Introduction to Power Electronics
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Lecture 33
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Part III. Magnetics

13  Basic Magnetics Theory
14  Inductor Design
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# Fundamentals of Power Electronics

## Chapter 13: Basic Magnetics Theory

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Chapter 13 Basic Magnetics Theory

13.5 Several Types of Magnetic Devices, Their $B$–$H$ Loops, and Core vs. Copper Loss

- 13.5.1 Filter inductor
- 13.5.2 AC inductor
- 13.5.3 Transformer
- 13.5.4 Coupled inductor
- 13.5.5 Flyback transformer

13.6 Summary of Key Points
13.1 Review of Basic Magnetics
13.1.1 Basic relationships

Faraday’s law

Terminal characteristics

Ampere’s law

Core characteristics

\( v(t) \) \( B(t), \Phi(t) \)

\( i(t) \) \( H(t), \mathcal{F}(t) \)
Basic quantities

**Magnetic quantities**

- Length $\ell$
- Magnetic field $H$
- MMF $\mathcal{F} = H\ell$
- Total flux $\Phi$
- Flux density $B$

**Electrical quantities**

- Length $\ell$
- Electric field $E$
- Voltage $V = E\ell$
- Total current $I$
- Current density $J$
Magnetic field $H$ and magnetomotive force $\mathcal{F}$

Magnetomotive force (MMF) $\mathcal{F}$ between points $x_1$ and $x_2$ is related to the magnetic field $H$ according to

$$\mathcal{F} = \int_{x_1}^{x_2} H \cdot dl$$

**Example: uniform magnetic field of magnitude $H$**

Length $\ell$

Magnetic field $H$

$\mathcal{F} = H\ell$

**Analogous to electric field of strength $E$, which induces voltage (EMF) $V$:**

Length $\ell$

Electric field $E$

Voltage $V = E\ell$
Flux density $B$ and total flux $\Phi$

The total magnetic flux $\Phi$ passing through a surface of area $A_c$ is related to the flux density $B$ according to

$$\Phi = \int_{\text{surface } S} B \cdot dA$$

**Example:** uniform flux density of magnitude $B$

Analogous to electrical conductor current density of magnitude $J$, which leads to total conductor current $I$:
Faraday’s law

Voltage $v(t)$ is induced in a loop of wire by change in the total flux $\Phi(t)$ passing through the interior of the loop, according to

$$v(t) = \frac{d\Phi(t)}{dt}$$

For uniform flux distribution, $\Phi(t) = B(t)A_c$ and hence

$$v(t) = A_c \frac{dB(t)}{dt}$$
**Lenz’s law**

The voltage $v(t)$ induced by the changing flux $\Phi(t)$ is of the polarity that tends to drive a current through the loop to counteract the flux change.

**Example: a shorted loop of wire**

- Changing flux $\Phi(t)$ induces a voltage $v(t)$ around the loop.
- This voltage, divided by the impedance of the loop conductor, leads to current $i(t)$.
- This current induces a flux $\Phi'(t)$, which tends to oppose changes in $\Phi(t)$. 
The net MMF around a closed path is equal to the total current passing through the interior of the path:

\[ \oint_{\text{closed path}} H \cdot dl = \text{total current passing through interior of path} \]

**Example: magnetic core. Wire carrying current } i(t) \text{ passes through core window.**

- Illustrated path follows magnetic flux lines around interior of core
- For uniform magnetic field strength } H(t) \text{, the integral (MMF) is } H(t)l_m. \text{ So}

\[ \mathcal{F}(t) = H(t)l_m = i(t) \]
Ampere’s law: discussion

- Relates magnetic field strength $H(t)$ to winding current $i(t)$
- We can view winding currents as sources of MMF
- Previous example: total MMF around core, $\mathcal{F}(t) = H(t)\ell_m$, is equal to the winding current MMF $i(t)$
- The total MMF around a closed loop, accounting for winding current MMF’s, is zero
Core material characteristics: the relation between $B$ and $H$

$\mu_0 = \text{permeability of free space} = 4\pi \cdot 10^{-7}\ \text{Henries per meter}$

Highly nonlinear, with hysteresis and saturation
Piecewise-linear modeling of core material characteristics

No hysteresis or saturation

$$ B = \mu H $$

$$ \mu = \mu_r \mu_0 $$

Typical $\mu_r = 10^3$ to $10^5$

Saturation, no hysteresis

$$ B_s = B_s + B_{sat} $$

$$ \mu $$

Typical $B_{sat} = 0.3$ to $0.5T$, ferrite

0.5 to $1T$, powdered iron

1 to $2T$, iron laminations
# Units

<table>
<thead>
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<th>quantity</th>
<th>MKS</th>
<th>unreationalized cgs</th>
<th>conversions</th>
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<td>core material equation</td>
<td>$B = \mu_0 \mu, H$</td>
<td>$B = \mu, H$</td>
<td>$1 \text{T} = 10^4 \text{G}$</td>
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<tr>
<td>$B$</td>
<td>Tesla</td>
<td>Gauss</td>
<td>$1 \text{A/m} = 4\pi \cdot 10^{-3} \text{Oe}$</td>
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<td>$H$</td>
<td>Ampere / meter</td>
<td>Oersted</td>
<td>$1 \text{Wb} = 10^8 \text{Mx}$</td>
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<tr>
<td>$\Phi$</td>
<td>Weber</td>
<td>Maxwell</td>
<td>$1 \text{T} = 1 \text{Wb} / \text{m}^2$</td>
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Example: a simple inductor

**Faraday’s law:**
For each turn of wire, we can write

\[ v_{\text{turn}}(t) = \frac{d\Phi(t)}{dt} \]

Total winding voltage is

\[ v(t) = n v_{\text{turn}}(t) = n \frac{d\Phi(t)}{dt} \]

Express in terms of the average flux density \( B(t) = \bar{\Phi}(t)/A_c \)

\[ v(t) = nA_c \frac{dB(t)}{dt} \]
Inductor example: Ampere’s law

Choose a closed path which follows the average magnetic field line around the interior of the core. Length of this path is called the mean magnetic path length $l_m$.

For uniform field strength $H(t)$, the core MMF around the path is $H l_m$.

Winding contains $n$ turns of wire, each carrying current $i(t)$. The net current passing through the path interior (i.e., through the core window) is $ni(t)$.

From Ampere’s law, we have

$$H(t) l_m = n i(t)$$
Inductor example: core material model

\[ B = \begin{cases} 
B_{\text{sat}} & \text{for } H \geq B_{\text{sat}}/\mu \\
\mu H & \text{for } |H| < B_{\text{sat}}/\mu \\
-B_{\text{sat}} & \text{for } H \leq -B_{\text{sat}}/\mu 
\end{cases} \]

Find winding current at onset of saturation: substitute \( i = I_{\text{sat}} \) and \( H = B_{\text{sat}}/\mu \) into equation previously derived via Ampere’s law. Result is

\[ I_{\text{sat}} = \frac{B_{\text{sat}} l_m}{\mu n} \]
Electrical terminal characteristics

We have:

\[ v(t) = nA_c \frac{dB(t)}{dt} \quad H(t) \ell_m = n i(t) \quad B = \begin{cases} B_{sat} & \text{for } H \geq B_{sat}/\mu \\ \mu H & \text{for } |H| < B_{sat}/\mu \\ -B_{sat} & \text{for } H \leq -B_{sat}/\mu \end{cases} \]

Eliminate \( B \) and \( H \), and solve for relation between \( v \) and \( i \). For \(|i| < I_{sat}\),

\[ v(t) = \mu n A_c \frac{dH(t)}{dt} \]

which is of the form

\[ v(t) = L \frac{di(t)}{dt} \quad \text{with} \quad L = \frac{\mu n^2 A_c}{\ell_m} \]

—an inductor

For \(|i| > I_{sat}\) the flux density is constant and equal to \( B_{sat} \). Faraday’s law then predicts

\[ v(t) = nA_c \frac{dB_{sat}}{dt} = 0 \quad \text{—saturation leads to short circuit} \]
13.1.2 Magnetic circuits

Uniform flux and magnetic field inside a rectangular element:

MMF between ends of element is

\[ \mathcal{F} = H \ell \]

Since \( H = B / \mu \) and \( B = \Phi / A_c \), we can express \( \mathcal{F} \) as

\[ \mathcal{F} = \Phi R \]

with

\[ R = \frac{\ell}{\mu A_c} \]

A corresponding model:

\[ \Phi \quad R \quad \mathcal{F} + \longrightarrow - \]

\( R \) = reluctance of element
Magnetic circuits: magnetic structures composed of multiple windings and heterogeneous elements

- Represent each element with reluctance
- Windings are sources of MMF
- MMF $\rightarrow$ voltage, flux $\rightarrow$ current
- Solve magnetic circuit using Kirchoff’s laws, etc.
Magnetic analog of Kirchoff’s current law

Divergence of $B = 0$

Flux lines are continuous and cannot end

Total flux entering a node must be zero
Magnetic analog of Kirchoff’s voltage law

Follows from Ampere’s law:

$$\oint_{\text{closed path}} H \cdot dl = \text{total current passing through interior of path}$$

Left-hand side: sum of MMF’s across the reluctances around the closed path

Right-hand side: currents in windings are sources of MMF’s. An \(n\)-turn winding carrying current \(i(t)\) is modeled as an MMF (voltage) source, of value \(ni(t)\).

Total MMF’s around the closed path add up to zero.
Example: inductor with air gap

\[ V(t) - \text{Air gap} \]

\[ n \text{ turns} \]

\[ i(t) \]

Core permeability $\mu$

Cross-sectional area $A_c$

Air gap $l_g$

Magnetic path length $l_m$
Magnetic circuit model

\[ \Phi_c + \Phi_g = ni \]

\[ ni = \Phi \left( R_c + R_g \right) \]

\[ R_c = \frac{\ell_c}{\mu A_c} \]

\[ R_g = \frac{\ell_g}{\mu_0 A_c} \]
Solution of model

Faraday’s law: \( v(t) = n \frac{d\Phi(t)}{dt} \)

Substitute for \( \Phi \) :

\[ v(t) = \frac{n^2}{R_c + R_g} \frac{di(t)}{dt} \]

Hence inductance is

\[ L = \frac{n^2}{R_c + R_g} \]
Effect of air gap

\[ ni = \Phi \left( R_c + R_g \right) \]
\[ L = \frac{n^2}{R_c + R_g} \]
\[ \Phi_{sat} = B_{sat} A_c \]
\[ I_{sat} = \frac{B_{sat} A_c}{n} \left( R_c + R_g \right) \]

Effect of air gap:
- decrease inductance
- increase saturation current
- inductance is less dependent on core permeability
13.2 Transformer modeling

Two windings, no air gap:

\[ R = \frac{\ell_m}{\mu A_c} \]

\[ \Phi_c = n_1 i_1 + n_2 i_2 \]

\[ \Phi R = n_1 i_1 + n_2 i_2 \]

Magnetic circuit model:
13.2.1 The ideal transformer

In the ideal transformer, the core reluctance $R_c$ approaches zero.

MMF $\mathcal{J}_c = \Phi R_c$ also approaches zero. We then obtain

$$0 = n_1 i_1 + n_2 i_2$$

Also, by Faraday’s law,

$$v_1 = n_1 \frac{d\Phi}{dt}$$
$$v_2 = n_2 \frac{d\Phi}{dt}$$

Eliminate $\Phi$ :

$$\frac{d\Phi}{dt} = \frac{v_1}{n_1} = \frac{v_2}{n_2}$$

Ideal transformer equations:

$$\frac{v_1}{n_1} = \frac{v_2}{n_2} \quad \text{and} \quad n_1 i_1 + n_2 i_2 = 0$$
13.2.2 The magnetizing inductance

For nonzero core reluctance, we obtain

\[ \Phi R = n_1 i_1 + n_2 i_2 \quad \text{with} \quad v_1 = n_1 \frac{d\Phi}{dt} \]

Eliminate \( \Phi \):

\[ v_1 = \frac{n_1^2}{R} \frac{d}{dt} \left[ i_1 + \frac{n_2}{n_1} i_2 \right] \]

This equation is of the form

\[ v_1 = L_M \frac{di_M}{dt} \]

with

\[ L_M = \frac{n_1^2}{R} \]
\[ i_M = i_1 + \frac{n_2}{n_1} i_2 \]
Magnetizing inductance: discussion

- Models magnetization of core material
- A real, physical inductor, that exhibits saturation and hysteresis
- If the secondary winding is disconnected:
  - we are left with the primary winding on the core
  - primary winding then behaves as an inductor
  - the resulting inductor is the magnetizing inductance, referred to the primary winding
- Magnetizing current causes the ratio of winding currents to differ from the turns ratio
Transformer saturation

- Saturation occurs when core flux density $B(t)$ exceeds saturation flux density $B_{\text{sat}}$.
- When core saturates, the magnetizing current becomes large, the impedance of the magnetizing inductance becomes small, and the windings are effectively shorted out.
- Large winding currents $i_1(t)$ and $i_2(t)$ do not necessarily lead to saturation. If
  \[ 0 = n_1 i_1 + n_2 i_2 \]
  then the magnetizing current is zero, and there is no net magnetization of the core.
- Saturation is caused by excessive applied volt-seconds
Saturation vs. applied volt-seconds

Magnetizing current depends on the integral of the applied winding voltage:

$$i_M(t) = \frac{1}{L_M} \int v_1(t) dt$$

Flux density is proportional:

$$B(t) = \frac{1}{n_1 A_c} \int v_1(t) dt$$

Flux density becomes large, and core saturates, when the applied volt-seconds $\lambda_1$ are too large, where

$$\lambda_1 = \int_{t_1}^{t_2} v_1(t) dt$$

limits of integration chosen to coincide with positive portion of applied voltage waveform
13.2.3 Leakage inductances
Transformer model, including leakage inductance

Terminal equations can be written in the form

\[
\begin{bmatrix}
  v_1(t) \\
  v_2(t)
\end{bmatrix} = \begin{bmatrix}
  L_{11} & L_{12} \\
  L_{12} & L_{22}
\end{bmatrix} \frac{d}{dt} \begin{bmatrix}
  i_1(t) \\
  i_2(t)
\end{bmatrix}
\]

Mutual inductance:

\[
L_{12} = \frac{n_1 n_2}{R} = \frac{n_2}{n_1} L_M
\]

Primary and secondary self-inductances:

\[
\begin{align*}
L_{11} &= L_{\ell 1} + \frac{n_1}{n_2} L_{12} \\
L_{22} &= L_{\ell 2} + \frac{n_2}{n_1} L_{12}
\end{align*}
\]

effective turns ratio

\[
n_e = \sqrt{\frac{L_{22}}{L_{11}}}
\]

coupling coefficient

\[
k = \frac{L_{12}}{\sqrt{L_{11}L_{22}}}
\]