Part II
Converter Dynamics and Control

7. AC equivalent circuit modeling
8. Converter transfer functions
9. Controller design
10. Input filter design
11. AC and DC equivalent circuit modeling of the discontinuous conduction mode
12. Current programmed control
Chapter 7. AC Equivalent Circuit Modeling

7.1 Introduction
7.2 The basic AC modeling approach
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7.1. Introduction

Objective: maintain $v(t)$ equal to an accurate, constant value $V$.

There are disturbances:
- in $v_g(t)$
- in $R$

There are uncertainties:
- in element values
- in $V_g$
- in $R$
Applications of control in power electronics

**DC-DC converters**
- Regulate dc output voltage.
- Control the duty cycle $d(t)$ such that $v(t)$ accurately follows a reference signal $v_{ref}$.

**DC-AC inverters**
- Regulate an ac output voltage.
- Control the duty cycle $d(t)$ such that $v(t)$ accurately follows a reference signal $v_{ref}(t)$.

**AC-DC rectifiers**
- Regulate the dc output voltage.
- Regulate the ac input current waveform.
- Control the duty cycle $d(t)$ such that $i_g(t)$ accurately follows a reference signal $i_{ref}(t)$, and $v(t)$ accurately follows a reference signal $v_{ref}$.
Converter Modeling

Applications
- Aerospace worst-case analysis
- Commercial high-volume production: design for reliability and yield

High quality design
- Ensure that the converter works well under worst-case conditions
  - Steady state (losses, efficiency, voltage regulation)
  - Small-signal ac (controller stability and transient response)

Engineering methodology
- Simulate model during preliminary design (design verification)
- Construct laboratory prototype converter system and make it work under nominal conditions
- Develop a converter model. Refine model until it predicts behavior of nominal laboratory prototype
- Use model to predict behavior under worst-case conditions
- Improve design until worst-case behavior meets specifications (or until reliability and production yield are acceptable)
Objective of Part II

Develop tools for modeling, analysis, and design of converter control systems

Need dynamic models of converters:

- How do ac variations in $v_g(t)$, $R$, or $d(t)$ affect the output voltage $v(t)$?
- What are the small-signal transfer functions of the converter?

- Extend the steady-state converter models of Chapters 2 and 3, to include CCM converter dynamics (Chapter 7)
- Construct converter small-signal transfer functions (Chapter 8)
- Design converter control systems (Chapter 9)
- Design input EMI filters that do not disrupt control system operation (Chapter 10)
- Model converters operating in DCM (Chapter 11)
- Current-programmed control of converters (Chapter 12)
Modeling

- Representation of physical behavior by mathematical means
- Model dominant behavior of system, ignore other insignificant phenomena
- Simplified model yields physical insight, allowing engineer to design system to operate in specified manner
- Approximations neglect small but complicating phenomena
- After basic insight has been gained, model can be refined (if it is judged worthwhile to expend the engineering effort to do so), to account for some of the previously neglected phenomena
Finding the response to ac variations
Neglecting the switching ripple

Suppose the control signal varies sinusoidally:
\[ v_c(t) = V_c + V_{cm} \cos \omega_m t \]

This causes the duty cycle to be modulated sinusoidally:
\[ d(t) = D + D_m \cos \omega_m t \]

Assume \( D \) and \( D_m \) are constants, \( |D_m| \ll D \), and the modulation frequency \( \omega_m \) is much smaller than the converter switching frequency \( \omega_s = 2\pi f_s \).
Output voltage spectrum with sinusoidal modulation of duty cycle

Contains frequency components at:
- Modulation frequency and its harmonics
- Switching frequency and its harmonics
- Sidebands of switching frequency

With small switching ripple, high-frequency components (switching harmonics and sidebands) are small.

If ripple is neglected, then only low-frequency components (modulation frequency and harmonics) remain.
Objective of ac converter modeling

- Predict how low-frequency variations in duty cycle induce low-frequency variations in the converter voltages and currents
- Ignore the switching ripple
- Ignore complicated switching harmonics and sidebands

Approach:
- Remove switching harmonics by averaging all waveforms over one switching period
Averaging to remove switching ripple

Average over one switching period to remove switching ripple:

\[ L \frac{d\langle i_L(t) \rangle_{T_s}}{dt} = \langle v_L(t) \rangle_{T_s} \]
\[ C \frac{d\langle v_C(t) \rangle_{T_s}}{dt} = \langle i_C(t) \rangle_{T_s} \]

where

\[ \langle x(t) \rangle_{T_s} = \frac{1}{T_s} \int_{t-T_s/2}^{t+T_s/2} x(\tau) d\tau \]

Note that, in steady-state,

\[ \langle v_L(t) \rangle_{T_s} = 0 \]
\[ \langle i_C(t) \rangle_{T_s} = 0 \]

by inductor volt-second balance and capacitor charge balance.
Nonlinear averaged equations

The averaged voltages and currents are, in general, nonlinear functions of the converter duty cycle, voltages, and currents. Hence, the averaged equations

\[ L \frac{d}{dt} \langle i_L(t) \rangle_{T_s} = \langle v_L(t) \rangle_{T_s} \]

\[ C \frac{d}{dt} \langle v_C(t) \rangle_{T_s} = \langle i_C(t) \rangle_{T_s} \]

constitute a system of nonlinear differential equations.

Hence, must linearize by constructing a small-signal converter model.
Small-signal modeling of the diode

Nonlinear diode, driven by current source having a DC and small AC component

\[ i = I + i \]
\[ v = V + \hat{v} \]

Small-signal AC model

\[ i \]
\[ r_D \]
\[ \hat{v} \]

Linearization of the diode \( i-v \) characteristic about a quiescent operating point

Actual nonlinear characteristic
Quiescent operating point
Linearized function
\[ \hat{i}(t) \]
\[ \hat{v}(t) \]
Buck-boost converter:
nonlinear static control-to-output characteristic

- Actual nonlinear characteristic
- Linearized function
- Quiescent operating point

\[ V = V_g \frac{D}{1 - D} \]

Example: linearization at the quiescent operating point
\[ D = 0.5 \]
Result of averaged small-signal ac modeling

Small-signal ac equivalent circuit model

*buck-boost example*
7.2. The basic ac modeling approach

*Buck-boost converter example*
Switch in position 1

Inductor voltage and capacitor current are:

\[ v_L(t) = L \frac{di(t)}{dt} = v_s(t) \]

\[ i_C(t) = C \frac{dv(t)}{dt} = -\frac{v(t)}{R} \]

Small ripple approximation: replace waveforms with their low-frequency averaged values:

\[ v_L(t) = L \frac{di(t)}{dt} \approx \langle v_s(t) \rangle_{T_s} \]

\[ i_C(t) = C \frac{dv(t)}{dt} \approx -\frac{\langle v(t) \rangle_{T_s}}{R} \]
Switch in position 2

Inductor voltage and capacitor current are:

\[ v_L(t) = L \frac{di(t)}{dt} = v(t) \]

\[ i_C(t) = C \frac{dv(t)}{dt} = -i(t) - \frac{v(t)}{R} \]

Small ripple approximation: replace waveforms with their low-frequency averaged values:

\[ v_L(t) = L \frac{di(t)}{dt} \approx \langle v(t) \rangle_{T_s} \]

\[ i_C(t) = C \frac{dv(t)}{dt} \approx -\langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R} \]
7.2.1 Averaging the inductor waveforms

Inductor voltage waveform

For the instantaneous inductor waveforms, we have:

\[ L \frac{di(t)}{dt} = v_L(t) \]

Can we write a similar equation for the averaged components? Let us investigate the derivative of the average inductor current:

\[ \frac{d\langle i(t) \rangle_{T_s}}{dt} = \frac{1}{T_s} \int_{t-T_s/2}^{t+T_s/2} \frac{di(\tau)}{d\tau} d\tau \]

On the right hand side, we are allowed to interchange the order of integration and differentiation because the inductor current is continuous and its derivative \( v/L \) has a finite number of discontinuities in the period of integration. Hence:

\[ \frac{d\langle i(t) \rangle_{T_s}}{dt} = \frac{1}{T_s} \int_{t-T_s/2}^{t+T_s/2} \frac{di(\tau)}{d\tau} d\tau \]
Averaging the inductor waveforms, p. 2

Now replace the derivative of the inductor current with $v/L$:

$$
\frac{d\langle i(t) \rangle_{T_s}}{dt} = \frac{1}{T_s} \int_{t-T_s/2}^{t+T_s/2} \frac{v_L(\tau)}{L} \, d\tau
$$

Finally, rearrange to obtain:

$$
L \frac{d\langle i(t) \rangle_{T_s}}{dt} = \langle v_L(t) \rangle_{T_s}
$$

So the averaged inductor current and voltage follow the same defining equation, with no additional terms and with the same value of $L$. A similar derivation for the capacitor leads to:

$$
C \frac{d\langle v(t) \rangle_{T_s}}{dt} = \langle i_C(t) \rangle_{T_s}
$$
7.2.2 The average inductor voltage and small ripple approximation

The actual inductor voltage waveform, sketched at some arbitrary time $t$. Let’s compute the average over the averaging interval $(t - T_s/2, t + T_s/2)$

While the transistor conducts, the inductor voltage is

$$v_L(t) = L \frac{di(t)}{dt} \approx \langle v_g(t) \rangle_{T_s}$$

*Small ripple approximation:* assume that $v_g$ has small ripple, and does not change significantly over the averaging interval.

While the diode conducts, the inductor voltage is

$$v_L(t) = L \frac{di(t)}{dt} \approx \langle v(t) \rangle_{T_s}$$

*Small ripple approximation:* assume that $v$ has small ripple, and does not change significantly over the averaging interval.
Average inductor voltage, p. 2

The average inductor voltage is therefore:

\[
\langle v_L(t) \rangle_{T_s} = \frac{1}{T_s} \int_{t-T_s/2}^{t+T_s/2} v_L(\tau) \, d\tau \approx d(t)\langle v_g(t) \rangle_{T_s} + d'(t)\langle v(t) \rangle_{T_s}
\]

Hence:

\[
L \frac{d\langle i(t) \rangle_{T_s}}{dt} = d(t)\langle v_g(t) \rangle_{T_s} + d'(t)\langle v(t) \rangle_{T_s}
\]
7.2.3 Discussion of the averaging approximation

- Averaging facilitates derivation of tractable equations describing the dynamics of the converter
- Averaging removes the waveform components at the switching frequency and its harmonics, while preserving the magnitude and phase of the waveform low-frequency components
- Averaging can be viewed as a form of low-pass filtering

\[
\langle x(t) \rangle_{T_s} = \frac{1}{T_s} \int_{t-T_s/2}^{t+T_s/2} x(\tau) d\tau
\]

*The averaging operator*
Simulation of converter waveforms and their averages

Computer-generated plots of inductor current and voltage, and comparison with their averages, for sinusoidal modulation of duty cycle.
The averaging operator as a low-pass filter

The averaging operator:

\[ \langle x(t) \rangle_{T_s} = \frac{1}{T_s} \int_{t-T_s/2}^{t+T_s/2} x(\tau) d\tau \]

Take Laplace transform:

\[ \langle x(s) \rangle_{T_s} = G_{av}(s)x(s) \]

\[ G_{av}(s) = \frac{e^{sT_s/2} - e^{-sT_s/2}}{sT_s} \]

- Notches at the switching frequency and its harmonics
- Zero phase shift
- Magnitude of –3 dB and half of the switching frequency
Averaging the capacitor waveforms

Switch position 1:

\[ i_c(t) = C \frac{dv(t)}{dt} = -\frac{v(t)}{R} \approx -\frac{\langle v(t) \rangle_{T_s}}{R} \]

Switch position 2:

\[ i_c(t) = C \frac{dv(t)}{dt} = -i(t) - \frac{v(t)}{R} \approx -\langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R} \]

Average:

\[ \langle i_c(t) \rangle_{T_s} = d(t) \left( -\frac{\langle v(t) \rangle_{T_s}}{R} \right) + d'(t) \left( -\langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R} \right) \]

Hence:

\[ C \frac{d\langle v(t) \rangle_{T_s}}{dt} = -d'(t) \langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R} \]

For the inductor voltage, the procedure was explained in most correct in general way for arbitrary \( t \). But result is the same when \( t \) coincides with beginning of switching period, as illustrated here.
7.2.4 The average input current

We found in Chapter 3 that it was sometimes necessary to write an equation for the average converter input current, to derive a complete dc equivalent circuit model. It is likewise necessary to do this for the ac model.

Buck-boost input current waveform is

\[ i_g(t) = \begin{cases} 
\langle i(t) \rangle_{T_s} & \text{during subinterval 1} \\
0 & \text{during subinterval 2} 
\end{cases} \]

Average value:

\[ \langle i_g(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s} \]
7.2.5. Perturbation and linearization

Converter averaged equations:

\[ L \frac{d\langle i(t) \rangle_{T_s}}{dt} = d(t) \langle v_s(t) \rangle_{T_s} + d'(t) \langle v(t) \rangle_{T_s} \]

\[ C \frac{d\langle v(t) \rangle_{T_s}}{dt} = -d'(t) \langle i(t) \rangle_{T_s} - \frac{\langle v(t) \rangle_{T_s}}{R} \]

\[ \langle i_s(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s} \]

—nonlinear because of multiplication of the time-varying quantity \( d(t) \) with other time-varying quantities such as \( i(t) \) and \( v(t) \).
Construct small-signal model:
Linearize about quiescent operating point

If the converter is driven with some steady-state, or quiescent, inputs

\[ d(t) = D \]

\[ \langle v_g(t) \rangle_{T_s} = V_g \]

then, from the analysis of Chapter 2, after transients have subsided
the inductor current, capacitor voltage, and input current

\[ \langle i(t) \rangle_{T_s}, \langle v(t) \rangle_{T_s}, \langle i_g(t) \rangle_{T_s} \]

reach the quiescent values \( I, V, \) and \( I_g, \) given by the steady-state
analysis as

\[ V = -\frac{D}{D'} V_g \]

\[ I = -\frac{V}{D' R} \]

\[ I_g = D I \]
Perturbation

So let us assume that the input voltage and duty cycle are equal to some given (dc) quiescent values, plus superimposed small ac variations:

\[
\begin{align*}
\langle v_g(t) \rangle_{T_s} &= V_g + \dot{v}_g(t) \\
\dot{d}(t) &= D + \tilde{d}(t)
\end{align*}
\]

In response, and after any transients have subsided, the converter dependent voltages and currents will be equal to the corresponding quiescent values, plus small ac variations:

\[
\begin{align*}
\langle i(t) \rangle_{T_s} &= I + \dot{i}(t) \\
\langle v(t) \rangle_{T_s} &= V + \hat{v}(t) \\
\langle i_s(t) \rangle_{T_s} &= I_s + \dot{i}_s(t)
\end{align*}
\]
The small-signal assumption

If the ac variations are much smaller in magnitude than the respective quiescent values,

\[
\begin{align*}
|\hat{v}_g(t)| & \ll |V_g| \\
|\hat{d}(t)| & \ll |D| \\
|i(t)| & \ll |I| \\
|\hat{v}(t)| & \ll |V| \\
|i_g(t)| & \ll |I_g|
\end{align*}
\]

then the nonlinear converter equations can be linearized.
Perturbation of inductor equation

Insert the perturbed expressions into the inductor differential equation:

\[ L \frac{d(I + \hat{i}(t))}{dt} = (D + \hat{d}(t)) \left( V_g + \hat{v}_g(t) \right) + \left( D' - \hat{d}(t) \right) \left( V + \hat{v}(t) \right) \]

note that \( d'(t) \) is given by

\[ d'(t) = (1 - d(t)) = 1 - \left( D + \hat{d}(t) \right) = D' - \hat{d}(t) \quad \text{with} \quad D' = 1 - D \]

Multiply out and collect terms:

\[ L \left( \frac{d^0 i(t)}{dt} + \frac{d i(t)}{dt} \right) = \left( D V_g + D' V \right) + \left( D \hat{v}_g(t) + D' \hat{v}(t) + \left( V_g - V \right) \hat{d}(t) \right) + \hat{d}(t) \left( \hat{v}_g(t) - \hat{v}(t) \right) \]

\[ \text{Dc terms} \quad 1^{\text{st}} \text{order ac terms (linear)} \quad 2^{\text{nd}} \text{order ac terms (nonlinear)} \]
The perturbed inductor equation

\[ L \left( \frac{d^0}{dt^0} + \frac{di(t)}{dt} \right) = \left( DV_g + D'V \right) + \left( D\dot{v}_g(t) + D'\dot{v}(t) + \left( V_g - V \right) \hat{d}(t) \right) + \hat{d}(t) \left( \dot{v}_g(t) - \dot{v}(t) \right) \]

\( \text{Dc terms} \quad 1^{st} \text{ order ac terms} \quad 2^{nd} \text{ order ac terms} \)

Since \( I \) is a constant (dc) term, its derivative is zero

The right-hand side contains three types of terms:

- Dc terms, containing only dc quantities
- First-order ac terms, containing a single ac quantity, usually multiplied by a constant coefficient such as a dc term. These are linear functions of the ac variations
- Second-order ac terms, containing products of ac quantities. These are nonlinear, because they involve multiplication of ac quantities
Neglect of second-order terms

\[
L \left( \frac{d^0}{dt} + \frac{di(t)}{dt} \right) = \left[ DV_g + D'V \right] + \left[ Dv_g(t) + D'v(t) + \left(V_g - V\right)\hat{d}(t) \right] + \hat{d}(t) \left[ \hat{v}_g(t) - \hat{v}(t) \right]
\]

Dc terms \hspace{1cm} 1^{st} order ac terms \hspace{1cm} 2^{nd} order ac terms
(linear) \hspace{1cm} (nonlinear)

Provided \[|\hat{v}_g(t)| << |V_g|\], \[|\hat{d}(t)| << |D|\], \[|i(t)| << |I|\], \[|\hat{v}(t)| << |V|\], \[|\hat{i}_g(t)| << |I_g|\] then the second-order ac terms are much smaller than the first-order terms. For example,
\[|\hat{d}(t)| \hat{v}_g(t)| << |D \hat{v}_g(t)|\] when \[|\hat{d}(t)| << D\]

So neglect second-order terms. Also, dc terms on each side of equation are equal.
Linearized inductor equation

Upon discarding second-order terms, and removing dc terms (which add to zero), we are left with

\[ L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t) \]

This is the desired result: a linearized equation which describes small-signal ac variations.

Note that the quiescent values \( D, D', V, V_g \), are treated as given constants in the equation.
Capacitor equation

Perturbation leads to

\[ C \frac{d[V + \varphi(t)]}{dt} = - \left( D' - \dot{d}(t) \right) \left( I + \dot{i}(t) \right) - \frac{[V + \varphi(t)]}{R} \]

Collect terms:

\[ C \left( \frac{dV}{dt} + \frac{d\varphi(t)}{dt} \right) = \left( -D'I - \frac{V}{R} \right) + \left( -D'i(t) - \frac{\dot{\varphi}(t)}{R} + I \ddot{d}(t) \right) + \ddot{d}(t)\dot{i}(t) \]

- **Dc terms**
- **1st order ac terms** (linear)
- **2nd order ac term** (nonlinear)

Neglect second-order terms. Dc terms on both sides of equation are equal. The following terms remain:

\[ C \frac{d\varphi(t)}{dt} = - D'i(t) - \frac{\dot{\varphi}(t)}{R} + I \ddot{d}(t) \]

This is the desired small-signal linearized capacitor equation.
Average input current

Perturbation leads to

\[ I_g + \dot{i}_g(t) = \left(D + \dot{d}(t)\right) \left(I + \dot{i}(t)\right) \]

Collect terms:

\[
\frac{I_g}{D} + \frac{\dot{i}_g(t)}{D\dot{I}} = \left[D\dot{I}\right] + \left[D\dot{i}(t) + I\dot{d}(t)\right] + \dot{d}(t)\dot{i}(t)
\]

\[ Dc \text{ term} \quad 1^{st} \text{ order ac term} \quad Dc \text{ term} \quad 1^{st} \text{ order ac terms} \quad 2^{nd} \text{ order ac term} \]

\[ \text{(linear)} \quad \text{(nonlinear)} \]

Neglect second-order terms. Dc terms on both sides of equation are equal. The following first-order terms remain:

\[ \dot{i}_g(t) = D\dot{i}(t) + I\dot{d}(t) \]

This is the linearized small-signal equation which described the converter input port.
7.2.6. Construction of small-signal equivalent circuit model

The linearized small-signal converter equations:

\[
L \frac{d\hat{i}(t)}{dt} = D\hat{V}_g(t) + D'\hat{\phi}(t) + \left[V_g - V\right]\hat{d}(t)
\]

\[
C \frac{d\hat{\phi}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)
\]

\[
\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)
\]

Reconstruct equivalent circuit corresponding to these equations, in manner similar to the process used in Chapter 3.
Inductor loop equation

\[ L \frac{d\hat{i}(t)}{dt} = D\hat{\psi}_g(t) + D'\hat{\psi}(t) + \left(V_g - V\right) \hat{d}(t) \]
Capacitor node equation

\[ C \frac{d\hat{v}(t)}{dt} = -D' \hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t) \]
Input port node equation

\[ i_g(t) = D\dot{i}(t) + I\ddot{i}(t) \]
Complete equivalent circuit

Collect the three circuits:

Replace dependent sources with ideal dc transformers:

Small-signal ac equivalent circuit model of the buck-boost converter
7.2.7. Results for several basic converters

**buck**

\[ V_g \dot{d}(t) \]

\[ \dot{i}(t) \]

\[ L \]

\[ C, R \]

\[ v(t) \]

\[ \Phi_g(t) \]

\[ I \dot{d}(t) \]

**boost**

\[ V \dot{d}(t) \]

\[ D' : I \]

\[ L \]

\[ C, R \]

\[ v(t) \]

\[ \Phi_g(t) \]

\[ I \dot{d}(t) \]
Results for several basic converters

*buck-boost*

\[
\begin{align*}
\frac{dL}{dt} &= (V_s - V) \hat{a}(t) \\
\frac{dv}{dt} &= i(t) \\
\frac{di}{dt} &= v(t) - R \hat{a}(t)
\end{align*}
\]
7.3. Example: a nonideal flyback converter

Flyback converter example

- MOSFET has on-resistance $R_{on}$
- Flyback transformer has magnetizing inductance $L$, referred to primary
Circuits during subintervals 1 and 2

Flyback converter, with transformer equivalent circuit

Subinterval 1

Subinterval 2
Subinterval 1

Circuit equations:

\[ v_L(t) = v_g(t) - i(t) \frac{R_{on}}{R} \]
\[ i_C(t) = -\frac{v(t)}{R} \]
\[ i_g(t) = i(t) \]

Small ripple approximation:

\[ v_L(t) = \left\langle v_g(t) \right\rangle_{T_s} - \left\langle i(t) \right\rangle_{T_s} \frac{R_{on}}{R} \]
\[ i_C(t) = -\frac{\left\langle v(t) \right\rangle_{T_s}}{R} \]
\[ i_g(t) = \left\langle i(t) \right\rangle_{T_s} \]

MOSFET conducts, diode is reverse-biased
Subinterval 2

Circuit equations:

\[ v_L(t) = -\frac{v(t)}{n} \]
\[ i_C(t) = -\frac{i(t)}{n} - \frac{v(t)}{R} \]
\[ i_g(t) = 0 \]

Small ripple approximation:

\[ v_L(t) = -\frac{\langle v(t) \rangle_{T_s}}{n} \]
\[ i_C(t) = -\frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R} \]
\[ i_g(t) = 0 \]

MOSFET is off, diode conducts
Inductor waveforms

Average inductor voltage:

\[
\langle v_L(t) \rangle_{T_s} = d(t) \left( \langle v_{g}(t) \rangle_{T_s} - \langle i(t) \rangle_{T_s} R_{on} \right) + d'(t) \left( -\frac{\langle v(t) \rangle_{T_s}}{n} \right)
\]

Hence, we can write:

\[
L \frac{d\langle i(t) \rangle_{T_s}}{dt} = d(t) \left( \langle v_{g}(t) \rangle_{T_s} - d(t) \langle i(t) \rangle_{T_s} R_{on} - d'(t) \frac{\langle v(t) \rangle_{T_s}}{n} \right)
\]
Capacitor waveforms

Average capacitor current:

\[ \langle i_C(t) \rangle_{T_s} = d(t) \left( -\frac{\langle v(t) \rangle_{T_s}}{R} \right) + d'(t) \left( \frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R} \right) \]

Hence, we can write:

\[ C \frac{d\langle v(t) \rangle_{T_s}}{dt} = d'(t) \left( \frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R} \right) \]
Input current waveform

Average input current:

\[ \langle i_s(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s} \]
The averaged converter equations

\[
L \frac{d\langle i(t) \rangle_{T_s}}{dt} = d(t) \langle v_g(t) \rangle_{T_s} - d(t) \langle i(t) \rangle_{T_s} R_{un} - d'(t) \frac{\langle v(t) \rangle_{T_s}}{n}
\]

\[
C \frac{d\langle v(t) \rangle_{T_s}}{dt} = d'(t) \frac{\langle i(t) \rangle_{T_s}}{n} - \frac{\langle v(t) \rangle_{T_s}}{R}
\]

\[
\langle i_s(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s}
\]

— a system of nonlinear differential equations

Next step: perturbation and linearization. Let

\[
\langle v_g(t) \rangle_{T_s} = V_g + \nu_g(t)
\]

\[
d(t) = D + \hat{d}(t)
\]

\[
\langle i(t) \rangle_{T_s} = I + \hat{i}(t)
\]

\[
\langle v(t) \rangle_{T_s} = V + \hat{v}(t)
\]

\[
\langle i_s(t) \rangle_{T_s} = I_s + \hat{i}_s(t)
\]
Perturbation of the averaged inductor equation

\[
L \frac{d\langle i(t) \rangle_{T_s}}{dt} = d(t) \left( V_g(t) \right)_{T_s} - d(t) \langle i(t) \rangle_{T_s} R_{on} - d'(t) \frac{\langle v(t) \rangle_{T_s}}{n}
\]

\[
L \frac{d\langle I + \hat{i}(t) \rangle}{dt} = \left( D + \hat{d}(t) \right) \left( V_g + \hat{v}_g(t) \right) - \left( D' - \hat{d}(t) \right) \frac{\left( V + \hat{v}(t) \right)}{n} - \left( D + \hat{d}(t) \right) \left( I + \hat{i}(t) \right) R_{on}
\]

\[
L \left( \frac{d^0}{dt} + \frac{d\hat{i}(t)}{dt} \right) = \left( D V_g - D' \frac{V}{n} - DR_{on} \right) I + \left( D\hat{v}_g(t) - D' \frac{\hat{v}(t)}{n} + \left( V_g + \frac{V}{n} - IR_{on} \right) \hat{d}(t) - DR_{on} \hat{i}(t) \right)
\]

\[
\begin{aligned}
Dc \ terms \quad & + \quad \left( \hat{d}(t) v_g(t) + \hat{d}(t) \frac{\hat{v}(t)}{n} - \hat{d}(t) \hat{i}(t) R_{on} \right) \\
1^{st} \ order \ ac \ terms \ (linear) \quad & + \quad \left( \hat{d}(t) \hat{v}_g(t) + \hat{d}(t) \frac{\hat{v}(t)}{n} - \hat{d}(t) \hat{i}(t) R_{on} \right) \\
2^{nd} \ order \ ac \ terms \ (nonlinear) \quad & + \quad \left( \hat{d}(t) \hat{v}_g(t) + \hat{d}(t) \frac{\hat{v}(t)}{n} - \hat{d}(t) \hat{i}(t) R_{on} \right)
\end{aligned}
\]
Linearization of averaged inductor equation

Dc terms:

\[ 0 = D V_s - D' \frac{V}{n} - D R_{on} I \]

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

\[ L \frac{d\hat{i}(t)}{dt} = D\hat{v}_s(t) - D\hat{v}(t) + \left( V_s + \frac{V}{n} - I R_{on} \right) \hat{d}(t) - D R_{on} \hat{i}(t) \]

This is the desired linearized inductor equation.
Perturbation of averaged capacitor equation

Original averaged equation:

\[ C \frac{d}{dt} \left\langle \frac{v(t)}{T_s} \right\rangle = d'\left(t\right) \frac{\langle i(t) \rangle}{n} - \frac{\langle v(t) \rangle}{R} \]

Perturb about quiescent operating point:

\[ C \frac{d\left[V + \hat{v}(t)\right]}{dt} = \left(D' - \hat{d}(t)\right) \frac{\left(I + \hat{i}(t)\right)}{n} - \frac{\left[V + \hat{v}(t)\right]}{R} \]

Collect terms:

\[ C \left( \frac{\hat{d}(0)}{dt} + \frac{d\hat{v}(t)}{dt} \right) = \left( D'H - \frac{V}{R} \right) + \left( D'\hat{i}(t) - \frac{\hat{v}(t)}{R} - I\hat{d}(t) \right) - \frac{\hat{d}(t)i(t)}{n} \]

- Dc terms
- 1st order ac terms (linear)
- 2nd order ac term (nonlinear)
Linearization of averaged capacitor equation

Dc terms:

\[ 0 = \left( \frac{D'y}{n} - \frac{V}{R} \right) \]

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

\[ C \frac{d\dot{v}(t)}{dt} = \frac{D'i(t)}{n} - \frac{\dot{v}(t)}{R} - \frac{I\ddot{d}(t)}{n} \]

This is the desired linearized capacitor equation.
Perturbation of averaged input current equation

Original averaged equation:

\[ \langle i_g(t) \rangle_{T_s} = d(t) \langle i(t) \rangle_{T_s} \]

Perturb about quiescent operating point:

\[ I_g + \dot{i}_g(t) = \left( D + \dot{d}(t) \right) \left( I + \dot{i}(t) \right) \]

Collect terms:

\[
\begin{align*}
I_g + \dot{i}_g(t) &= \underbrace{\left( DI \right)}_{Dc \, term} + \underbrace{\left( D\dot{i}(t) + I\dot{d}(t) \right)}_{1^{st} \, order \, ac \, term} + \underbrace{\dot{d}(t)\dot{i}(t)}_{2^{nd} \, order \, ac \, term} \\
\end{align*}
\]
Linearization of averaged input current equation

Dc terms:

\[ I_g = DI \]

Second-order terms are small when the small-signal assumption is satisfied. The remaining first-order terms are:

\[ \dot{i}_g(t) = D\dot{i}(t) + I\ddot{i}(t) \]

This is the desired linearized input current equation.
Summary: dc and small-signal ac converter equations

Dc equations:

\[ 0 = DV_g - D'I \frac{V}{n} - DR_{on}I \]
\[ 0 = \left( \frac{D'I}{n} - \frac{V}{R} \right) \]
\[ I_g = DI \]

Small-signal ac equations:

\[ L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\hat{v}(t) + \left( V_g + \frac{V}{n} - IR_{on} \right) \hat{d}(t) - DR_{on}\hat{i}(t) \]
\[ C \frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n} \]
\[ \hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t) \]

Next step: construct equivalent circuit models.
Small-signal ac equivalent circuit: inductor loop

\[ L \frac{d\bar{i}(t)}{dt} = D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\bar{d}(t) - DR_{on}\bar{i}(t) \]
Small-signal ac equivalent circuit: capacitor node

\[ C \frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{a}(t)}{n} \]
Small-signal ac equivalent circuit: converter input node

\[ i_g(t) = D\dot{i}(t) + I\ddot{d}(t) \]
Small-signal ac model, nonideal flyback converter example

Combine circuits:

Replace dependent sources with ideal transformers:

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Chapter 7: AC equivalent circuit modeling