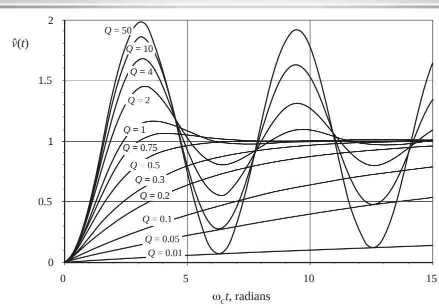
$$\omega_1,\ \omega_2 = \frac{\omega_c}{2Q}\left(1 \pm \sqrt{1-4Q^2}\right)$$

For $Q > 0.5$ , the peak value is

$$peak\ \hat{v}(t) = 1 + e^{-\pi/\sqrt{4Q^2-1}}$$



# Transient response vs. damping factor



## 9.5. Regulator design

Typical specifications:

- Effect of load current variations on output voltage regulation
  This is a limit on the maximum allowable output impedance
- Effect of input voltage variations on the output voltage regulation
  This limits the maximum allowable line-to-output transfer function
- Transient response time
  This requires a sufficiently high crossover frequency
- Overshoot and ringing
  An adequate phase margin must be obtained

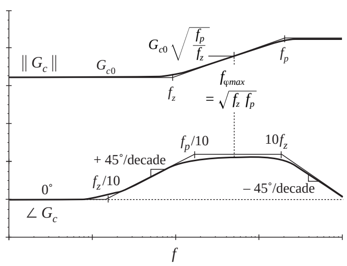
The regulator design problem: add compensator network $G_c(s)$ to modify $T(s)$ such that all specifications are met.



## 9.5.1. Lead (PD) compensator

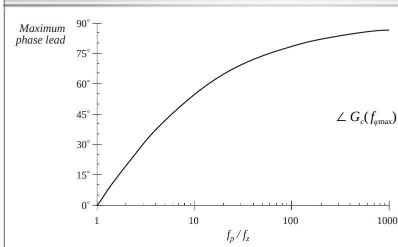
$$G_c(s) = G_{c0}\frac{\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{\omega_p}\right)}$$

Improves phase margin






## Lead compensator: maximum phase lead




$$f_{\varphi max} = \sqrt{f_z f_p}$$

$$\angle G_c(f_{\varphi max}) = \tan^{-1}\left(\frac{\sqrt{\frac{f_p}{f_z}} - \sqrt{\frac{f_z}{f_p}}}{2}\right)$$

$$\frac{f_p}{f_z} = \frac{1 + \sin(\theta)}{1 - \sin(\theta)}$$

## Lead compensator design

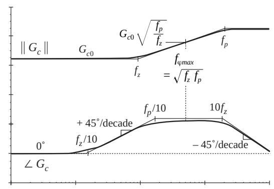
To optimally obtain a compensator phase lead of θ at frequency $f_c$, the pole and zero frequencies should be chosen as follows:

$$f_z = f_c \sqrt{\frac{1 - \sin(\theta)}{1 + \sin(\theta)}}$$

$$f_p = f_c \sqrt{\frac{1 + \sin(\theta)}{1 - \sin(\theta)}}$$

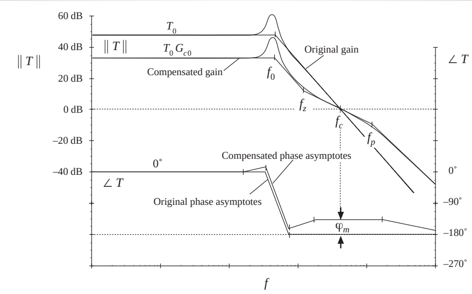
If it is desired that the magnitude of the compensator gain at $f_c$ be unity, then $G_{c0}$ should be chosen as

$$G_{c0} = \sqrt{\frac{f_z}{f_p}}$$






## Example: lead compensation

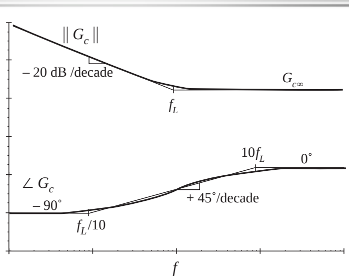





## 9.5.2. Lag (PI) compensation

$$G_c(s) = G_{c\infty}\left(1 + \frac{\omega_L}{s}\right)$$

Improves low-frequency loop gain and regulation



## Example: lag compensation

original (uncompensated) loop gain is

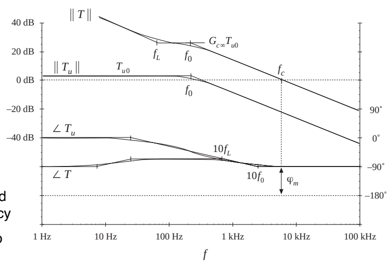
$$T_u(s) = \frac{T_{u0}}{\left(1 + \frac{s}{\omega_0}\right)}$$

compensator:

$$G_c(s) = G_{c\infty}\left(1 + \frac{\omega_L}{s}\right)$$

Design strategy: choose

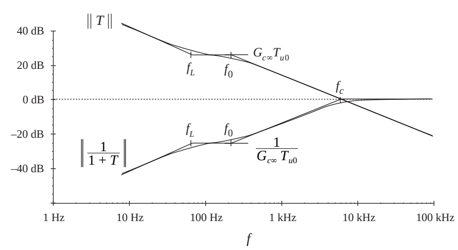
- $G_{c\infty}$ to obtain desired crossover frequency
- $\omega_L$ sufficiently low to maintain adequate phase margin






## Example, continued

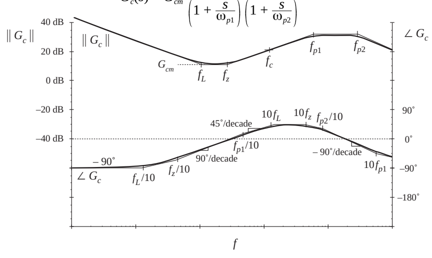
*Construction of 1/(1+T), lag compensator example:*



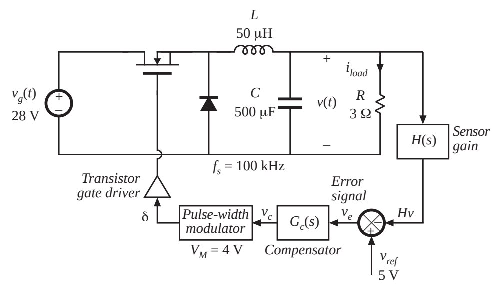



## 9.5.3. Combined (PID) compensator

$$G_c(s) = G_{cm}\frac{\left(1 + \frac{\omega_L}{s}\right)\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{\omega_{p1}}\right)\left(1 + \frac{s}{\omega_{p2}}\right)}$$



## 9.5.4. Design example



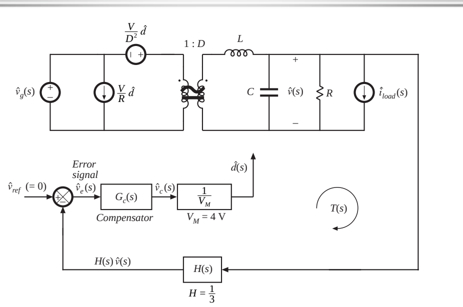



## Quiescent operating point

| | |
|---|---|
| Input voltage | $V_g = 28\text{V}$ |
| Output | $V = 15\text{V},\ I_{load} = 5\text{A},\ R = 3\Omega$ |
| Quiescent duty cycle | $D = 15/28 = 0.536$ |
| Reference voltage | $V_{ref} = 5\text{V}$ |
| Quiescent value of control voltage | $V_c = DV_M = 2.14\text{V}$ |
| Gain $H(s)$ | $H = V_{ref}/V = 5/15 = 1/3$ |



## Small-signal model



## Open-loop control-to-output transfer function $G_{vd}(s)$

$$G_{vd}(s) = \frac{V}{D} \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

standard form:

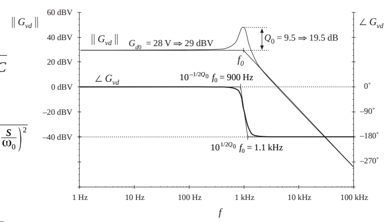
$$G_{vd}(s) = G_{d0} \frac{1}{1 + \frac{s}{Q_0 \omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

salient features:

$$G_{d0} = \frac{V}{D} = 28\text{V}$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 1\text{kHz}$$

$$Q_0 = R\sqrt{\frac{C}{L}} = 9.5 \Rightarrow 19.5\text{dB}$$






## Open-loop line-to-output transfer function and output impedance

$$G_{vg}(s) = D \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

—same poles as control-to-output transfer function
standard form:

$$G_{vg}(s) = G_{g0} \frac{1}{1 + \frac{s}{Q_0 \omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Output impedance:

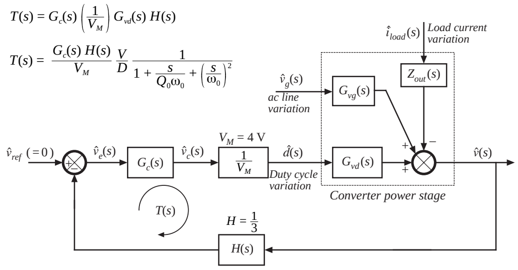
$$Z_{out}(s) = R \parallel \frac{1}{sC} \parallel sL = \frac{sL}{1 + s\frac{L}{R} + s^2 LC}$$



## System block diagram

$$T(s) = G_c(s) \left(\frac{1}{V_M}\right) G_{vd}(s)\, H(s)$$

$$T(s) = \frac{G_c(s)\, H(s)}{V_M} \frac{V}{D} \frac{1}{1 + \frac{s}{Q_0 \omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

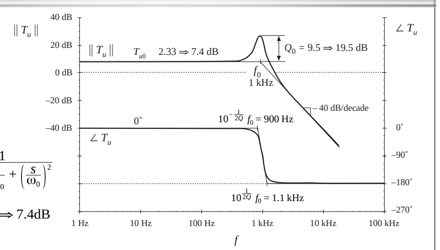


## Uncompensated loop gain (with $G_c = 1$)

With $G_c = 1$, the loop gain is

$$T_u(s) = T_{u0} \frac{1}{1 + \frac{s}{Q_0 \omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

$$T_{u0} = \frac{H}{D} \frac{V}{V_M} = 2.33 \Rightarrow 7.4\text{dB}$$




$f_c = 1.8$ kHz, $\varphi_m = 5^\circ$



## Lead compensator design

- Obtain a crossover frequency of 5 kHz, with phase margin of 52˚
- $T_u$ has phase of approximately – 180˚ at 5 kHz, hence lead (PD) compensator is needed to increase phase margin.
- Lead compensator should have phase of + 52˚ at 5 kHz
- $T_u$ has magnitude of – 20.6 dB at 5 kHz
- Lead compensator gain should have magnitude of + 20.6 dB at 5 kHz
- Lead compensator pole and zero frequencies should be

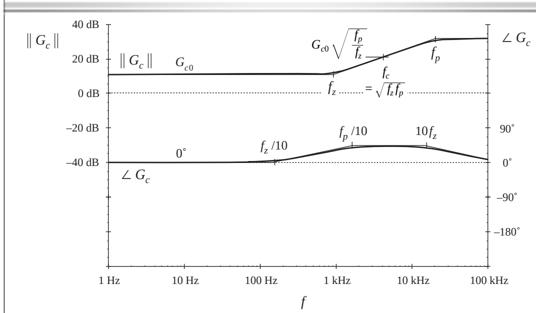
$$f_z = (5\text{kHz}) \sqrt{\frac{1 - \sin(52°)}{1 + \sin(52°)}} = 1.7\text{kHz}$$

$$f_p = (5\text{kHz}) \sqrt{\frac{1 + \sin(52°)}{1 - \sin(52°)}} = 14.5\text{kHz}$$

- Compensator dc gain should be $G_{c0} = \left(\frac{f_c}{f_0}\right)^2 \frac{1}{T_{u0}} \sqrt{\frac{f_z}{f_p}} = 3.7 \Rightarrow 11.3\text{dB}$

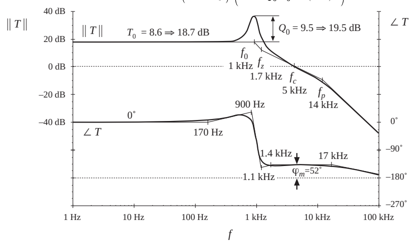


## Lead compensator Bode plot



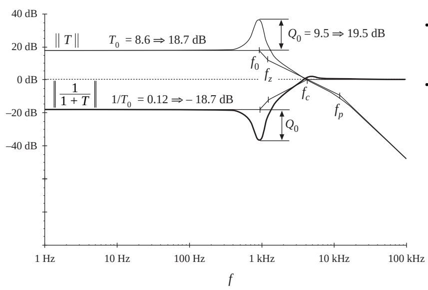
## Loop gain, with lead compensator

$$T(s) = T_{u0}\, G_{c0} \frac{\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{\omega_p}\right)\left(1 + \frac{s}{Q_0\omega_0} + \left(\frac{s}{\omega_0}\right)^2\right)}$$






## 1/(1+T), with lead compensator

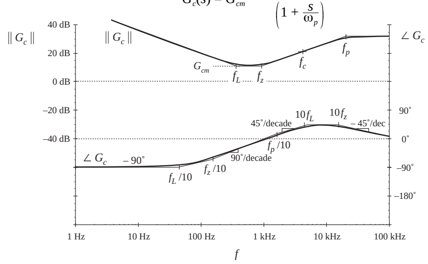



- need more low-frequency loop gain
- hence, add inverted zero (PID controller)



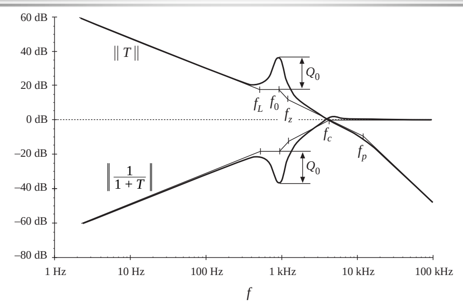
## Improved compensator (PID)

$$G_c(s) = G_{cm} \frac{\left(1 + \frac{s}{\omega_z}\right)\left(1 + \frac{\omega_L}{s}\right)}{\left(1 + \frac{s}{\omega_p}\right)}$$



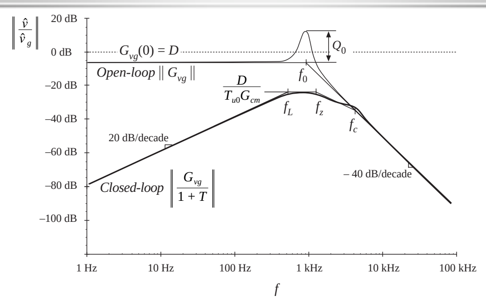

- add inverted zero to PD compensator, without changing dc gain or corner frequencies
- choose $f_L$ to be $f_c$/10, so that phase margin is unchanged

## $T(s)$ and $1/(1+T(s))$, with PID compensator



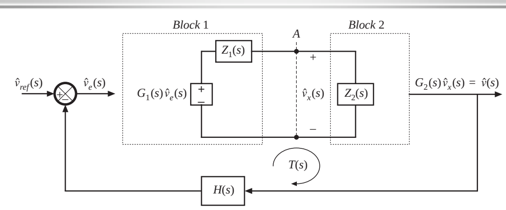



## Line-to-output transfer function






## 9.6. Measurement of loop gains

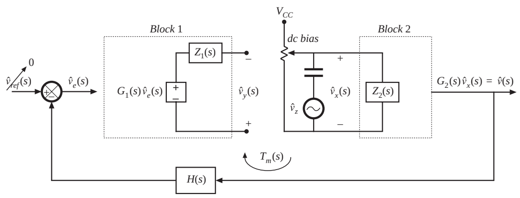



Objective: experimentally determine loop gain $T(s)$, by making measurements at point $A$

Correct result is

$$T(s) = G_1(s)\left(\frac{Z_2(s)}{Z_1(s)+Z_2(s)}\right)G_2(s)\,H(s)$$

## Conventional approach: break loop, measure $T(s)$ as conventional transfer function




measured gain is

$$T_m(s) = \left.\frac{\hat{v}_y(s)}{\hat{v}_x(s)}\right|_{\substack{\hat{v}_{ref}=0 \\ \hat{v}_g=0}} \qquad T_m(s) = G_1(s)\, G_2(s)\, H(s)$$



## Measured vs. actual loop gain

Actual loop gain:

$$T(s) = G_1(s)\left(\frac{Z_2(s)}{Z_1(s) + Z_2(s)}\right) G_2(s)\, H(s)$$

Measured loop gain:

$$T_m(s) = G_1(s)\, G_2(s)\, H(s)$$

Express $T_m$ as function of $T$:

$$T_m(s) = T(s)\left(1 + \frac{Z_1(s)}{Z_2(s)}\right)$$

$$T_m(s) \approx T(s) \quad \text{provided that} \quad \| Z_2 \| >> \| Z_1 \|$$



## Discussion

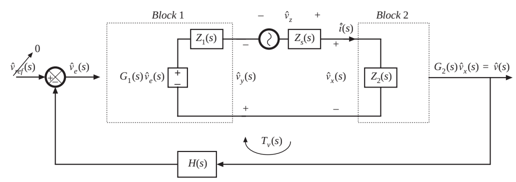
- Breaking the loop disrupts the loading of block 2 on block 1.

  A suitable injection point must be found, where loading is not significant.
- Breaking the loop disrupts the dc biasing and quiescent operating point.

  A potentiometer must be used, to correctly bias the input to block 2.

  In the common case where the dc loop gain is large, it is very difficult to correctly set the dc bias.
- It would be desirable to avoid breaking the loop, such that the biasing circuits of the system itself set the quiescent operating point.

## 9.6.1. Voltage injection



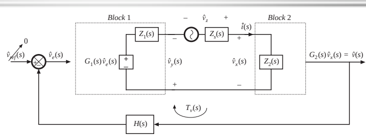

- Ac injection source $v_z$ is connected between blocks 1 and 2
- Dc bias is determined by biasing circuits of the system itself
- Injection source does modify loading of block 2 on block 1



## Voltage injection: measured transfer function $T_v(s)$

Network analyzer measures

$$T_v(s) = \left.\frac{\hat{v}_y(s)}{\hat{v}_x(s)}\right|_{\substack{\hat{v}_{ref}=0\\ \hat{v}_g=0}}$$



Solve block diagram:

$$\hat{v}_e(s) = -H(s)\,G_2(s)\,\hat{v}_x(s)$$

$$-\hat{v}_y(s) = G_1(s)\,\hat{v}_e(s) - \hat{i}(s)\,Z_1(s)$$

Hence

$$-\hat{v}_y(s) = -\hat{v}_x(s)\,G_2(s)\,H(s)\,G_1(s) - \hat{i}(s)\,Z_1(s)$$

with

$$\hat{i}(s) = \frac{\hat{v}_x(s)}{Z_2(s)}$$

Substitute:

$$\hat{v}_y(s) = \hat{v}_x(s)\left(G_1(s)\,G_2(s)\,H(s) + \frac{Z_1(s)}{Z_2(s)}\right)$$

which leads to the measured gain

$$T_v(s) = G_1(s)\,G_2(s)\,H(s) + \frac{Z_1(s)}{Z_2(s)}$$



## Comparison of $T_v(s)$ with $T(s)$

Actual loop gain is

$$T(s) = G_1(s)\left(\frac{Z_2(s)}{Z_1(s)+Z_2(s)}\right)G_2(s)\,H(s)$$

Gain measured via voltage injection:

$$T_v(s) = G_1(s)\,G_2(s)\,H(s) + \frac{Z_1(s)}{Z_2(s)}$$

Express $Tv(s)$ in terms of $T(s)$:

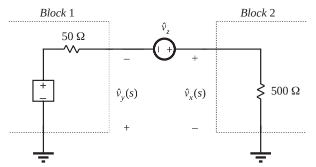
$$T_v(s) = T(s)\left(1 + \frac{Z_1(s)}{Z_2(s)}\right) + \frac{Z_1(s)}{Z_2(s)}$$

Condition for accurate measurement:

$$T_v(s) \approx T(s) \quad \text{provided} \quad (i)\ \left\|Z_1(s)\right\| << \left\|Z_2(s)\right\|, \quad \text{and}$$

$$(ii)\ \left\|T(s)\right\| >> \left\|\frac{Z_1(s)}{Z_2(s)}\right\|$$

## Example: voltage injection



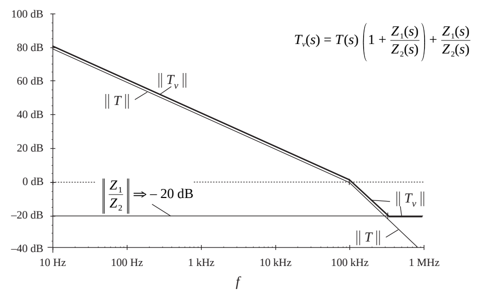

$$Z_1(s) = 50\Omega$$
$$Z_2(s) = 500\Omega$$
$$\frac{Z_1(s)}{Z_2(s)} = 0.1 \Rightarrow -20\text{dB}$$
$$\left(1 + \frac{Z_1(s)}{Z_2(s)}\right) = 1.1 \Rightarrow 0.83\text{dB}$$

$$\text{suppose actual } \; T(s) = \frac{10^4}{\left(1 + \frac{s}{2\pi\ 10\text{Hz}}\right)\left(1 + \frac{s}{2\pi\ 100\text{kHz}}\right)}$$



## Example: measured $T_v(s)$ and actual $T(s)$

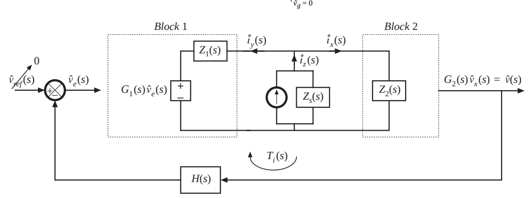
$$T_v(s) = T(s)\left(1 + \frac{Z_1(s)}{Z_2(s)}\right) + \frac{Z_1(s)}{Z_2(s)}$$






## 9.6.2. Current injection

$$T_i(s) = \left.\frac{\hat{i}_y(s)}{\hat{i}_x(s)}\right|_{\substack{\hat{v}_{ref} = 0 \\ \hat{v}_g = 0}}$$



## Current injection

It can be shown that

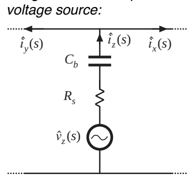
$$T_i(s) = T(s)\left(1 + \frac{Z_2(s)}{Z_1(s)}\right) + \frac{Z_2(s)}{Z_1(s)}$$

Conditions for obtaining accurate measurement:

(*i*) $\left\| Z_2(s) \right\| << \left\| Z_1(s) \right\|$, and

(*ii*) $\left\| T(s) \right\| >> \left\| \frac{Z_2(s)}{Z_1(s)} \right\|$

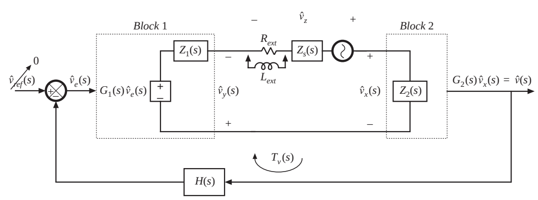
*Injection source impedance $Z_s$ is irrelevant. We could inject using a Thevenin-equivalent voltage source:*






## 9.6.3. Measurement of unstable systems

- Injection source impedance $Z_s$ does not affect measurement
- Increasing $Z_s$ reduces loop gain of circuit, tending to stabilize system
- Original (unstable) loop gain is measured (not including $Z_s$ ), while circuit operates stabily






## 9.7. Summary of key points

1. Negative feedback causes the system output to closely follow the reference input, according to the gain $1/H(s)$. The influence on the output of disturbances and variation of gains in the forward path is reduced.
2. The loop gain $T(s)$ is equal to the products of the gains in the forward and feedback paths. The loop gain is a measure of how well the feedback system works: a large loop gain leads to better regulation of the output. The crossover frequency $f_c$ is the frequency at which the loop gain $T$ has unity magnitude, and is a measure of the bandwidth of the control system.

## Summary of key points

3. The introduction of feedback causes the transfer functions from disturbances to the output to be multiplied by the factor $1/(1+T(s))$. At frequencies where $T$ is large in magnitude (i.e., below the crossover frequency), this factor is approximately equal to $1/T(s)$. Hence, the influence of low-frequency disturbances on the output is reduced by a factor of $1/T(s)$. At frequencies where $T$ is small in magnitude (i.e., above the crossover frequency), the factor is approximately equal to 1. The feedback loop then has no effect. Closed-loop disturbance-to-output transfer functions, such as the line-to-output transfer function or the output impedance, can easily be constructed using the algebra-on-the-graph method.
4. Stability can be assessed using the phase margin test. The phase of $T$ is evaluated at the crossover frequency, and the stability of the important closed-loop quantities $T/(1+T)$ and $1/(1+T)$ is then deduced. Inadequate phase margin leads to ringing and overshoot in the system transient response, and peaking in the closed-loop transfer functions.



## Summary of key points

5. Compensators are added in the forward paths of feedback loops to shape the loop gain, such that desired performance is obtained. Lead compensators, or *PD* controllers, are added to improve the phase margin and extend the control system bandwidth. *PI* controllers are used to increase the low-frequency loop gain, to improve the rejection of low-frequency disturbances and reduce the steady-state error.
6. Loop gains can be experimentally measured by use of voltage or current injection. This approach avoids the problem of establishing the correct quiescent operating conditions in the system, a common difficulty in systems having a large dc loop gain. An injection point must be found where interstage loading is not significant. Unstable loop gains can also be measured.