Introduction to Power Electronics
ECEN 4797/5797

Lecture 29
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9.5.4. Design example

![Diagram of a control system with a voltage source, inductor, capacitor, resistor, and various control elements including a pulse-width modulator, transistor gate driver, and compensator.]

- \( v_g(t) \) (28 V)
- \( v_f(s) = 100 \text{ kHz} \)
- \( L \) (50 \( \mu \)H)
- \( C \) (500 \( \mu \)F)
- \( R \) (3 \( \Omega \))
- \( i_{load} \)
- \( H(s) \) (Sensor gain)
- \( G_c(s) \) (Compensator)
- \( V_M = 4 \text{ V} \)
- \( v_c \)
- \( v_e \)
- \( v_{ref} = 5 \text{ V} \)
- Design example

\[
\frac{v(t)}{v_{ref}} = \frac{H(s)}{1 + G_c(s)H(s)}
\]

Fundamentals of Power Electronics
**Quiescent operating point**

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input voltage</td>
<td>$V_g = 28V$</td>
</tr>
<tr>
<td>Output</td>
<td>$V = 15V$, $I_{load} = 5A$, $R = 3\Omega$</td>
</tr>
<tr>
<td>Quiescent duty cycle</td>
<td>$D = 15/28 = 0.536$</td>
</tr>
<tr>
<td>Reference voltage</td>
<td>$V_{ref} = 5V$</td>
</tr>
<tr>
<td>Quiescent value of control voltage</td>
<td>$V_c = DV_M = 2.14V$</td>
</tr>
<tr>
<td>Gain $H(s)$</td>
<td>$H = V_{ref}/V = 5/15 = 1/3$</td>
</tr>
</tbody>
</table>
Small-signal model

\[ V_{	ext{d}} \frac{d}{ds} \hat{d} \]

\[ \frac{V}{R} \hat{d} \]

\[ 1 : D \]

\[ L \]

\[ C \]

\[ R \]

\[ \hat{v}_g(s) \]

\[ \hat{v}_e(s) \]

\[ G_c(s) \]

\[ \frac{1}{V_M} \]

\[ V_M = 4 \text{ V} \]

\[ \hat{v}(s) \]

\[ \hat{i}_{\text{load}}(s) \]

\[ \hat{v}_{\text{ref}}(=0) \]

\[ H(s) \hat{v}(s) \]

\[ H = \frac{1}{3} \]

\[ T(s) \]
Open-loop control-to-output transfer function $G_{vd}(s)$

$$G_{vd}(s) = \frac{V}{D} \frac{1}{1 + s\frac{L}{R} + s^2LC}$$

standard form:

$$G_{vd}(s) = G_{d0} \frac{1}{1 + s\frac{s}{Q_0\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

salient features:

$$G_{d0} = \frac{V}{D} = 28V$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 1kHz$$

$$Q_0 = R \sqrt{\frac{C}{L}} = 9.5 \Rightarrow 19.5dB$$
Open-loop line-to-output transfer function and output impedance

\[ G_{vg}(s) = D \frac{1}{1 + s \frac{L}{R} + s^2 LC} \]

—same poles as control-to-output transfer function

\[ G_{vg}(s) = G_{g0} \frac{1}{1 + s \frac{s}{Q_0 \omega_0} + \left(\frac{s}{\omega_0}\right)^2} \]

Output impedance:

\[ Z_{out}(s) = R \parallel \frac{1}{sC} \parallel sL = \frac{sL}{1 + s \frac{L}{R} + s^2 LC} \]
\[ T(s) = G_c(s) \left( \frac{1}{V_M} \right) G_vd(s) H(s) \]

\[ T(s) = \frac{G_c(s) H(s)}{V_M} \frac{V}{D} \frac{1}{1 + \frac{s}{Q_0 \omega_0} + \left( \frac{s}{\omega_0} \right)^2} \]

System block diagram

\[ V_M = 4 \text{ V} \]

Duty cycle variation

\[ H = \frac{1}{3} \]

Load current variation
Uncompensated loop gain (with $G_c = 1$)

With $G_c = 1$, the loop gain is

$$T_u(s) = T_{u0} \frac{1}{1 + \frac{s}{Q_0 \omega_0} + \left( \frac{s}{\omega_0} \right)^2}$$

$$T_{u0} = \frac{H V}{D V_M} = 2.33 \Rightarrow 7.4 \text{ dB}$$

$Q_0 = 9.5 \Rightarrow 19.5 \text{ dB}$

$f_c = 1.8 \text{ kHz}$, $\varphi_m = 5^\circ$
Lead compensator design

- Obtain a crossover frequency of 5 kHz, with phase margin of 52°
- $T_u$ has phase of approximately $-180°$ at 5 kHz, hence lead (PD) compensator is needed to increase phase margin.
- Lead compensator should have phase of $+52°$ at 5 kHz
- $T_u$ has magnitude of $-20.6$ dB at 5 kHz
- Lead compensator gain should have magnitude of $+20.6$ dB at 5 kHz
- Lead compensator pole and zero frequencies should be
  
  $f_z = (5\text{kHz}) \sqrt{\frac{1 - \sin(52°)}{1 + \sin(52°)}} = 1.7\text{kHz}$
  
  $f_p = (5\text{kHz}) \sqrt{\frac{1 + \sin(52°)}{1 - \sin(52°)}} = 14.5\text{kHz}$

- Compensator dc gain should be
  
  $G_{c0} = \left(\frac{f_z}{f_0}\right)^2 \frac{1}{T_{u0}} \sqrt{\frac{f_z}{f_p}} = 3.7 \Rightarrow 11.3$ dB
$Q$ vs. $\phi_m$

- $Q = 1 \Rightarrow 0$ dB
- $Q = 0.5 \Rightarrow -6$ dB
- $\phi_m = 52^\circ$
- $\phi_m = 76^\circ$
9.5.1. Lead (PD) compensator

\[ G_c(s) = G_{c0} \frac{1 + \frac{s}{\omega_z}}{1 + \frac{s}{\omega_p}} \]

Improves phase margin

\[ f_{p} / 10 \quad 10 f_{z} \]

\[ f_{z} / 10 \quad + 45^\circ/\text{decade} \]

\[ - 45^\circ/\text{decade} \]

\[ \frac{f_{p}}{f_{z}} = \sqrt{\frac{f_{p}}{f_{z}}} \]

\[ f_{q_{\text{max}}} \]

\( f_p \) and \( f_z \)
Lead compensator: maximum phase lead

\[ f_{q\text{max}} = \sqrt{f_z f_p} \]

\[ \angle G_c(f_{q\text{max}}) = \tan^{-1}\left(\frac{\sqrt{\frac{f_p}{f_z}} - \sqrt{\frac{f_z}{f_p}}}{2}\right) \]

\[ \frac{f_p}{f_z} = \frac{1 + \sin(\theta)}{1 - \sin(\theta)} \]
Lead compensator design

To optimally obtain a compensator phase lead of $\theta$ at frequency $f_c$, the pole and zero frequencies should be chosen as follows:

$$f_z = f_c \sqrt{\frac{1 - \sin(\theta)}{1 + \sin(\theta)}}$$

$$f_p = f_c \sqrt{\frac{1 + \sin(\theta)}{1 - \sin(\theta)}}$$

If it is desired that the magnitude of the compensator gain at $f_c$ be unity, then $G_{c0}$ should be chosen as

$$G_{c0} = \sqrt{\frac{f_z}{f_p}}$$
Lead compensator Bode plot

\[ G_{c0} \sqrt{\frac{f_p}{f_z}} \]

\[ f_z = \sqrt{f_z f_p} \]

\[ f_z /10 \quad f_p /10 \quad 10f_z \]

\[ 0^\circ \quad 90^\circ \quad 180^\circ \]

\[ 1 \text{ Hz} \quad 10 \text{ Hz} \quad 100 \text{ Hz} \quad 1 \text{ kHz} \quad 10 \text{ kHz} \quad 100 \text{ kHz} \]
Loop gain, with lead compensator

\[ T(s) = T_{u0} G_{c0} \frac{\left(1 + \frac{s}{\omega_c}\right)}{\left(1 + \frac{s}{\omega_p}\right)} \left(1 + \frac{s}{Q_0\omega_0} + \left(\frac{s}{\omega_0}\right)^2\right) \]

\begin{align*}
\|T\| & \quad T_0 = 8.6 \Rightarrow 18.7 \text{ dB} \\
\angle T & \quad Q_0 = 9.5 \Rightarrow 19.5 \text{ dB}
\end{align*}

\begin{align*}
\frac{f_0}{1 \text{ kHz}} & \quad \frac{f_z}{1.7 \text{ kHz}} \\
\frac{f_c}{5 \text{ kHz}} & \quad \frac{f_p}{14 \text{ kHz}}
\end{align*}

\begin{align*}
\phi_m & \quad \phi_m = 52^\circ
\end{align*}
$\frac{1}{1+T}$, with lead compensator

- need more low-frequency loop gain
- hence, add inverted zero (PID controller)
Improved compensator (PID)

\[
G_c(s) = G_{cm} \frac{\left(1 + \frac{s}{\omega_z}\right) \left(1 + \frac{\omega_L}{s}\right)}{\left(1 + \frac{s}{\omega_p}\right)}
\]

- add inverted zero to PD compensator, without changing dc gain or corner frequencies
- choose \( f_L \) to be \( f_c/10 \), so that phase margin is unchanged
$T(s)$ and $1/(1+T(s))$, with PID compensator
Open-loop line-to-output transfer function and output impedance

\[ G_{vg}(s) = D \frac{1}{1 + s \frac{L}{R} + s^2 LC} \]

—same poles as control-to-output transfer function
standard form:

\[ G_{vg}(s) = G_{g0} \frac{1}{1 + s \frac{1}{Q_0 \omega_0} + \left( \frac{s}{\omega_0} \right)^2} \]

Output impedance:

\[ Z_{out}(s) = R \parallel \frac{1}{sC} \parallel sL = \frac{sL}{1 + s \frac{L}{R} + s^2 LC} \]
Line-to-output transfer function

\[ G_{vg}(0) = D \]

Open-loop \[ || G_{vg} || \]

Closed-loop \[ \frac{G_{vg}}{1 + T} \]
9.6. Measurement of loop gains

Objective: experimentally determine loop gain $T(s)$, by making measurements at point $A$

Correct result is

$$T(s) = G_1(s) \left( \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \right) G_2(s) H(s)$$
Conventional approach: break loop, measure $T(s)$ as conventional transfer function

$$T_m(s) = G_1(s)G_2(s)H(s)$$

measured gain is

$$T_m(s) = \frac{\hat{v}_y(s)}{\hat{v}_x(s)} \bigg|_{\hat{v}_{ref} = 0, \hat{v}_g = 0}$$
Measured vs. actual loop gain

Actual loop gain:
\[ T(s) = G_1(s) \left( \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \right) G_2(s) H(s) \]

Measured loop gain:
\[ T_m(s) = G_1(s) G_2(s) H(s) \]

Express \( T_m \) as function of \( T \):
\[ T_m(s) = T(s) \left( 1 + \frac{Z_1(s)}{Z_2(s)} \right) \]

\[ T_m(s) \approx T(s) \quad \text{provided that} \quad \| Z_2 \| >> \| Z_1 \| \]
Discussion

• Breaking the loop disrupts the loading of block 2 on block 1.
  A suitable injection point must be found, where loading is not significant.

• Breaking the loop disrupts the dc biasing and quiescent operating point.
  A potentiometer must be used, to correctly bias the input to block 2.
  In the common case where the dc loop gain is large, it is very difficult to correctly set the dc bias.

• It would be desirable to avoid breaking the loop, such that the biasing circuits of the system itself set the quiescent operating point.
9.6.1. Voltage injection

- Ac injection source $v_z$ is connected between blocks 1 and 2
- Dc bias is determined by biasing circuits of the system itself
- Injection source does modify loading of block 2 on block 1
Voltage injection: measured transfer function $T_v(s)$

Network analyzer measures

$$T_v(s) = \left. \frac{\hat{v}_y(s)}{\hat{v}_x(s)} \right|_{\hat{v}_{ref} = 0 \atop \hat{v}_g = 0}$$

Solve block diagram:

$$\hat{v}_e(s) = - H(s) G_2(s) \hat{v}_x(s)$$

$$- \hat{v}_y(s) = G_1(s) \hat{v}_e(s) - \hat{i}(s) Z_1(s)$$

Hence

$$- \hat{v}_y(s) = - \hat{v}_x(s) G_2(s) H(s) G_1(s) - \hat{i}(s) Z_1(s)$$

with

$$\hat{i}(s) = \frac{\hat{v}_x(s)}{Z_2(s)}$$

Substitute:

$$\hat{v}_y(s) = \hat{v}_x(s) \left( G_1(s) G_2(s) H(s) + \frac{Z_1(s)}{Z_2(s)} \right)$$

which leads to the measured gain

$$T_v(s) = G_1(s) G_2(s) H(s) + \frac{Z_1(s)}{Z_2(s)}$$
Comparison of $T_v(s)$ with $T(s)$

Actual loop gain is

$$T(s) = G_1(s) \left( \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \right) G_2(s) H(s)$$

Gain measured via voltage injection:

$$T_v(s) = G_1(s) G_2(s) H(s) + \frac{Z_1(s)}{Z_2(s)}$$

Express $T_v(s)$ in terms of $T(s)$:

$$T_v(s) = T(s) \left( 1 + \frac{Z_1(s)}{Z_2(s)} \right) + \frac{Z_1(s)}{Z_2(s)}$$

Condition for accurate measurement:

$$T_v(s) \approx T(s) \quad \text{provided} \quad (i) \quad \left\| Z_1(s) \right\| < < \left\| Z_2(s) \right\|, \quad \text{and}$$

$$(ii) \quad \left\| T(s) \right\| >> \left\| \frac{Z_1(s)}{Z_2(s)} \right\|$$
Example: voltage injection

\[ Z_1(s) = 50 \Omega \]
\[ Z_2(s) = 500 \Omega \]
\[ \frac{Z_1(s)}{Z_2(s)} = 0.1 \Rightarrow -20 \text{dB} \]
\[ \left( 1 + \frac{Z_1(s)}{Z_2(s)} \right) = 1.1 \Rightarrow 0.83 \text{dB} \]

\[ T(s) = \frac{10^4}{\left( 1 + \frac{s}{2\pi \times 10 \text{Hz}} \right) \left( 1 + \frac{s}{2\pi \times 100 \text{kHz}} \right)} \]
Example: measured $T_v(s)$ and actual $T(s)$

$$T_v(s) = T(s) \left( 1 + \frac{Z_1(s)}{Z_2(s)} \right) + \frac{Z_1(s)}{Z_2(s)}$$
9.6.2. Current injection

\[ T_i(s) = \frac{\dot{i}_y(s)}{\dot{i}_x(s)} \bigg|_{v_{ref} = 0, v_g = 0} \]

\[ \dot{i}_y(s) = G_1(s) \hat{v}_e(s) + \dot{i}_z(s) \]

Block 1

\[ G_1(s) \hat{v}_e(s) \quad + \]

\[ Z_1(s) \quad \dot{i}_y(s) \quad \dot{i}_z(s) \]

Block 2

\[ Z_2(s) \quad \dot{i}_x(s) \quad \hat{v}_e(s) \quad G_2(s) \hat{v}_x(s) = \hat{v}(s) \]

\[ H(s) \quad T_i(s) \]
Current injection

It can be shown that

\[ T_i(s) = T(s) \left( 1 + \frac{Z_2(s)}{Z_1(s)} \right) + \frac{Z_2(s)}{Z_1(s)} \]

Conditions for obtaining accurate measurement:

(i) \[ \| Z_2(s) \| \ll \| Z_1(s) \| , \text{ and} \]

(ii) \[ \| T(s) \| \gg \left\| \frac{Z_2(s)}{Z_1(s)} \right\| \]

Injection source impedance \( Z_s \) is irrelevant. We could inject using a Thevenin-equivalent voltage source:
9.6.3. Measurement of unstable systems

- Injection source impedance $Z_s$ does not affect measurement.
- Increasing $Z_s$ reduces loop gain of circuit, tending to stabilize system.
- Original (unstable) loop gain is measured (not including $Z_s$), while circuit operates stably.

![Diagram](image.png)
9.7. Summary of key points

1. Negative feedback causes the system output to closely follow the reference input, according to the gain $1/H(s)$. The influence on the output of disturbances and variation of gains in the forward path is reduced.

2. The loop gain $T(s)$ is equal to the products of the gains in the forward and feedback paths. The loop gain is a measure of how well the feedback system works: a large loop gain leads to better regulation of the output. The crossover frequency $f_c$ is the frequency at which the loop gain $T$ has unity magnitude, and is a measure of the bandwidth of the control system.
Summary of key points

3. The introduction of feedback causes the transfer functions from disturbances to the output to be multiplied by the factor $1/(1+T(s))$. At frequencies where $T$ is large in magnitude (i.e., below the crossover frequency), this factor is approximately equal to $1/T(s)$. Hence, the influence of low-frequency disturbances on the output is reduced by a factor of $1/T(s)$. At frequencies where $T$ is small in magnitude (i.e., above the crossover frequency), the factor is approximately equal to 1. The feedback loop then has no effect. Closed-loop disturbance-to-output transfer functions, such as the line-to-output transfer function or the output impedance, can easily be constructed using the algebra-on-the-graph method.

4. Stability can be assessed using the phase margin test. The phase of $T$ is evaluated at the crossover frequency, and the stability of the important closed-loop quantities $T/(1+T)$ and $1/(1+T)$ is then deduced. Inadequate phase margin leads to ringing and overshoot in the system transient response, and peaking in the closed-loop transfer functions.
Summary of key points

5. Compensators are added in the forward paths of feedback loops to shape the loop gain, such that desired performance is obtained. Lead compensators, or PD controllers, are added to improve the phase margin and extend the control system bandwidth. PI controllers are used to increase the low-frequency loop gain, to improve the rejection of low-frequency disturbances and reduce the steady-state error.

6. Loop gains can be experimentally measured by use of voltage or current injection. This approach avoids the problem of establishing the correct quiescent operating conditions in the system, a common difficulty in systems having a large dc loop gain. An injection point must be found where interstage loading is not significant. Unstable loop gains can also be measured.