9.3. Construction of the important quantities \(1/(1+T)\) and \(T/(1+T)\)

Example

\[ T(s) = T_0 \frac{\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{Q\omega_p} + \left(\frac{s}{\omega_p}\right)^2\right)\left(1 + \frac{s}{\omega_p^2}\right)} \]

At the crossover frequency \(f_c\), \(\| T \| = 1\)
Transient response vs. damping factor

\[ \hat{v}(t) \]

vs.

\[ \omega_c t, \text{ radians} \]
9.4. Stability

Even though the original open-loop system is stable, the closed-loop transfer functions can be unstable and contain right half-plane poles. Even when the closed-loop system is stable, the transient response can exhibit undesirable ringing and overshoot, due to the high $Q$-factor of the closed-loop poles in the vicinity of the crossover frequency.

When feedback destabilizes the system, the denominator $(1+T(s))$ terms in the closed-loop transfer functions contain roots in the right half-plane (i.e., with positive real parts). If $T(s)$ is a rational fraction of the form $N(s) / D(s)$, where $N(s)$ and $D(s)$ are polynomials, then we can write

\[
\frac{T(s)}{1 + T(s)} = \frac{\frac{N(s)}{D(s)}}{1 + \frac{N(s)}{D(s)}} = \frac{N(s)}{N(s) + D(s)}
\]

\[
\frac{1}{1 + T(s)} = \frac{1}{1 + \frac{N(s)}{D(s)}} = \frac{D(s)}{N(s) + D(s)}
\]

- Could evaluate stability by evaluating $N(s) + D(s)$, then factoring to evaluate roots. This is a lot of work, and is not very illuminating.
9.4. Stability

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- Could evaluate stability by evaluating $N(s) + D(s)$, then factoring to evaluate roots. This is a lot of work, and is not very illuminating.

**Effect of feedback on transfer function poles**

Feedback moves the poles of the system transfer functions

- Good news: we can use feedback to alter the poles and improve the frequency response

- Bad news: if you’re not careful, feedback can move the poles into the right half of the complex $s$-plane (poles have positive real parts), leading to an unstable system

![Open loop](image1)

![Closed loop](image2)
Example

The gain $G(s)$ below has three poles at $s = -1$

$$G(s) = \frac{100}{(1 + s)^3}$$

Add a simple feedback loop:

How does the feedback change the poles?

For our simple example, the closed-loop transfer function is

$$\frac{v_{out}}{v_{in}} = \frac{1}{H} \frac{T}{1 + T} \frac{G}{1 + G} = \frac{100}{1 + 100} \frac{(1 + s)^3}{1 + \frac{100}{1 + s}} = \frac{100}{101 + 3s + 3s^2 + s^3}$$

Factor denominator numerically:

$$\frac{v_{out}}{v_{in}} = \frac{100}{101 + 3s + 3s^2 + s^3} = \frac{100}{(s + 5.64)(s - 1.32 - j4.07)(s - 1.32 + j4.07)}$$

which has poles at $s = -5.64$ (LHP)

and at $s = +1.32 \pm j4.07$ (RHP)

The RHP poles indicate that the closed-loop system is unstable.
Example

The gain $G(s)$ below has three poles at $s = -1$

$$G(s) = \frac{100}{(1 + s)^3}$$

Add a simple feedback loop:

$$G(s) = \frac{100}{1 + s}$$

$G(s)$

$\hat{v}_{in}(s)$

$\hat{v}(s)$

$\hat{v}_{out}(s)$

$T(s)$

$H(s) = 1$

How does the feedback change the poles?

Exact closed-loop transfer function

For our simple example, the closed-loop transfer function is

$$\frac{\hat{v}_{out}}{\hat{v}_{in}} = \frac{1}{H} \frac{T}{1 + T} = \frac{G}{1 + G} = \frac{100}{1 + \frac{100}{1 + s}} = \frac{100}{1 + 3s + 3s^2 + s^3}$$

Factor denominator numerically:

$$\frac{\hat{v}_{out}}{\hat{v}_{in}} = \frac{100}{101 + 3s + 3s^2 + s^3} = \frac{100}{(s + 5.64)(s - 1.32 - j4.07)(s - 1.32 + j4.07)}$$

which has poles at $s = -5.64$ (LHP)
and at $s = +1.32 \pm j4.07$ (RHP)
The RHP poles indicate that the closed-loop system is unstable.
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Transient response of closed-loop system

One can take the inverse Laplace Transform to find the output waveform \( \hat{v}_{out}(t) \) for a given input. The resulting expression has terms that depend on the poles, of the form

\[
\hat{v}_{out}(t) = K_1 e^{-5.64t} + K_2 e^{(1.32 - j4.07)t} + K_2^* e^{(1.32 + j4.07)t}
\]

The terms with positive real exponents, corresponding to the RHP poles, lead to growing oscillations that are unstable responses.

Reason: the inverse Laplace transform of \( K_2 e^{(1.32 - j4.07)t} + K_2^* e^{(1.32 + j4.07)t} \) is

\[
|K_2| e^{1.32t} \cos\left(4.07t + \angle K_2\right)
\]

Determination of stability directly from \( T(s) \)

- Nyquist stability theorem: general result.
- A special case of the Nyquist stability theorem: the phase margin test
  
  Allows determination of closed-loop stability (i.e., whether \( 1/(1+T(s)) \) contains RHP poles) directly from the magnitude and phase of \( T(s) \).
  
  A good design tool: yields insight into how \( T(s) \) should be shaped, to obtain good performance in transfer functions containing \( 1/(1+T(s)) \) terms.
Determination of stability directly from $T(s)$

- Nyquist stability theorem: general result.
- A special case of the Nyquist stability theorem: the phase margin test

  Allows determination of closed-loop stability (i.e., whether $1/(1+T(s))$ contains RHP poles) directly from the magnitude and phase of $T(s)$.

  A good design tool: yields insight into how $T(s)$ should be shaped, to obtain good performance in transfer functions containing $1/(1+T(s))$ terms.
9.4.1. The phase margin test

A test on $T(s)$, to determine whether $1/(1+T(s))$ contains RHP poles.

The crossover frequency $f_c$ is defined as the frequency where

$$\| T(j2\pi f_c) \| = 1 \Rightarrow 0\text{dB}$$

The phase margin $\varphi_m$ is determined from the phase of $T(s)$ at $f_c$, as follows:

$$\varphi_m = 180^\circ + \angle T(j2\pi f_c)$$

If there is exactly one crossover frequency, and if $T(s)$ contains no RHP poles, then

the quantities $T(s)/(1+T(s))$ and $1/(1+T(s))$ contain no RHP poles whenever the phase margin $\varphi_m$ is positive.
Example: a loop gain leading to a stable closed-loop system

\[ \angle T(j2\pi f_c) = -112^\circ \]

\[ \varphi_m = 180^\circ - 112^\circ = +68^\circ \]
Example: a loop gain leading to an unstable closed-loop system

\[ \angle T(j2\pi f_c) = -230^\circ \]

\[ \varphi_m = 180^\circ - 230^\circ = -50^\circ \]
9.4.2. The relation between phase margin and closed-loop damping factor

How much phase margin is required?

A small positive phase margin leads to a stable closed-loop system having complex poles near the crossover frequency with high $Q$. The transient response exhibits overshoot and ringing.

Increasing the phase margin reduces the $Q$. Obtaining real poles, with no overshoot and ringing, requires a large phase margin.

The relation between phase margin and closed-loop $Q$ is quantified in this section.
A simple second-order system

Consider the case where $T(s)$ can be well-approximated in the vicinity of the crossover frequency as

$$T(s) = \frac{1}{\left(\frac{s}{\omega_0}\right) \left(1 + \frac{s}{\omega_2}\right)}$$
Closed-loop response

If
\[ T(s) = \left( \frac{s}{\omega_0} \right) \left( 1 + \frac{s}{\omega_2} \right) \]

Then
\[ \frac{T(s)}{1 + T(s)} = \frac{1}{1 + \frac{1}{T(s)}} = \frac{1}{1 + \frac{s}{\omega_0} + \frac{s^2}{\omega_0 \omega_2}} \]

or,
\[ \frac{T(s)}{1 + T(s)} = \frac{1}{1 + \frac{s}{Q \omega_c} + \left( \frac{s}{\omega_c} \right)^2} \]

where
\[ \omega_c = \sqrt{\omega_0 \omega_2} = 2\pi f_c \quad Q = \frac{\omega_0}{\omega_c} = \sqrt{\frac{\omega_0}{\omega_2}} \]
Low-\(Q\) case

\[ Q = \frac{\omega_0}{\omega_c} = \sqrt{\frac{\omega_0}{\omega_2}} \]

low-\(Q\) approximation:

\[ Q \omega_c = \omega_0 \quad \frac{\omega_c}{Q} = \omega_2 \]

\[ f_0 \]

\[ f_2 \]

\[ \frac{f_0}{f} \]

\[ \frac{T}{1 + T} \]

\[ f_c = \sqrt{f_0 f_2} \]

\[ Q = \frac{f_0}{f_c} \]

\[ -20 \text{ dB/decade} \]

\[ -40 \text{ dB/decade} \]
High-\(Q\) case

\[ \omega_c = \sqrt{\omega_0 \omega_2} = 2\pi f_c \]

\[ Q = \frac{\omega_0}{\omega_c} = \sqrt{\frac{\omega_0}{\omega_2}} \]
$Q$ vs. $\varphi_m$

Solve for exact crossover frequency, evaluate phase margin, express as function of $\varphi_m$. Result is:

$$Q = \sqrt{\frac{\cos \varphi_m}{\sin \varphi_m}}$$

$$\varphi_m = \tan^{-1} \sqrt{\frac{1 + \sqrt{1 + 4Q^4}}{2Q^4}}$$
$Q$ vs. $\varphi_m$

\[ Q = 1 \Rightarrow 0 \text{ dB} \]
\[ Q = 0.5 \Rightarrow -6 \text{ dB} \]
\[ \varphi_m = 52^\circ \]
\[ \varphi_m = 76^\circ \]
9.4.3. Transient response vs. damping factor

Unit-step response of second-order system $T(s)/(1+T(s))$

$$\hat{v}(t) = 1 + \frac{2Q e^{-\omega_c t/2Q}}{\sqrt{4Q^2 - 1}} \sin \left[ \frac{\sqrt{4Q^2 - 1}}{2Q} \omega_c t + \tan^{-1} \left( \sqrt{4Q^2 - 1} \right) \right] \quad Q > 0.5$$

$$\hat{v}(t) = 1 - \frac{\omega_2}{\omega_2 - \omega_1} e^{-\omega_1 t} - \frac{\omega_1}{\omega_1 - \omega_2} e^{-\omega_2 t} \quad Q < 0.5$$

$$\omega_1, \omega_2 = \frac{\omega_c}{2Q} \left( 1 \pm \sqrt{1 - 4Q^2} \right)$$

For $Q > 0.5$, the peak value is

$$\text{peak } \hat{v}(t) = 1 + e^{-\pi / \sqrt{4Q^2 - 1}}$$
Transient response vs. damping factor

\[ \hat{v}(t) \]

\[ \omega_c t, \text{ radians} \]

Graph showing the transient response \( \hat{v}(t) \) versus the damping factor \( \omega_c t \). The graph includes curves for different values of \( Q \) as indicated on the graph.
9.5. Regulator design

Typical specifications:

- Effect of load current variations on output voltage regulation
  This is a limit on the maximum allowable output impedance
- Effect of input voltage variations on the output voltage regulation
  This limits the maximum allowable line-to-output transfer function
- Transient response time
  This requires a sufficiently high crossover frequency
- Overshoot and ringing
  An adequate phase margin must be obtained

The regulator design problem: add compensator network $G_c(s)$ to modify $T(s)$ such that all specifications are met.
9.5.1. Lead (PD) compensator

\[ G_c(s) = G_{c0} \frac{1 + \frac{s}{\omega_z}}{1 + \frac{s}{\omega_p}} \]

Improves phase margin

\[ f_{\phi_{max}} = \frac{f_p}{f_z} + 45^\circ/\text{decade} \quad -45^\circ/\text{decade} \]

\[ G_{c0} \sqrt{\frac{f_p}{f_z}} = \sqrt{f_z f_p} \]
Lead compensator: maximum phase lead

\[ f_{\text{qmax}} = \sqrt{f_z f_p} \]

\[ \angle G_c(f_{\text{qmax}}) = \tan^{-1} \left( \frac{\sqrt{\frac{f_p}{f_z}} - \sqrt{\frac{f_z}{f_p}}}{2} \right) \]

\[ \frac{f_p}{f_z} = \frac{1 + \sin(\theta)}{1 - \sin(\theta)} \]
Lead compensator design

To optimally obtain a compensator phase lead of $\theta$ at frequency $f_c$, the pole and zero frequencies should be chosen as follows:

$$f_z = f_c \sqrt{\frac{1 - \sin(\theta)}{1 + \sin(\theta)}}$$

$$f_p = f_c \sqrt{\frac{1 + \sin(\theta)}{1 - \sin(\theta)}}$$

If it is desired that the magnitude of the compensator gain at $f_c$ be unity, then $G_{c0}$ should be chosen as

$$G_{c0} = \sqrt{\frac{f_z}{f_p}}$$
Example: lead compensation

![Graph showing lead compensation with key frequencies and asymptotes.](image)
9.5.2. Lag (PI) compensation

\[ G_c(s) = G_c\infty \left( 1 + \frac{\omega_L}{s} \right) \]

Improves low-frequency loop gain and regulation

\[ \| G_c \| \]

\[ \angle G_c \]

- 20 dB /decade

- 90°

\[ f_L \]

\[ 10f_L \]

+ 45°/decade

0°
Example: lag compensation

Original (uncompensated) loop gain is

\[ T_u(s) = \frac{T_{u0}}{1 + \frac{s}{\omega_0}} \]

Compensator:

\[ G_c(s) = G_{c\infty}\left(1 + \frac{\omega_L}{s}\right) \]

Design strategy:

Choose \( G_{c\infty} \) to obtain desired crossover frequency \( \omega_L \) sufficiently low to maintain adequate phase margin

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Example, continued

Construction of $\frac{1}{1+T}$, lag compensator example:

![Graph showing the construction of $1/(1+T)$ with frequency response across different frequencies from 1 Hz to 100 kHz, illustrating the gain and phase characteristics.]
9.5.3. Combined (PID) compensator

\[ G_c(s) = G_{cm} \frac{\left(1 + \frac{\omega_L}{s}\right)\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{\omega_{p1}}\right)\left(1 + \frac{s}{\omega_{p2}}\right)} \]
9.5.4. Design example

\[ v_g(t) = 28 \text{ V} \]

\[ v_c(s) \]

\[ G_c(s) \]

\[ v_e \]

\[ v_{ref} = 5 \text{ V} \]

\[ f_s = 100 \text{ kHz} \]

\[ s \]

\[ L = 50 \mu\text{H} \]

\[ C = 500 \mu\text{F} \]

\[ R = 3 \Omega \]

\[ i_{load} \]

\[ v(t) \]

Compensator

Transistor gate driver

Pulse-width modulator

Sensor gain

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### Quiescent operating point

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input voltage</td>
<td>$V_g = 28 \text{V}$</td>
</tr>
<tr>
<td>Output</td>
<td>$V = 15 \text{V}$, $I_{load} = 5 \text{A}$, $R = 3 \Omega$</td>
</tr>
<tr>
<td>Quiescent duty cycle</td>
<td>$D = 15/28 = 0.536$</td>
</tr>
<tr>
<td>Reference voltage</td>
<td>$V_{ref} = 5 \text{V}$</td>
</tr>
<tr>
<td>Quiescent value of control voltage</td>
<td>$V_c = DV_M = 2.14 \text{V}$</td>
</tr>
<tr>
<td>Gain $H(s)$</td>
<td>$H = V_{ref}/V = 5/15 = 1/3$</td>
</tr>
</tbody>
</table>
Small-signal model

\[ \frac{V}{D^2} \hat{d} \]

\[ \frac{V}{R} \hat{d} \]

\[ 1 : D \]

\[ L \]

\[ C \]

\[ \hat{v}(s) \]

\[ R \]

\[ \hat{i}_{\text{load}}(s) \]

\[ \hat{v}_g(s) \]

\[ \hat{v}_e(s) \]

\[ G_c(s) \]

\[ \frac{1}{V_M} \]

\[ V_M = 4 \text{ V} \]

\[ \hat{v}_{\text{ref}} (= 0) \]

\[ \hat{v}(s) \]

\[ H(s) \]

\[ H = \frac{1}{3} \]

\[ T(s) \]
Open-loop control-to-output transfer function $G_{vd}(s)$

$$G_{vd}(s) = \frac{V}{D} \frac{1}{1 + s\frac{L}{R} + s^2LC}$$

standard form:

$$G_{vd}(s) = G_{d0} \frac{1}{1 + \frac{s}{Q_0\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

salient features:

$$G_{d0} = \frac{V}{D} = 28V$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 1kHz$$

$$Q_0 = R \sqrt{\frac{C}{L}} = 9.5 \Rightarrow 19.5dB$$
Open-loop line-to-output transfer function and output impedance

\[ G_{vg}(s) = D \frac{1}{1 + s\frac{L}{R} + s^2LC} \]

—same poles as control-to-output transfer function

standard form:

\[ G_{vg}(s) = G_{g0} \frac{1}{1 + s \frac{s}{Q_0\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \]

Output impedance:

\[ Z_{out}(s) = R \parallel \frac{1}{sC} \parallel sL = \frac{sL}{1 + s\frac{L}{R} + s^2LC} \]
System block diagram

\[ T(s) = G_c(s) \left( \frac{1}{V_M} \right) G_{vd}(s) H(s) \]

\[ T(s) = \frac{G_c(s) H(s) V}{V_M D} \frac{1}{1 + \frac{s}{Q_0 \omega_0} + \left( \frac{s}{\omega_0} \right)^2} \]

\[ \hat{V}_{ref} (=0) \rightarrow \hat{V}_e(s) \rightarrow G_c(s) \rightarrow \hat{V}_c(s) \rightarrow \frac{1}{V_M} \rightarrow \hat{d}(s) \rightarrow \hat{i}_{load}(s) \rightarrow \hat{\nu}(s) \]

\[ \hat{V}_d(s) \rightarrow G_{vd}(s) \rightarrow Z_{out}(s) \]

\[ \hat{V}_e(s) \rightarrow \frac{1}{V_M} \rightarrow \hat{d}(s) \]

\[ \hat{d}(s) \rightarrow \frac{1}{3} \rightarrow H(s) \]

\[ \hat{d}(s) \rightarrow G_{vd}(s) \rightarrow Z_{out}(s) \]

\[ \hat{\nu}(s) \rightarrow G_{vg}(s) \rightarrow Z_{out}(s) \]

\[ \hat{\nu}(s) \rightarrow G_{vg}(s) \rightarrow Z_{out}(s) \]

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Uncompensated loop gain (with \( G_c = 1 \))

With \( G_c = 1 \), the loop gain is

\[
T_u(s) = T_{u0} \frac{1}{1 + \frac{s}{Q_0 \omega_0} + \left(\frac{s}{\omega_0}\right)^2}
\]

\[
T_{u0} = \frac{H V}{D V_M} = 2.33 \Rightarrow 7.4 \text{ dB}
\]

\[
f_c = 1.8 \text{ kHz}, \phi_m = 5^\circ
\]