Introduction to Power Electronics

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Lecture 18
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6.3.4. Flyback converter

*buck-boost converter:*

\[ V_g \quad L \quad V \]

construct inductor winding using two parallel wires:

\[ V_g \quad L \quad 1:1 \quad V \]
Derivation of flyback converter, cont.

Isolate inductor windings: the flyback converter

Flyback converter having a $1:n$ turns ratio and positive output:
The “flyback transformer”

- A two-winding inductor
- Symbol is same as transformer, but function differs significantly from ideal transformer
- Energy is stored in magnetizing inductance
- Magnetizing inductance is relatively small

- Current does not simultaneously flow in primary and secondary windings
- Instantaneous winding voltages follow turns ratio
- Instantaneous (and rms) winding currents do not follow turns ratio
- Model as (small) magnetizing inductance in parallel with ideal transformer
Subinterval 1

\[ v_L = V_g \]
\[ i_C = -\frac{V}{R} \]
\[ i_g = I \]

CCM: small ripple approximation leads to

Q₁ on, D₁ off

Transformer model

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Chapter 6: Converter circuits
Subinterval 2

\[ v_L = -\frac{v}{n} \]
\[ i_C = \frac{i}{n} - \frac{v}{R} \]
\[ i_g = 0 \]

CCM: small ripple approximation leads to

\[ v_L = -\frac{V}{n} \]
\[ i_C = \frac{I}{n} - \frac{V}{R} \]
\[ i_g = 0 \]

Q\textsubscript{1} off, D\textsubscript{1} on
CCM Flyback waveforms and solution

Volt-second balance:
\[ \langle v_L \rangle = D \langle V_g \rangle + D' \left( -\frac{V}{R} \right) = 0 \]

Conversion ratio is
\[ M(D) = \frac{V}{V_g} = n \frac{D}{D'} \]

Charge balance:
\[ \langle i_C \rangle = D \left( -\frac{V}{R} \right) + D' \left( \frac{I}{n} - \frac{V}{R} \right) = 0 \]

Dc component of magnetizing current is
\[ I = \frac{nV}{D'R} \]

Dc component of source current is
\[ I_s = \langle i_s \rangle = D \langle I \rangle + D' \langle 0 \rangle \]
Equivalent circuit model: CCM Flyback

\[ \langle v_L \rangle = D(V_g) + D\left( -\frac{V}{n} \right) = 0 \]

\[ \langle i_C \rangle = D\left( -\frac{V}{R} \right) + D\left( \frac{I}{n} - \frac{V}{R} \right) = 0 \]

\[ I_g = \langle i_g \rangle = D(I) + D'(0) \]
Discussion: Flyback converter

- Widely used in low power and/or high voltage applications
- Low parts count
- Multiple outputs are easily obtained, with minimum additional parts
- Cross regulation is inferior to buck-derived isolated converters
- Often operated in discontinuous conduction mode
- DCM analysis: DCM buck-boost with turns ratio
6.3.5. Boost-derived isolated converters

- A wide variety of boost-derived isolated dc-dc converters can be derived, by inversion of source and load of buck-derived isolated converters:
  - full-bridge and half-bridge isolated boost converters
  - inverse of forward converter: the “reverse” converter
  - push-pull boost-derived converter

Of these, the full-bridge and push-pull boost-derived isolated converters are the most popular, and are briefly discussed here.
Full-bridge transformer-isolated boost-derived converter

- Circuit topologies are equivalent to those of nonisolated boost converter
- With 1:1 turns ratio, inductor current $i(t)$ and output current $i_o(t)$ waveforms are identical to nonisolated boost converter
Transformer reset mechanism

- As in full-bridge buck topology, transformer volt-second balance is obtained over two switching periods.

- During first switching period: transistors $Q_1$ and $Q_4$ conduct for time $DT_s$, applying volt-seconds $VD_T$ to secondary winding.

- During next switching period: transistors $Q_2$ and $Q_3$ conduct for time $DT_s$, applying volt-seconds $-VD_T$ to secondary winding.
**Conversion ratio $M(D)$**

Application of volt-second balance to inductor voltage waveform:

$$\langle v_L \rangle = D(V_g) + D(V_g - \frac{V}{n}) = 0$$

Solve for $M(D)$:

$$M(D) = \frac{V}{V_g} = \frac{n}{D}$$

—boost with turns ratio $n$
Push-pull boost-derived converter

\[ M(D) = \frac{V}{V_g} = \frac{n}{D} \]
Push-pull converter based on Watkins-Johnson converter
6.3.6. Isolated versions of the SEPIC and Cuk converter

Basic nonisolated SEPIC

Isolated SEPIC

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Chapter 6: Converter circuits
Isolated SEPIC

\[ M(D) = \frac{V}{V_g} = \frac{nD}{D} \]
Inverse SEPIC

Nonisolated inverse SEPIC

Isolated inverse SEPIC
Obtaining isolation in the Cuk converter

**Nonisolated Cuk converter**

**Split capacitor \( C_1 \) into series capacitors \( C_{1a} \) and \( C_{1b} \)**

![Cuk converter circuit diagram]

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Chapter 6: Converter circuits
Isolated Cuk converter

Insert transformer between capacitors $C_{1a}$ and $C_{1b}$

$$M(D) = \frac{V}{V_g} = \frac{nD}{D'}$$

Discussion

- Capacitors $C_{1a}$ and $C_{1b}$ ensure that no dc voltage is applied to transformer primary or secondary windings
- Transformer functions in conventional manner, with small magnetizing current and negligible energy storage within the magnetizing inductance
Part II
Converter Dynamics and Control

7. AC equivalent circuit modeling
8. Converter transfer functions
9. Controller design
10. Ac and dc equivalent circuit modeling of the discontinuous conduction mode
11. Current programmed control
Chapter 7. AC Equivalent Circuit Modeling

7.1. Introduction
7.2. The basic ac modeling approach
7.3. Example: A nonideal flyback converter
7.4. State-space averaging
7.5. Circuit averaging and averaged switch modeling
7.6. The canonical circuit model
7.7. Modeling the pulse-width modulator
7.8. Summary of key points
7.1. Introduction

Objective: maintain $v(t)$ equal to an accurate, constant value $V$.

There are disturbances:
- in $v_g(t)$
- in $R$

There are uncertainties:
- in element values
- in $V_g$
- in $R$

A simple dc-dc regulator system, employing a buck converter
Applications of control in power electronics

*Dc-dc converters*

Regulate dc output voltage.

Control the duty cycle $d(t)$ such that $v(t)$ accurately follows a reference signal $v_{\text{ref}}$.

*Dc-ac inverters*

Regulate an ac output voltage.

Control the duty cycle $d(t)$ such that $v(t)$ accurately follows a reference signal $v_{\text{ref}}(t)$.

*Ac-dc rectifiers*

Regulate the dc output voltage.

Regulate the ac input current waveform.

Control the duty cycle $d(t)$ such that $i_g(t)$ accurately follows a reference signal $i_{\text{ref}}(t)$, and $v(t)$ accurately follows a reference signal $v_{\text{ref}}$. 
Objective of Part II

Develop tools for modeling, analysis, and design of converter control systems

Need dynamic models of converters:

How do ac variations in $v_g(t)$, $R$, or $d(t)$ affect the output voltage $v(t)$?

What are the small-signal transfer functions of the converter?

• Extend the steady-state converter models of Chapters 2 and 3, to include CCM converter dynamics (Chapter 7)
• Construct converter small-signal transfer functions (Chapter 8)
• Design converter control systems (Chapter 9)
• Model converters operating in DCM (Chapter 10)
• Current-programmed control of converters (Chapter 11)
Modeling

• Representation of physical behavior by mathematical means
• Model dominant behavior of system, ignore other insignificant phenomena
• Simplified model yields physical insight, allowing engineer to design system to operate in specified manner
• Approximations neglect small but complicating phenomena
• After basic insight has been gained, model can be refined (if it is judged worthwhile to expend the engineering effort to do so), to account for some of the previously neglected phenomena
Neglecting the switching ripple

Suppose the duty cycle is modulated sinusoidally:

\[ d(t) = D + D_m \cos \omega_m t \]

where \( D \) and \( D_m \) are constants, \( |D_m| \ll D \), and the modulation frequency \( \omega_m \) is much smaller than the converter switching frequency \( \omega_s = 2\pi f_s \).

The resulting variations in transistor gate drive signal and converter output voltage:

$\text{gate drive}$

$\text{actual waveform } v(t) \text{ including ripple}$

$\text{averaged waveform } <v(t)>_{T_s}$

$\text{with ripple neglected}$

$\text{t}$
Output voltage spectrum with sinusoidal modulation of duty cycle

Contains frequency components at:
- Modulation frequency and its harmonics
- Switching frequency and its harmonics
- Sidebands of switching frequency

With small switching ripple, high-frequency components (switching harmonics and sidebands) are small.

If ripple is neglected, then only low-frequency components (modulation frequency and harmonics) remain.
Objective of ac converter modeling

• Predict how low-frequency variations in duty cycle induce low-frequency variations in the converter voltages and currents

• Ignore the switching ripple

• Ignore complicated switching harmonics and sidebands

Approach:

• Remove switching harmonics by averaging all waveforms over one switching period
Averaging to remove switching ripple

Average over one switching period to remove switching ripple:

\[
L \frac{d}{dt} \langle i_l(t) \rangle_{T_s} = \langle v_L(t) \rangle_{T_s}
\]

\[
C \frac{d}{dt} \langle v_c(t) \rangle_{T_s} = \langle i_c(t) \rangle_{T_s}
\]

where

\[
\langle x_L(t) \rangle_{T_s} = \frac{1}{T_s} \int_{jT_s}^{jT_s + T_s} x(\tau) d\tau
\]

Note that, in steady-state,

\[
\langle v_L(t) \rangle_{T_s} = 0
\]

\[
\langle i_c(t) \rangle_{T_s} = 0
\]

by inductor volt-second balance and capacitor charge balance.
Nonlinear averaged equations

The averaged voltages and currents are, in general, nonlinear functions of the converter duty cycle, voltages, and currents. Hence, the averaged equations

\[
L \frac{d \langle i_L(t) \rangle_{T_s}}{dt} = \langle v_L(t) \rangle_{T_s}
\]

\[
C \frac{d \langle v_C(t) \rangle_{T_s}}{dt} = \langle i_C(t) \rangle_{T_s}
\]

constitute a system of nonlinear differential equations.

Hence, must linearize by constructing a small-signal converter model.
Small-signal modeling of the BJT

Nonlinear Ebers-Moll model

Linearized small-signal model, active region

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Chapter 7: AC equivalent circuit modeling
Buck-boost converter: nonlinear static control-to-output characteristic

Example: linearization at the quiescent operating point

$D = 0.5$
Result of averaged small-signal ac modeling

Small-signal ac equivalent circuit model

\[ v_g(t) \rightarrow I \hat{a}(t) \rightarrow L \rightarrow \frac{(V_s - V) \hat{a}(t)}{D} \rightarrow D:1 \rightarrow \]

\[ \rightarrow \frac{1}{D} \rightarrow \frac{1}{L} \rightarrow R \rightarrow \frac{1}{C} \rightarrow \hat{v}(t) \]

buck-boost example
7.2. The basic ac modeling approach

Buck-boost converter example

![Buck-boost converter circuit diagram]
Switch in position 1

Inductor voltage and capacitor current are:

\[ v_L(t) = L \frac{di(t)}{dt} = v_s(t) \]

\[ i_C(t) = C \frac{dv(t)}{dt} = -\frac{v(t)}{R} \]

Small ripple approximation: replace waveforms with their low-frequency averaged values:

\[ v_L(t) = L \frac{di(t)}{dt} \approx \left< v_s(t) \right>_T \]

\[ i_C(t) = C \frac{dv(t)}{dt} \approx -\left< \frac{v(t)}{T_s} \right> \]
Switch in position 2

Inductor voltage and capacitor current are:

\[ v_L(t) = L \frac{di(t)}{dt} = v(t) \]

\[ i_C(t) = C \frac{dv(t)}{dt} = -i(t) - \frac{v(t)}{R} \]

Small ripple approximation: replace waveforms with their low-frequency averaged values:

\[ v_L(t) = L \frac{di(t)}{dt} \approx \langle v(t) \rangle_{T_s} \]

\[ i_C(t) = C \frac{dv(t)}{dt} \approx -\langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R} \]
7.2.1 Averaging the inductor waveforms

Inductor voltage waveform

Low-frequency average is found by evaluation of

\[
\langle x_L(t) \rangle_{T_s} = \frac{1}{T_s} \int_{t}^{t+T_s} x(\tau) d\tau
\]

Average the inductor voltage in this manner:

\[
\langle v_L(t) \rangle_{T_s} = \frac{1}{T_s} \int_{t}^{t+T_s} v_L(\tau) d\tau = d(t) \langle v_g(t) \rangle_{T_s} + d'(t) \langle v(t) \rangle_{T_s}
\]

Insert into Eq. (7.2):

\[
L \frac{d\langle i(t) \rangle_{T_s}}{dt} = d(t) \langle v_g(t) \rangle_{T_s} + d'(t) \langle v(t) \rangle_{T_s}
\]

This equation describes how the low-frequency components of the inductor waveforms evolve in time.
7.2.2 Discussion of the averaging approximation

Use of the average inductor voltage allows us to determine the net change in inductor current over one switching period, while neglecting the switching ripple.

In steady-state, the average inductor voltage is zero (volt-second balance), and hence the inductor current waveform is periodic: \( i(t + T_s) = i(t) \).

There is no net change in inductor current over one switching period.

During transients or ac variations, the average inductor voltage is not zero in general, and this leads to net variations in inductor current.

Inductor voltage and current waveforms
Net change in inductor current is correctly predicted by the average inductor voltage

Inductor equation: 

\[ L \frac{di(t)}{dt} = v_L(t) \]

Divide by \( L \) and integrate over one switching period:

\[ \int_t^{t+T_s} di = \frac{1}{L} \int_t^{t+T_s} v_L(\tau) d\tau \]

Left-hand side is the change in inductor current. Right-hand side can be related to average inductor voltage by multiplying and dividing by \( T_s \) as follows:

\[ i(t + T_s) - i(t) = \frac{1}{L} T_s \langle v_L(t) \rangle_{T_s} \]

So the net change in inductor current over one switching period is exactly equal to the period \( T_s \) multiplied by the average slope \( \langle v_L \rangle_{T_s} / L \).
Average inductor voltage correctly predicts average slope of $i_L(t)$

The net change in inductor current over one switching period is exactly equal to the period $T_s$ multiplied by the average slope $\langle v_L \rangle_{T_s} / L$. 

$$d \langle v_g(t) \rangle_{T_s} + d' \langle v(t) \rangle_{T_s} = \frac{i(T_s) - i(0)}{L}$$
\[
\frac{d\langle i(t) \rangle_{T_s}}{dt}
\]

We have
\[
\frac{i(t + T_s) - i(t)}{T_s} = \frac{1}{L} T_s \langle v_L(t) \rangle_{T_s}
\]

Rearrange:
\[
L \frac{i(t + T_s) - i(t)}{T_s} = \langle v_L(t) \rangle_{T_s}
\]

Define the derivative of \( \langle i \rangle_{T_s} \) as (Euler formula):
\[
\frac{d\langle i(t) \rangle_{T_s}}{dt} = \frac{i(t + T_s) - i(t)}{T_s}
\]

Hence,
\[
L \frac{d\langle i(t) \rangle_{T_s}}{dt} = \langle v_L(t) \rangle_{T_s}
\]
Computing how the inductor current changes over one switching period

Let’s compute the actual inductor current waveform, using the linear ripple approximation.

With switch in position 1:

\[ i(dT_s) = i(0) + \frac{\langle v_s \rangle_{T_s}}{L} dT_s \]

(final value) = (initial value) + (length of interval) (average slope)

With switch in position 2:

\[ i(T_s) = i(dT_s) + \frac{\langle v(t) \rangle_{T_s}}{L} d'T_s \]

(final value) = (initial value) + (length of interval) (average slope)
Net change in inductor current over one switching period

Eliminate $i(dT_s)$, to express $i(T_s)$ directly as a function of $i(0)$:

$$i(T_s) = i(0) + \frac{T_s}{L} \left\{ d(t) \left\langle v_g(t) \right\rangle_{T_s} + d'(t) \left\langle v(t) \right\rangle_{T_s} \right\} \left\langle v_L(t) \right\rangle_{T_s}$$

The intermediate step of computing $i(dT_s)$ is eliminated.

The final value $i(T_s)$ is equal to the initial value $i(0)$, plus the switching period $T_s$ multiplied by the average slope $\left\langle v_L \right\rangle_{T_s}/L$. 

![Diagram illustrating the net change in inductor current over one switching period. The diagram shows the actual waveform including ripple and the averaged waveform. The initial value $i(0)$, the final value $i(T_s)$, and the intermediate step $dT_s$ are highlighted.]
7.2.3 Averaging the capacitor waveforms

Average capacitor current:

\[
\langle i_c(t) \rangle_{T_s} = d(t) \left( -\frac{\langle v(t) \rangle_{T_s}}{R} \right) + d'(t) \left( -\langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R} \right)
\]

Collect terms, and equate to \( C \frac{d\langle v \rangle_{T_s}}{dt} \):

\[
C \frac{d\langle v(t) \rangle_{T_s}}{dt} = -d'(t) \langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R}
\]
7.2.4 The average input current

We found in Chapter 3 that it was sometimes necessary to write an equation for the average converter input current, to derive a complete dc equivalent circuit model. It is likewise necessary to do this for the ac model.

Buck-boost input current waveform is

\[
i_g(t) = \begin{cases} 
\langle i(t) \rangle_{T_s} & \text{during subinterval 1} \\
0 & \text{during subinterval 2}
\end{cases}
\]

Average value:

\[
\langle i_g(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s}
\]
7.2.5. Perturbation and linearization

Converter averaged equations:

\[ L \frac{d}{dt} \langle i(t) \rangle_{T_s} = d(t) \langle v_{g}(t) \rangle_{T_s} + d'(t) \langle v(t) \rangle_{T_s} \]
\[ C \frac{d}{dt} \langle v(t) \rangle_{T_s} = -d'(t) \langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R} \]
\[ \langle i_s(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s} \]

—nonlinear because of multiplication of the time-varying quantity \( d(t) \) with other time-varying quantities such as \( i(t) \) and \( v(t) \).
Construct small-signal model:
Linearize about quiescent operating point

If the converter is driven with some steady-state, or quiescent, inputs

\[ d(t) = D \]
\[ \langle v_g(t) \rangle_{T_s} = V_g \]

then, from the analysis of Chapter 2, after transients have subsided
the inductor current, capacitor voltage, and input current

\[ \langle i(t) \rangle_{T_s}, \langle v(t) \rangle_{T_s}, \langle i_g(t) \rangle_{T_s} \]

reach the quiescent values \( I, V, \) and \( I_g, \) given by the steady-state
analysis as

\[ V = - \frac{D}{D'} V_g \]
\[ I = - \frac{V}{D' R} \]
\[ I_g = D I \]
Perturbation

So let us assume that the input voltage and duty cycle are equal to some given (dc) quiescent values, plus superimposed small ac variations:

\[
\begin{align*}
\langle v_g(t) \rangle_{T_s} &= V_g + \hat{v}_g(t) \\
\langle d(t) \rangle &= D + \hat{d}(t)
\end{align*}
\]

In response, and after any transients have subsided, the converter dependent voltages and currents will be equal to the corresponding quiescent values, plus small ac variations:

\[
\begin{align*}
\langle i(t) \rangle_{T_s} &= I + \hat{i}(t) \\
\langle v(t) \rangle_{T_s} &= V + \hat{v}(t) \\
\langle i_g(t) \rangle_{T_s} &= I_g + \hat{i}_g(t)
\end{align*}
\]
The small-signal assumption

If the ac variations are much smaller in magnitude than the respective quiescent values,

\[
\begin{align*}
|\dot{v}_g(t)| & \ll |V_g| \\
|\dot{d}(t)| & \ll |D| \\
|i(t)| & \ll |I| \\
|\dot{v}(t)| & \ll |V| \\
|i_g(t)| & \ll |I_g|
\end{align*}
\]

then the nonlinear converter equations can be linearized.
Perturbation of inductor equation

Insert the perturbed expressions into the inductor differential equation:

\[ L \frac{d\left[I + i(t)\right]}{dt} = \left[D + \hat{d}(t)\right]\left[V_g + \hat{v}_g(t)\right] + \left[D' - \hat{d}(t)\right]\left[V + \hat{v}(t)\right] \]

note that \(d'(t)\) is given by

\[ d'(t) = 1 - d(t) = 1 - \left[D + \hat{d}(t)\right] = D' - \hat{d}(t) \quad \text{with} \quad D' = 1 - D \]

Multiply out and collect terms:

\[ L \left(\frac{d^0 i(t)}{dt} + \frac{d i(t)}{dt}\right) = \left[D V_g + D' V\right] + \left[D\hat{v}_g(t) + D' \hat{v}(t) + \left(V_g - V\right) \hat{d}(t)\right] + \hat{d}(t) \left[\hat{v}_g(t) - \hat{v}(t)\right] \]

\( Dc \) terms \hspace{1cm} 1\textsuperscript{st} order ac terms \hspace{1cm} 2\textsuperscript{nd} order ac terms

(linear) \hspace{1cm} (nonlinear)
The perturbed inductor equation

\[
L \left( \frac{dI}{dt} + \frac{d\hat{i}(t)}{dt} \right) = \left( DV_g + D'V \right) + \left( D\hat{v}_g(t) + D'\hat{v}(t) + \left( V_g - V \right) \hat{d}(t) \right) + \hat{a}(t) \left( \hat{v}_g(t) - \hat{v}(t) \right)
\]

\[
\begin{align*}
\text{Dc terms} & \\
1^{st} \text{ order ac terms} & \quad \text{(linear)} \\
2^{nd} \text{ order ac terms} & \quad \text{(nonlinear)}
\end{align*}
\]

Since \( I \) is a constant (dc) term, its derivative is zero.

The right-hand side contains three types of terms:

- Dc terms, containing only dc quantities
- First-order ac terms, containing a single ac quantity, usually multiplied by a constant coefficient such as a dc term. These are linear functions of the ac variations.
- Second-order ac terms, containing products of ac quantities. These are nonlinear, because they involve multiplication of ac quantities.
Neglect of second-order terms

\[
L \left( \frac{0}{dt} + \frac{d i(t)}{dt} \right) = \left[ D V_g + D' V \right] + \left[ D \hat{v}_g(t) + D' \hat{v}(t) + \left( V_g - V \right) \hat{d}(t) \right] + \hat{d}(t) \left[ \hat{v}_g(t) - \hat{v}(t) \right]
\]

Dc terms \hspace{1cm} 1^{st} \text{ order ac terms (linear)} \hspace{1cm} 2^{nd} \text{ order ac terms (nonlinear)}

Provided
\[
\begin{align*}
|\hat{v}_g(t)| &<< |V_g| \\
|\hat{d}(t)| &<< |D| \\
|i(t)| &<< |I| \\
|\hat{v}(t)| &<< |V| \\
|i_g(t)| &<< |I_g|
\end{align*}
\]
then the second-order ac terms are much smaller than the first-order terms. For example,
\[
|\hat{d}(t) \hat{v}_g(t)| << |D \hat{v}_g(t)| \quad \text{when} \quad |\hat{d}(t)| << |D|
\]
So neglect second-order terms. Also, dc terms on each side of equation are equal.
Linearized inductor equation

Upon discarding second-order terms, and removing dc terms (which add to zero), we are left with

\[ L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left( V_g - V \right) \hat{d}(t) \]

This is the desired result: a linearized equation which describes small-signal ac variations.

Note that the quiescent values \( D, D', V, V_g \), are treated as given constants in the equation.
Capacitor equation

Perturbation leads to

\[
C \left( \frac{d(V + \varphi(t))}{dt} \right) = \left( D' - \hat{d}(t) \right) \left( I + \hat{i}(t) \right) - \frac{V + \varphi(t)}{R}
\]

Collect terms:

\[
C \left( \frac{dV}{dt} + \frac{d\varphi(t)}{dt} \right) = \left( -D'I - \frac{V}{R} \right) + \left( -D'i(t) - \frac{\varphi(t)}{R} + I\hat{d}(t) \right) + \hat{d}(t)\hat{i}(t)
\]

\[
\text{Dc terms} \quad \text{1}\text{st order ac terms} \quad \text{2}\text{nd order ac term}
\]

(linear) \quad \text{(nonlinear)}

Neglect second-order terms. Dc terms on both sides of equation are equal. The following terms remain:

\[
C \frac{d\varphi(t)}{dt} = - D'i(t) - \frac{\varphi(t)}{R} + I\hat{d}(t)
\]

This is the desired small-signal linearized capacitor equation.
Average input current

Perturbation leads to

\[ I_g + \dot{i}_g(t) = \left( D + \dot{d}(t) \right) \left( I + \dot{i}(t) \right) \]

Collect terms:

\[
\frac{I_g}{D} + \frac{\dot{i}_g(t)}{I} = \frac{\left( D I \right)}{D} + \frac{\left( D \dot{i}(t) + I \dot{d}(t) \right)}{I} + \frac{\dot{d}(t) \dot{i}(t)}{I}
\]

DC term \hspace{1cm} 1^{st} order AC term \hspace{1cm} DC term \hspace{1cm} 1^{st} order AC terms \hspace{1cm} 2^{nd} order AC term
(linear) \hspace{1cm} (nonlinear)

Neglect second-order terms. DC terms on both sides of equation are equal. The following first-order terms remain:

\[ \dot{i}_g(t) = D \dot{i}(t) + I \dot{d}(t) \]

This is the linearized small-signal equation which described the converter input port.
7.2.6. Construction of small-signal equivalent circuit model

The linearized small-signal converter equations:

\[
L \frac{d \dot{i}(t)}{dt} = D \dot{v}(t) + D' \ddot{v}(t) + \left[ V_g - V \right] \ddot{d}(t)
\]

\[
C \frac{d \ddot{v}(t)}{dt} = -D' \dot{i}(t) - \frac{\dddot{v}(t)}{R} + I \ddot{d}(t)
\]

\[
\dot{i}_g(t) = D \dot{i}(t) + I \ddot{d}(t)
\]

Reconstruct equivalent circuit corresponding to these equations, in manner similar to the process used in Chapter 3.
Inductor loop equation

\[ L \frac{d\hat{i}(t)}{dt} = D\hat{\phi}_g(t) + D'\hat{\phi}(t) + \left[V_g - V\right]\hat{d}(t) \]
Capacitor node equation

\[ C \frac{d\hat{v}(t)}{dt} = -D' \hat{i}(t) - \frac{\hat{\varphi}(t)}{R} + I\hat{d}(t) \]
Input port node equation

\[ i'_{g}(t) = D i(t) + I \hat{d}(t) \]
Complete equivalent circuit

Collect the three circuits:

Replace dependent sources with ideal dc transformers:

Small-signal ac equivalent circuit model of the buck-boost converter
7.2.7. Results for several basic converters

**buck**

\[ v_g(t) \]
\[ I \hat{a}(t) \]
\[ \frac{1}{D} \]
\[ V_g \hat{a}(t) \]
\[ L \]
\[ i(t) \]
\[ C \]
\[ R \]
\[ v(t) \]

**boost**

\[ v_g(t) \]
\[ i(t) \]
\[ L \]
\[ V \hat{a}(t) \]
\[ \frac{1}{D} \]
\[ I \hat{a}(t) \]
\[ C \]
\[ R \]
\[ v(t) \]
Results for several basic converters

*buck-boost*

\[ v_g(t) \quad i(t) \quad \frac{1}{D} \quad L \quad \frac{V_g - V}{D} \quad \frac{1}{D'} \quad i(t) \quad \frac{1}{C} \quad \frac{1}{R} \quad v(t) \quad \frac{1}{-} \]
7.3. Example: a nonideal flyback converter

**Flyback converter example**

- MOSFET has on-resistance $R_{on}$
- Flyback transformer has magnetizing inductance $L$, referred to primary

![Flyback converter diagram]
Circuits during subintervals 1 and 2

**Flyback converter, with transformer equivalent circuit**

Subinterval 1

Subinterval 2

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Subinterval 1

Circuit equations:

\[ v_L(t) = v_g(t) - i(t) R_{on} \]
\[ i_C(t) = - \frac{v(t)}{R} \]
\[ i_g(t) = i(t) \]

Small ripple approximation:

\[ v_L(t) = \left< v_g(t) \right>_T - \left< i(t) \right>_T R_{on} \]
\[ i_C(t) = - \frac{\left< v(t) \right>_T}{R} \]
\[ i_g(t) = \left< i(t) \right>_T \]

MOSFET conducts, diode is reverse-biased.
Subinterval 2

Circuit equations:

\[ v_L(t) = -\frac{v(t)}{n} \]
\[ i_C(t) = -\frac{i(t)}{n} - \frac{v(t)}{R} \]
\[ i_g(t) = 0 \]

Small ripple approximation:

\[ v_L(t) = -\frac{\langle v(t) \rangle_{T_s}}{n} \]
\[ i_C(t) = -\frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R} \]
\[ i_g(t) = 0 \]

MOSFET is off, diode conducts
Inductor waveforms

Average inductor voltage:

\[ \langle v_L(t) \rangle_{T_s} = d(t) \left( \langle v_g(t) \rangle_{T_s} - \langle i(t) \rangle_{T_s} R_{on} \right) + d'(t) \left( \frac{-\langle v(t) \rangle_{T_s}}{n} \right) \]

Hence, we can write:

\[ L \frac{d \langle i(t) \rangle_{T_s}}{d t} = d(t) \langle v_g(t) \rangle_{T_s} - d(t) \langle i(t) \rangle_{T_s} R_{on} - d'(t) \frac{\langle v(t) \rangle_{T_s}}{n} \]
Capacitor waveforms

Average capacitor current:

$$\langle i_c(t) \rangle_{T_s} = d(t) \left( -\frac{\langle v(t) \rangle_{T_s}}{R} \right) + d'(t) \left( \frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R} \right)$$

Hence, we can write:

$$C \frac{d\langle v(t) \rangle_{T_s}}{dt} = d'(t) \frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R}$$
Input current waveform

Average input current:
\[
\langle i_s(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s}
\]
The averaged converter equations

\[
L \frac{d\left\langle i(t) \right\rangle_{T_s}}{dt} = d(t) \left\langle v_g(t) \right\rangle_{T_s} - d(t) \left\langle i(t) \right\rangle_{T_s} R_{on} - d'(t) \frac{\left\langle v(t) \right\rangle_{T_s}}{n}
\]

\[
C \frac{d\left\langle v(t) \right\rangle_{T_s}}{dt} = d'(t) \frac{\left\langle i(t) \right\rangle_{T_s}}{n} - \frac{\left\langle v(t) \right\rangle_{T_s}}{R}
\]

\[
\left\langle i_g(t) \right\rangle_{T_s} = d(t) \left\langle i(t) \right\rangle_{T_s}
\]

— a system of nonlinear differential equations

Next step: perturbation and linearization. Let

\[
\left\langle v_g(t) \right\rangle_{T_s} = V_g + \nu_g(t)
\]

\[
d(t) = D + \hat{d}(t)
\]

\[
\left\langle i(t) \right\rangle_{T_s} = I + \hat{i}(t)
\]

\[
\left\langle v(t) \right\rangle_{T_s} = V + \hat{v}(t)
\]

\[
\left\langle i_g(t) \right\rangle_{T_s} = I_g + \hat{i}_g(t)
\]
Perturbation of the averaged inductor equation

\[
L \frac{d\langle i(t) \rangle_{Ts}}{dt} = d(t) \left\langle v_s(t) \right\rangle_{Ts} - d(t) \left\langle i(t) \right\rangle_{Ts} R_{on} - d'(t) \left\langle v(t) \right\rangle_{Ts} n
\]

\[
L \frac{d(I + \hat{i}(t))}{dt} = \left( D + \hat{d}(t) \right) \left[ V_g + \hat{v}_s(t) \right] - \left( D' - \hat{a}(t) \right) \left( \frac{V + \hat{v}(t)}{n} \right) - \left( D - \hat{a}(t) \right) \left[ I + \hat{i}(t) \right] R_{on}
\]

\[
L \left( \frac{d\hat{i}(t)}{dt} \right) = \left( D\hat{v}_g(t) - \hat{d}\frac{V}{n} - DR_{on}I \right) + \left( D\hat{v}_g(t) - \hat{d}\frac{V}{n} + \left( V_g + \frac{V}{n} - IR_{on} \right) \hat{d} - DR_{on}\hat{i}(t) \right)
\]

\[\text{Dc terms} \quad 1^{st} \text{ order ac terms (linear)} \quad 2^{nd} \text{ order ac terms (nonlinear)}\]
Linearization of averaged inductor equation

Dc terms:

\[ 0 = DV_g - D' \frac{V}{N} - DR_{on}I \]

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

\[ L \frac{d\hat{i}(t)}{dt} = D\hat{v}_s(t) - D\hat{v}(t) + \left( V_g + \frac{V}{N} - IR_{on} \right) \hat{a}(t) - DR_{on}\hat{i}(t) \]

This is the desired linearized inductor equation.
Perturbation of averaged capacitor equation

Original averaged equation:

\[
C \frac{d\langle v(t) \rangle_{T_s}}{dt} = d'(t) \frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R}
\]

Perturb about quiescent operating point:

\[
C \frac{d[V + \hat{v}(t)]}{dt} = \left(D' - \hat{d}(t)\right) \frac{[I + \hat{i}(t)]}{n} - \frac{[V + \hat{v}(t)]}{R}
\]

Collect terms:

\[
C \left(\frac{\hat{d}(t)\hat{i}(t)}{n} + \frac{d\hat{v}(t)}{dt} + \left[\frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}\right]\right) = \frac{\hat{d}(t)\hat{i}(t)}{n}
\]

\[
\text{Dc terms} \quad \text{1st order ac terms (linear)} \quad \text{2nd order ac term (nonlinear)}
\]
Linearization of averaged capacitor equation

Dc terms:

\[ 0 = \left( \frac{D' I}{n} - \frac{V}{R} \right) \]

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

\[ C \frac{d\hat{v}(t)}{dt} = \frac{D' \hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I \hat{d}(t)}{n} \]

This is the desired linearized capacitor equation.
Perturbation of averaged input current equation

Original averaged equation:

\[
\langle i_g(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s}
\]

Perturb about quiescent operating point:

\[
I_g + \dot{i}_g(t) = (D + \hat{d}(t)) \left( I + \dot{i}(t) \right)
\]

Collect terms:

\[
\underbrace{I_g}_{\text{Dc term}} + \underbrace{\dot{i}_g(t)}_{1^{st} \text{ order ac term}} = \underbrace{(D I)}_{\text{Dc term}} + \underbrace{(D \dot{i}(t) + I \hat{d}(t))}_{1^{st} \text{ order ac terms (linear)}} + \underbrace{\hat{d}(t)\dot{i}(t)}_{2^{nd} \text{ order ac term (nonlinear)}}
\]
Linearization of averaged input current equation

Dc terms:

\[ I_g = DI \]

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

\[ i_g(t) = D\dot{i}(t) + I\ddot{d}(t) \]

This is the desired linearized input current equation.
Summary: dc and small-signal ac converter equations

Dc equations:

\[ 0 = DV_g - D'I_n - DR_{on}I \]
\[ 0 = \left( \frac{D'I_n}{n} - \frac{V}{R} \right) \]
\[ I_g = DI \]

Small-signal ac equations:

\[ L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\hat{v}(t) + \left( V_g + \frac{V}{n} - IR_{on} \right) \hat{d}(t) - DR_{on}\hat{i}(t) \]
\[ C \frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n} \]
\[ \hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t) \]

Next step: construct equivalent circuit models.
Small-signal ac equivalent circuit: inductor loop

$$L \frac{di(t)}{dt} = D\hat{v}_g(t) - D'\hat{v}(t) + \left( V_g + \frac{V}{n} - IR_{on} \right) \hat{d}(t) - DR_{on}i(t)$$
Small-signal ac equivalent circuit: capacitor node

\[ C \frac{d\hat{v}(t)}{dt} = \frac{D' \hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I \hat{a}(t)}{n} \]
Small-signal ac equivalent circuit: converter input node

\[ \hat{i}_g(t) = D \hat{i}(t) + I \hat{d}(t) \]
Small-signal ac model, nonideal flyback converter example

Combine circuits:

Replace dependent sources with ideal transformers:

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