Chapter 2
Principles of Steady-State Converter Analysis

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2.1 Introduction
Buck converter

**SPDT switch changes dc component**

**Switch output voltage waveform**

Duty cycle $D$:
$0 \leq D \leq 1$

*complement $D'$: $D' = 1 - D$*
Dc component of switch output voltage

\[ \langle v_s \rangle = \frac{1}{T_s} \int_0^{T_s} v_s(t) \, dt \]

\[ \langle v_s \rangle = \frac{1}{T_s} (DT_s V_g) = DV_g \]

Fourier analysis: Dc component = average value

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Insertion of low-pass filter to remove switching harmonics and pass only dc component

\[ v \approx \langle v_s \rangle = D V_g \]
Three basic dc-dc converters

Buck

Boost

Buck-boost
Objectives of this chapter

- Develop techniques for easily determining output voltage of an arbitrary converter circuit
- Derive the principles of *inductor volt-second balance* and *capacitor charge (amp-second) balance*
- Introduce the key *small ripple approximation*
- Develop simple methods for selecting filter element values
- Illustrate via examples
2.2. Inductor volt-second balance, capacitor charge balance, and the small ripple approximation

**Actual output voltage waveform, buck converter**

**Buck converter containing practical low-pass filter**

Actual output voltage waveform

\[ v(t) = V + v_{\text{ripple}}(t) \]

Actual waveform

\[ v(t) = V + v_{\text{ripple}}(t) \]

**dc component V**
The small ripple approximation

In a well-designed converter, the output voltage ripple is small. Hence, the waveforms can be easily determined by ignoring the ripple:

\[ v(t) = V + v_{\text{ripple}}(t) \]

\[ v(t) \approx V \]

\[ \|v_{\text{ripple}}\| < V \]
Details:
The small ripple approximation

1. For reasons that will become apparent as the course progresses, the small ripple approximation is useful only for continuous waveforms that have small ripple. Specifically, it is applied only to:
   - Inductor currents
   - Capacitor voltages

2. The small ripple approximation must not be applied to discontinuous waveforms, i.e., waveforms that switch.

3. The small ripple approximation is used to simplify the solution of the filter element waveforms. Instead of exponential and damped sinusoidal solutions for the circuit differential equations, the small ripple approximation allows approximation of the solutions as linear functions. The approximation is valid provided that the switching period $T_s$ is short compared to the natural time constants of the circuit.
Buck converter analysis: inductor current waveform

Original converter

Switch in position 1

Switch in position 2
Inductor voltage and current
Subinterval 1: switch in position 1

**Inductor voltage**

\[ v_L = V_g - v(t) \]

*Small ripple approximation:*

\[ v_L \approx V_g - V \]

Knowing the inductor voltage, we can now find the inductor current via

\[ v_L(t) = L \frac{di_L(t)}{dt} \]

**Solve for the slope:**

\[ \frac{di_L(t)}{dt} = \frac{v_L(t)}{L} \approx \frac{V_g - V}{L} \]

⇒ *The inductor current changes with an essentially constant slope*
Inductor voltage and current
Subinterval 2: switch in position 2

Inductor voltage

\[ v_L(t) = -v(t) \]

Small ripple approximation:

\[ v_L(t) \approx -V \]

Knowing the inductor voltage, we can again find the inductor current via

\[ v_L(t) = L \frac{di_L(t)}{dt} \]

Solve for the slope:

\[ \frac{di_L(t)}{dt} \approx -\frac{V}{L} \]

⇒ The inductor current changes with an essentially constant slope
Inductor voltage and current waveforms

\[ v_L(t) = L \frac{di_L(t)}{dt} \]

Switch position:

1. \( V_g - V \)
2. \( -V \)
3. \( V_L \)

\[ i_L(0) \]
\[ i_L(DT_s) \]
\[ \frac{V_R - V}{L} \]
\[ \frac{-V}{L} \]

\[ v_L(t) = V_g - V \]
\[ DT_s \]
\[ DT_s \]

\[ \Delta i_L \]
Determination of inductor current ripple magnitude

\[
\text{(change in } i_L) = \text{(slope) (length of subinterval)}
\]

\[
2\Delta i_L = \left( \frac{V_g - V}{L} \right) DT_s
\]

\[
\Rightarrow \quad \Delta i_L = \frac{V_g - V}{2L} DT_s \quad \quad L = \frac{V_g - V}{2\Delta i_L} DT_s
\]
When the converter operates in equilibrium:

\[ i_L((n + 1)T_s) = i_L(nT_s) \]
The principle of inductor volt-second balance: Derivation

Inductor defining relation:

\[ v_L(t) = L \frac{di_L(t)}{dt} \]

Integrate over one complete switching period:

\[ i_L(T_s) - i_L(0) = \frac{1}{L} \int_0^{T_s} v_L(t) \, dt \]

In periodic steady state, the net change in inductor current is zero:

\[ 0 = \int_0^{T_s} v_L(t) \, dt \]

Hence, the total area (or volt-seconds) under the inductor voltage waveform is zero whenever the converter operates in steady state.

An equivalent form:

\[ 0 = \frac{1}{T_s} \int_0^{T_s} v_L(t) \, dt = \langle v_L \rangle \]

The average inductor voltage is zero in steady state.
Inductor volt-second balance: Buck converter example

Integral of voltage waveform is area of rectangles:
\[ \lambda = \int_{0}^{T_s} v_L(t) \, dt = (V_g - V)(DT_s) + (-V)(D'T_s) \]

Average voltage is
\[ \langle v_L \rangle = \frac{\lambda}{T_s} = D(V_g - V) + D'( - V) \]

Equate to zero and solve for \( V \):
\[ 0 = DV_g - (D + D')V = DV_g - V \quad \Rightarrow \quad V = DV_g \]
The principle of capacitor charge balance: Derivation

Capacitor defining relation:

\[ i_c(t) = C \frac{dv_c(t)}{dt} \]

Integrate over one complete switching period:

\[ v_c(T_s) - v_c(0) = \frac{1}{C} \int_0^{T_s} i_c(t) \, dt \]

In periodic steady state, the net change in capacitor voltage is zero:

\[ 0 = \frac{1}{T_s} \int_0^{T_s} i_c(t) \, dt = \langle i_c \rangle \]

Hence, the total area (or charge) under the capacitor current waveform is zero whenever the converter operates in steady state. The average capacitor current is then zero.
2.3 Boost converter example

Boost converter with ideal switch

Realization using power MOSFET and diode
Boost converter analysis

Original converter

Switch in position 1

Switch in position 2

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Subinterval 1: switch in position 1

**Inductor voltage and capacitor current**

\[
\begin{align*}
  v_L &= V_g \\
  i_C &= -\frac{v}{R}
\end{align*}
\]

**Small ripple approximation:**

\[
\begin{align*}
  v_L &= V_g \\
  i_C &= -\frac{V}{R}
\end{align*}
\]
Subinterval 2: switch in position 2

**Inductor voltage and capacitor current**

\[ v_L = V_g - v \]
\[ i_C = i_L - v / R \]

**Small ripple approximation:**

\[ v_L = V_g - V \]
\[ i_C = I - V / R \]
Inductor voltage and capacitor current waveforms

\[ v_L(t) \]

\[ V_g - V \]

\[ DT_s \]

\[ D'T_s \]

\[ i_C(t) \]

\[ I - V/R \]

\[ -V/R \]
Inductor volt-second balance

Net volt-seconds applied to inductor over one switching period:

\[ \int_0^{T_s} v_L(t) \, dt = (V_g) \, DT_s + (V_g - V) \, D'T_s \]

Equate to zero and collect terms:

\[ V_g (D + D') - V \, D' = 0 \]

Solve for \( V \):

\[ V = \frac{V_g}{D'} \]

The voltage conversion ratio is therefore:

\[ M(D) = \frac{V}{V_g} = \frac{1}{D'} = \frac{1}{1 - D} \]
Conversion ratio $M(D)$ of the boost converter

$$M(D) = \frac{1}{D'} = \frac{1}{1 - D}$$
Determination of inductor current dc component

Capacitor charge balance:

\[ \int_{0}^{T_s} i_C(t) \, dt = \left( -\frac{V}{R} \right) DT_s + (I - \frac{V}{R}) D'T_s \]

Collect terms and equate to zero:

\[-\frac{V}{R} (D + D') + I D' = 0 \]

Solve for \( I \):

\[ I = \frac{V}{D'R} \]

Eliminate \( V \) to express in terms of \( V_g \):

\[ I = \frac{V_g}{D'T^2 R} \]
Determination of inductor current ripple

Inductor current slope during subinterval 1:
\[ \frac{di_L(t)}{dt} = \frac{v_L(t)}{L} = \frac{V_g}{L} \]

Inductor current slope during subinterval 2:
\[ \frac{di_L(t)}{dt} = \frac{v_L(t)}{L} = \frac{V_g - V}{L} \]

Change in inductor current during subinterval 1 is (slope) (length of subinterval):
\[ 2\Delta i_L = \frac{V_g}{L} DT_s \]

Solve for peak ripple:
\[ \Delta i_L = \frac{V_g}{2L} DT_s \]

- Choose \( L \) such that desired ripple magnitude is obtained
Determination of capacitor voltage ripple

Capacitor voltage slope during subinterval 1:
\[
\frac{dv_c(t)}{dt} = \frac{i_c(t)}{C} = -\frac{V}{RC}
\]

Capacitor voltage slope during subinterval 2:
\[
\frac{dv_c(t)}{dt} = \frac{i_c(t)}{C} = \frac{I}{C} - \frac{V}{RC}
\]

Change in capacitor voltage during subinterval 1 is (slope) (length of subinterval):
\[
-2\Delta v = -\frac{V}{RC} DT_s
\]

Solve for peak ripple:
\[
\Delta v = \frac{V}{2RC} DT_s
\]

- Choose C such that desired voltage ripple magnitude is obtained
- In practice, capacitor equivalent series resistance (esr) leads to increased voltage ripple

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Chapter 2: Principles of steady-state converter analysis
2.4 Cuk converter example

Cuk converter, with ideal switch

\[ + \quad L_1 \quad C_1 \quad L_2 \quad + \]

\[ i_1 \quad + v_1 \quad - \quad 2 \quad i_2 \]

\[ C_2 \quad v_2 \quad R \]

Cuk converter: practical realization using MOSFET and diode

\[ + \quad L_1 \quad C_1 \quad L_2 \quad + \]

\[ i_1 \quad + v_1 \quad - \quad 2 \quad i_2 \]

\[ V_g \quad Q_1 \quad D_1 \quad C_2 \quad v_2 \quad R \]
Analysis strategy

This converter has two inductor currents and two capacitor voltages, that can be expressed as

\[ i_1(t) = I_1 + i_{1-\text{ripple}}(t) \]
\[ i_2(t) = I_2 + i_{2-\text{ripple}}(t) \]
\[ v_1(t) = V_1 + v_{1-\text{ripple}}(t) \]
\[ v_2(t) = V_2 + v_{2-\text{ripple}}(t) \]

To solve the converter in steady state, we want to find the dc components \( I_1, I_2, V_1, \) and \( V_2, \) when the ripples are small.

**Strategy:**
- Apply volt-second balance to each inductor voltage
- Apply charge balance to each capacitor current
- Simplify using the small ripple approximation
- Solve the resulting four equations for the four unknowns \( I_1, I_2, V_1, \) and \( V_2. \)
Cuk converter circuit
with switch in positions 1 and 2

Switch in position 1:
MOSFET conducts
Capacitor $C_1$ releases energy to output

Switch in position 2:
diode conducts
Capacitor $C_1$ is charged from input
Waveforms during subinterval 1
MOSFET conduction interval

Inductor voltages and capacitor currents:

\[ v_{L1} = V_g \]
\[ v_{L2} = -v_1 - v_2 \]
\[ i_{C1} = i_2 \]
\[ i_{C2} = i_2 - \frac{v_2}{R} \]

Small ripple approximation for subinterval 1:

\[ v_{L1} = V_g \]
\[ v_{L2} = -V_1 - V_2 \]
\[ i_{C1} = I_2 \]
\[ i_{C2} = I_2 - \frac{V_2}{R} \]
Waveforms during subinterval 2
Diode conduction interval

Inductor voltages and capacitor currents:

\[ v_{L1} = V_g - v_1 \]
\[ v_{L2} = -v_2 \]
\[ i_{C1} = i_1 \]
\[ i_{C2} = i_2 - \frac{v_2}{R} \]

Small ripple approximation for subinterval 2:

\[ v_{L1} = V_g - V_1 \]
\[ v_{L2} = -V_2 \]
\[ i_{C1} = I_1 \]
\[ i_{C2} = I_2 - \frac{V_2}{R} \]
Equate average values to zero

The principles of inductor volt-second and capacitor charge balance state that the average values of the periodic inductor voltage and capacitor current waveforms are zero, when the converter operates in steady state. Hence, to determine the steady-state conditions in the converter, let us sketch the inductor voltage and capacitor current waveforms, and equate their average values to zero.

**Waveforms:**

**Inductor voltage** $v_{L1}(t)$

![Inductor voltage waveform](image)

**Volt-second balance on $L_1$:**

$$\left\langle v_{L1} \right\rangle = DV_g + D'(V_g - V_1) = 0$$
Equate average values to zero

**Inductor $L_2$ voltage**

$$v_{L2}(t) - V_2$$ 

$$-V_1 - V_2$$ 

$$DT_s$$ 

$$DT_i$$ 

$t$

Average the waveforms:

$$\langle v_{L2} \rangle = D(-V_1 - V_2) + D'(-V_2) = 0$$

**Capacitor $C_1$ current**

$$i_{C1}(t) = I_1$$ 

$$I_2$$ 

$$DT_s$$ 

$$DT_i$$ 

$t$

$$\langle i_{C1} \rangle = DI_2 + D'I_1 = 0$$
Equate average values to zero

Capacitor current $i_{c2}(t)$ waveform

\[
\langle i_{c2} \rangle = I_2 - \frac{V_2}{R} = 0
\]

Note: during both subintervals, the capacitor current $i_{c2}$ is equal to the difference between the inductor current $i_2$ and the load current $V_2/R$. When ripple is neglected, $i_{c2}$ is constant and equal to zero.
Solve for steady-state inductor currents and capacitor voltages

The four equations obtained from volt-sec and charge balance:

\[
\begin{align*}
\langle v_{L1} \rangle &= DV_g + D'(V_g - V_1) = 0 \\
\langle v_{L2} \rangle &= D(-V_1 - V_2) + D'(-V_2) = 0 \\
\langle i_{C1} \rangle &= DI_2 + DI_1 = 0 \\
\langle i_{C2} \rangle &= I_2 - \frac{V_2}{R} = 0
\end{align*}
\]

Solve for the dc capacitor voltages and inductor currents, and express in terms of the known \( V_g, D, \) and \( R \):

\[
\begin{align*}
V_1 &= \frac{V_g}{D} \\
V_2 &= -\frac{D}{D'} V_g \\
I_1 &= -\frac{D}{D'} I_2 = \left(\frac{D}{D'}\right)^2 V_g \\
I_2 &= \frac{V_2}{R} = -\frac{D}{D'} V_g \\
\end{align*}
\]
Cuk converter conversion ratio \( M = \frac{V}{V_g} \)

\[
M(D) = \frac{V_2}{V_g} = -\frac{D}{1-D}
\]
Inductor current waveforms

Interval 1 slopes, using small ripple approximation:

\[
\frac{di_1(t)}{dt} = \frac{v_{L1}(t)}{L_1} = \frac{V_g}{L_1} \\
\frac{di_2(t)}{dt} = \frac{v_{L2}(t)}{L_2} = \frac{-V_1 - V_2}{L_2}
\]

Interval 2 slopes:

\[
\frac{di_1(t)}{dt} = \frac{v_{L1}(t)}{L_1} = \frac{V_g - V_1}{L_1} \\
\frac{di_2(t)}{dt} = \frac{v_{L2}(t)}{L_2} = \frac{-V_2}{L_2}
\]
Capacitor $C_1$ waveform

Subinterval 1:
\[
\frac{dv_1(t)}{dt} = \frac{i_{C1}(t)}{C_1} = \frac{I_2}{C_1}
\]

Subinterval 2:
\[
\frac{dv_1(t)}{dt} = \frac{i_{C1}(t)}{C_1} = \frac{I_1}{C_1}
\]
Ripple magnitudes

Analysis results

\[ \Delta i_1 = \frac{V_s DT_s}{2L_1} \]

\[ \Delta i_2 = \frac{V_1 + V_2}{2L_2} DT_s \]

\[ \Delta v_1 = \frac{-I_2 DT_s}{2C_1} \]

Use dc converter solution to simplify:

\[ \Delta i_1 = \frac{V_s DT_s}{2L_1} \]

\[ \Delta i_2 = \frac{V_s DT_s}{2L_2} \]

\[ \Delta v_1 = \frac{V_s D^2 T_s}{2D'RC_1} \]

Q: How large is the output voltage ripple?
2.5 Estimating ripple in converters containing two-pole low-pass filters

Buck converter example: Determine output voltage ripple

Inductor current waveform.
What is the capacitor current?
Capacitor current and voltage, buck example

Must not neglect inductor current ripple!

If the capacitor voltage ripple is small, then essentially all of the ac component of inductor current flows through the capacitor.
Estimating capacitor voltage ripple $\Delta v$

Current $i_C(t)$ is positive for half of the switching period. This positive current causes the capacitor voltage $v_C(t)$ to increase between its minimum and maximum extrema. During this time, the total charge $q$ is deposited on the capacitor plates, where

$$q = C (2\Delta v)$$

$$(\text{change in charge}) = C (\text{change in voltage})$$
Estimating capacitor voltage ripple $\Delta v$

The total charge $q$ is the area of the triangle, as shown:

$$q = \frac{1}{2} \Delta i_L \frac{T_s}{2}$$

Eliminate $q$ and solve for $\Delta v$:

$$\Delta v = \frac{\Delta i_L T_s}{8 C}$$

Note: in practice, capacitor equivalent series resistance (esr) further increases $\Delta v$. 
Inductor current ripple in two-pole filters

Example: problem 2.9

The total flux linkage can use similar arguments, with

\[ \lambda = L(2\Delta i) \]

\( \lambda \) = inductor flux linkages

= inductor volt-seconds
2.6 Summary of Key Points

1. The dc component of a converter waveform is given by its average value, or the integral over one switching period, divided by the switching period. Solution of a dc-dc converter to find its dc, or steady-state, voltages and currents therefore involves averaging the waveforms.

2. The linear ripple approximation greatly simplifies the analysis. In a well-designed converter, the switching ripples in the inductor currents and capacitor voltages are small compared to the respective dc components, and can be neglected.

3. The principle of inductor volt-second balance allows determination of the dc voltage components in any switching converter. In steady-state, the average voltage applied to an inductor must be zero.
Summary of Chapter 2

4. The principle of capacitor charge balance allows determination of the dc components of the inductor currents in a switching converter. In steady-state, the average current applied to a capacitor must be zero.

5. By knowledge of the slopes of the inductor current and capacitor voltage waveforms, the ac switching ripple magnitudes may be computed. Inductance and capacitance values can then be chosen to obtain desired ripple magnitudes.

6. In converters containing multiple-pole filters, continuous (nonpulsating) voltages and currents are applied to one or more of the inductors or capacitors. Computation of the ac switching ripple in these elements can be done using capacitor charge and/or inductor flux-linkage arguments, without use of the small-ripple approximation.

7. Converters capable of increasing (boost), decreasing (buck), and inverting the voltage polarity (buck-boost and Cuk) have been described. Converter circuits are explored more fully in a later chapter.