Introduction to Power Electronics
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Lecture 34
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Part III. Magnetics

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13.1 Review of Basic Magnetics
13.1.1 Basic relationships

Faraday’s law

Terminal characteristics

Ampere’s law

Core characteristics

\[ v(t) \rightarrow B(t), \Phi(t) \]

\[ i(t) \rightarrow H(t), \mathcal{F}(t) \]
Basic quantities

**Magnetic quantities**
- Length $\ell$
- Magnetic field $H$
- MMF $\mathcal{F} = H\ell$
- Total flux $\Phi$
- Flux density $B$

**Electrical quantities**
- Length $\ell$
- Electric field $E$
- Voltage $V = E\ell$
- Total current $I$
- Current density $J$
Magnetic field $H$ and magnetomotive force $\mathcal{F}$

Magnetomotive force (MMF) $\mathcal{F}$ between points $x_1$ and $x_2$ is related to the magnetic field $H$ according to

$$\mathcal{F} = \int_{x_1}^{x_2} H \cdot dl$$

**Example: uniform magnetic field of magnitude $H$**

- Length $\ell$
- Magnetic field $H$
- MMF $\mathcal{F} = H\ell$

**Analogous to electric field of strength $E$, which induces voltage (EMF) $V$:***

- Length $\ell$
- Electric field $E$
- Voltage $V = E\ell$
Flux density $B$ and total flux $\Phi$

The total magnetic flux $\Phi$ passing through a surface of area $A_c$ is related to the flux density $B$ according to

$$\Phi = \int_{\text{surface } S} B \cdot dA$$

**Example: uniform flux density of magnitude $B$**

Total flux $\Phi$

Total current $I$

**Analogous to electrical conductor current density of magnitude $J$, which leads to total conductor current $I$:**

Flux density $B$

Current density $J$
Faraday’s law

Voltage $v(t)$ is induced in a loop of wire by change in the total flux $\Phi(t)$ passing through the interior of the loop, according to

$$v(t) = \frac{d\Phi(t)}{dt}$$

For uniform flux distribution, $\Phi(t) = B(t)A_c$ and hence

$$v(t) = A_c \frac{dB(t)}{dt}$$
Lenz’s law

The voltage $v(t)$ induced by the changing flux $\Phi(t)$ is of the polarity that tends to drive a current through the loop to counteract the flux change.

**Example: a shorted loop of wire**

- Changing flux $\Phi(t)$ induces a voltage $v(t)$ around the loop.
- This voltage, divided by the impedance of the loop conductor, leads to current $i(t)$.
- This current induces a flux $\Phi'(t)$, which tends to oppose changes in $\Phi(t)$.
Ampere’s law

The net MMF around a closed path is equal to the total current passing through the interior of the path:

\[ \oint_{\text{closed path}} \mathbf{H} \cdot d\mathbf{l} = \text{total current passing through interior of path} \]

Example: magnetic core. Wire carrying current \( i(t) \) passes through core window.

- Illustrated path follows magnetic flux lines around interior of core.
- For uniform magnetic field strength \( H(t) \), the integral (MMF) is \( H(t)l_m \). So

\[ \mathcal{F}(t) = H(t)l_m = i(t) \]
Ampere’s law: discussion

• Relates magnetic field strength $H(t)$ to winding current $i(t)$
• We can view winding currents as sources of MMF
• Previous example: total MMF around core, $\mathcal{F}(t) = H(t)\ell_m$, is equal to the winding current MMF $i(t)$
• The total MMF around a closed loop, accounting for winding current MMF’s, is zero
Core material characteristics: the relation between $B$ and $H$

Free space

$B = \mu_0 H$

$\mu_0 = \text{permeability of free space}$

$= 4\pi \cdot 10^{-7} \text{ Henrys per meter}$

A magnetic core material

Highly nonlinear, with hysteresis and saturation
Piecewise-linear modeling of core material characteristics

No hysteresis or saturation

![Graph showing no hysteresis or saturation with typical $\mu_r = 10^3$ to $10^5$]

Saturation, no hysteresis

![Graph showing saturation, no hysteresis with typical $B_{sat} = 0.3$ to $0.5$T, ferrite $0.5$ to $1$T, powdered iron $1$ to $2$T, iron laminations]

Typical $B_{sat} = 0.3$ to $0.5$T, ferrite
$0.5$ to $1$T, powdered iron
$1$ to $2$T, iron laminations
## Units

### Table 12.1. Units for magnetic quantities

<table>
<thead>
<tr>
<th>quantity</th>
<th>MKS</th>
<th>unrationlized cgs</th>
<th>conversions</th>
</tr>
</thead>
<tbody>
<tr>
<td>core material equation</td>
<td>$B = \mu_0 \mu_r H$</td>
<td>$B = \mu_r H$</td>
<td></td>
</tr>
<tr>
<td>$B$</td>
<td>Tesla</td>
<td>Gauss</td>
<td>$1\text{T} = 10^4\text{G}$</td>
</tr>
<tr>
<td>$H$</td>
<td>Ampere / meter</td>
<td>Oersted</td>
<td>$1\text{A/m} = 4\pi \cdot 10^{-3} \text{Oe}$</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Weber</td>
<td>Maxwell</td>
<td>$1\text{Wb} = 10^8 \text{Mx}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$1\text{T} = 1\text{Wb} / \text{m}^2$</td>
</tr>
</tbody>
</table>
Example: a simple inductor

Faraday’s law:
For each turn of wire, we can write

\[ v_{\text{turn}}(t) = \frac{d\Phi(t)}{dt} \]

Total winding voltage is

\[ v(t) = nv_{\text{turn}}(t) = n \frac{d\Phi(t)}{dt} \]

Express in terms of the average flux density \( B(t) = \bar{\Phi}(t)/A_c \)

\[ v(t) = nA_c \frac{dB(t)}{dt} \]
Inductor example: Ampere’s law

Choose a closed path which follows the average magnetic field line around the interior of the core. Length of this path is called the mean magnetic path length $l_m$.

For uniform field strength $H(t)$, the core MMF around the path is $Hl_m$.

Winding contains $n$ turns of wire, each carrying current $i(t)$. The net current passing through the path interior (i.e., through the core window) is $ni(t)$.

From Ampere’s law, we have

$$H(t) l_m = ni(t)$$
Inductor example: core material model

\[ B = \begin{cases} 
B_{\text{sat}} & \text{for } H \geq B_{\text{sat}}/\mu \\
\mu H & \text{for } |H| < B_{\text{sat}}/\mu \\
- B_{\text{sat}} & \text{for } H \leq -B_{\text{sat}}/\mu 
\end{cases} \]

Find winding current at onset of saturation: substitute \( i = I_{\text{sat}} \) and \( H = B_{\text{sat}}/\mu \) into equation previously derived via Ampere’s law. Result is

\[ I_{\text{sat}} = \frac{B_{\text{sat}} l_m}{\mu n} \]
Electrical terminal characteristics

We have:

\[ v(t) = nA_c \frac{dB(t)}{dt} \quad H(t) \ell_m = n \, i(t) \]

\[ B = \begin{cases} 
B_{sat} & \text{for } H \geq B_{sat}/\mu \\
\mu H & \text{for } |H| < B_{sat}/\mu \\
-B_{sat} & \text{for } H \leq -B_{sat}/\mu 
\end{cases} \]

Eliminate \( B \) and \( H \), and solve for relation between \( v \) and \( i \). For \( |i| < I_{sat} \),

\[ v(t) = \mu n A_c \frac{dH(t)}{dt} \quad \rightarrow \quad v(t) = \frac{\mu n^2 A_c}{\ell_m} \frac{di(t)}{dt} \]

which is of the form

\[ v(t) = L \frac{di(t)}{dt} \quad \text{with} \quad L = \frac{\mu n^2 A_c}{\ell_m} \]

—an inductor

For \( |i| > I_{sat} \) the flux density is constant and equal to \( B_{sat} \). Faraday’s law then predicts

\[ v(t) = nA_c \frac{dB_{sat}}{dt} = 0 \quad \text{—saturation leads to short circuit} \]
13.1.2 Magnetic circuits

Uniform flux and magnetic field inside a rectangular element:

MMF between ends of element is

\[ \mathcal{F} = H \ell \]

Since \( H = B / \mu \) and \( B = \Phi / A_c \), we can express \( \mathcal{F} \) as

\[ \mathcal{F} = \Phi R \]

A corresponding model:

\[ R = \frac{\ell}{\mu A_c} \]

\( R \) = reluctance of element
Magnetic circuits: magnetic structures composed of multiple windings and heterogeneous elements

- Represent each element with reluctance
- Windings are sources of MMF
- MMF $\Rightarrow$ voltage, flux $\Rightarrow$ current
- Solve magnetic circuit using Kirchoff's laws, etc.
Magnetic analog of Kirchoff’s current law

Divergence of $\mathbf{B} = 0$

Flux lines are continuous and cannot end

Total flux entering a node must be zero
Magnetic analog of Kirchoff’s voltage law

Follows from Ampere’s law:

$$\oint_{\text{closed path}} H \cdot dl = \text{total current passing through interior of path}$$

Left-hand side: sum of MMF’s across the reluctances around the closed path

Right-hand side: currents in windings are sources of MMF’s. An $n$-turn winding carrying current $i(t)$ is modeled as an MMF (voltage) source, of value $ni(t)$.

Total MMF’s around the closed path add up to zero.
Example: inductor with air gap

![Inductor Diagram](image)

- Core permeability \( \mu \)
- Cross-sectional area \( A_c \)
- Air gap \( \ell_g \)
- Magnetic path length \( \ell_m \)
- Current \( i(t) \)
- Voltage \( v(t) \)
- Turns \( n \)
Magnetic circuit model

\[ \Phi = n i(t) \]

\[ R_c = \frac{l_c}{\mu A_c} \]

\[ R_g = \frac{l_g}{\mu_0 A_c} \]
Solution of model

Faraday’s law: \( v(t) = n \frac{d\Phi(t)}{dt} \)

Substitute for \( \Phi \): \( v(t) = \frac{n^2}{R_c + R_g} \frac{di(t)}{dt} \)

Hence inductance is

\[
L = \frac{n^2}{R_c + R_g}
\]
Effect of air gap

- decrease inductance
- increase saturation current
- inductance is less dependent on core permeability

\[ ni = \Phi \left( R_c + R_g \right) \]
\[ L = \frac{n^2}{R_c + R_g} \]
\[ \Phi_{sat} = B_{sat}A_c \]
\[ I_{sat} = \frac{B_{sat}A_c}{n} \left( R_c + R_g \right) \]
13.2 Transformer modeling

Two windings, no air gap:

\[ R = \frac{L_m}{\mu A_c} \]

\[ \tilde{\mathcal{F}}_c = n_1 i_1 + n_2 i_2 \]

\[ \Phi R = n_1 i_1 + n_2 i_2 \]

Magnetic circuit model:
13.2.1 The ideal transformer

In the ideal transformer, the core reluctance $R$ approaches zero. MMF $\mathcal{F}_c = \Phi R$ also approaches zero. We then obtain

$$0 = n_1i_1 + n_2i_2$$

Also, by Faraday’s law,

$$v_1 = n_1 \frac{d\Phi}{dt}$$
$$v_2 = n_2 \frac{d\Phi}{dt}$$

Eliminate $\Phi$:

$$\frac{d\Phi}{dt} = \frac{v_1}{n_1} = \frac{v_2}{n_2}$$

Ideal transformer equations:

$$\frac{v_1}{n_1} = \frac{v_2}{n_2} \quad \text{and} \quad n_1i_1 + n_2i_2 = 0$$
13.2.2 The magnetizing inductance

For nonzero core reluctance, we obtain

\[ \Phi \mathcal{R} = n_1 i_1 + n_2 i_2 \quad \text{with} \quad v_1 = n_1 \frac{d\Phi}{dt} \]

Eliminate \( \Phi \):

\[ v_1 = \frac{n_2^2}{\mathcal{R}} \frac{d}{dt} \left[ i_1 + \frac{n_2}{n_1} i_2 \right] \]

This equation is of the form

\[ v_1 = L_M \frac{di_M}{dt} \]

with

\[ L_M = \frac{n_1^2}{\mathcal{R}} \]

\[ i_M = i_1 + \frac{n_2}{n_1} i_2 \]
Magnetizing inductance: discussion

• Models magnetization of core material
• A real, physical inductor, that exhibits saturation and hysteresis
• If the secondary winding is disconnected:
  we are left with the primary winding on the core
  primary winding then behaves as an inductor
  the resulting inductor is the magnetizing inductance, referred to
  the primary winding
• Magnetizing current causes the ratio of winding currents to differ
  from the turns ratio
Transformer saturation

- Saturation occurs when core flux density $B(t)$ exceeds saturation flux density $B_{sat}$.
- When core saturates, the magnetizing current becomes large, the impedance of the magnetizing inductance becomes small, and the windings are effectively shorted out.
- Large winding currents $i_1(t)$ and $i_2(t)$ do not necessarily lead to saturation. If
  \[ 0 = n_1i_1 + n_2i_2 \]
  then the magnetizing current is zero, and there is no net magnetization of the core.
- Saturation is caused by excessive applied volt-seconds
Saturation vs. applied volt-seconds

Magnetizing current depends on the integral of the applied winding voltage:

\[ i_M(t) = \frac{1}{L_M} \int v_1(t) \, dt \]

Flux density is proportional:

\[ B(t) = \frac{1}{n_1 A_c} \int v_1(t) \, dt \]

Flux density becomes large, and core saturates, when the applied volt-seconds \( \lambda_1 \) are too large, where

\[ \lambda_1 = \int_{t_1}^{t_2} v_1(t) \, dt \]

limits of integration chosen to coincide with positive portion of applied voltage waveform
13.2.3 Leakage inductances
Transformer model, including leakage inductance

Terminal equations can be written in the form

\[
\begin{bmatrix}
    v_1(t) \\
    v_2(t)
\end{bmatrix} =
\begin{bmatrix}
    L_{11} & L_{12} \\
    L_{12} & L_{22}
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
    i_1(t) \\
    i_2(t)
\end{bmatrix}
\]

mutual inductance:

\[
L_{12} = \frac{n_1 n_2}{R} = \frac{n_2}{n_1} L_M
\]

primary and secondary self-inductances:

\[
\begin{align*}
L_{11} &= L_{\ell 1} + \frac{n_1}{n_2} L_{12} \\
L_{22} &= L_{\ell 2} + \frac{n_2}{n_1} L_{12}
\end{align*}
\]

effective turns ratio

\[
n_e = \sqrt{\frac{L_{22}}{L_{11}}}
\]

coupling coefficient

\[
k = \frac{L_{12}}{\sqrt{L_{11} L_{22}}}
\]