Introduction to Power Electronics
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Lecture 4
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2.4 Cuk converter example

Cuk converter, with ideal switch

Cuk converter: practical realization using MOSFET and diode
This converter has two inductor currents and two capacitor voltages, that can be expressed as

\[ i_1(t) = I_1 + i_{1\text{-ripple}}(t) \]
\[ i_2(t) = I_2 + i_{2\text{-ripple}}(t) \]
\[ v_1(t) = V_1 + v_{1\text{-ripple}}(t) \]
\[ v_2(t) = V_2 + v_{2\text{-ripple}}(t) \]

To solve the converter in steady state, we want to find the dc components \( I_1, I_2, V_1, \) and \( V_2, \) when the ripples are small.

**Strategy:**
- Apply volt-second balance to each inductor voltage
- Apply charge balance to each capacitor current
- Simplify using the small ripple approximation
- Solve the resulting four equations for the four unknowns \( I_1, I_2, V_1, \) and \( V_2. \)
Cuk converter circuit
with switch in positions 1 and 2

Switch in position 1:
MOSFET conducts
Capacitor $C_1$ releases energy to output

Switch in position 2:
diode conducts
Capacitor $C_1$ is charged from input
Waveforms during subinterval 1
MOSFET conduction interval

Inductor voltages and capacitor currents:

\[ v_{L1} = V_g \]
\[ v_{L2} = -v_1 - v_2 \]
\[ i_{C1} = i_2 \]
\[ i_{C2} = i_2 - \frac{v_2}{R} \]

Small ripple approximation for subinterval 1:

\[ v_{L1} = V_g \]
\[ v_{L2} = -V_1 - V_2 \]
\[ i_{C1} = I_2 \]
\[ i_{C2} = I_2 - \frac{V_2}{R} \]
Waveforms during subinterval 2
Diode conduction interval

Inductor voltages and capacitor currents:

\[ v_{L1} = V_g - v_1 \]
\[ v_{L2} = -v_2 \]
\[ i_{C1} = i_1 \]
\[ i_{C2} = i_2 - \frac{v_2}{R} \]

Small ripple approximation for subinterval 2:

\[ v_{L1} = V_g - V_1 \]
\[ v_{L2} = -V_2 \]
\[ i_{C1} = I_1 \]
\[ i_{C2} = I_2 - \frac{V_2}{R} \]
Equate average values to zero

The principles of inductor volt-second and capacitor charge balance state that the average values of the periodic inductor voltage and capacitor current waveforms are zero, when the converter operates in steady state. Hence, to determine the steady-state conditions in the converter, let us sketch the inductor voltage and capacitor current waveforms, and equate their average values to zero.

**Waveforms:**

Inductor voltage $v_{L1}(t)$

Volt-second balance on $L_1$:

$$\langle v_{L1} \rangle = D V_g + D'(V_g - V_1) = 0$$
Equate average values to zero

**Inductor \( L_2 \) voltage**

\[
\left< v_{L2} \right> = D( -V_1 - V_2 ) + D'( -V_2 ) = 0
\]

**Capacitor \( C_1 \) current**

\[
\left< i_{C1} \right> = DI_2 + D'I_1 = 0
\]
Equate average values to zero

Capacitor current $i_{C2}(t)$ waveform

\[ i_{C2}(t) \]

\[ L_2 - \frac{V_2}{R} = 0 \]

\[ DT_s \quad \text{and} \quad DT_s \]

\[ \langle i_{C2} \rangle = I_2 - \frac{V_2}{R} = 0 \]

Note: during both subintervals, the capacitor current $i_{C2}$ is equal to the difference between the inductor current $i_2$ and the load current $\frac{V_2}{R}$. When ripple is neglected, $i_{C2}$ is constant and equal to zero.
Solve for steady-state inductor currents and capacitor voltages

The four equations obtained from volt-sec and charge balance:

\[
\begin{align*}
\langle v_{L1} \rangle &= DV_g + D' (V_g - V_1) = 0 \\
\langle v_{L2} \rangle &= D (-V_1 - V_2) + D' (-V_2) = 0 \\
\langle i_{C1} \rangle &= DI_2 + D'I_1 = 0 \\
\langle i_{C2} \rangle &= I_2 - \frac{V_2}{R} = 0
\end{align*}
\]

Solve for the dc capacitor voltages and inductor currents, and express in terms of the known \( V_g, D, \) and \( R \):

\[
\begin{align*}
V_1 &= \frac{V_g}{D} \\
V_2 &= -\frac{D'}{D} V_g \\
I_1 &= -\frac{D}{D'} I_2 = \left(\frac{D'}{D}\right)^2 \frac{V_g}{R} \\
I_2 &= \frac{V_2}{R} = -\frac{D}{D'} \frac{V_g}{R}
\end{align*}
\]
Cuk converter conversion ratio $M = \frac{V}{V_g}$

$M(D) = \frac{V_2}{V_g} = -\frac{D}{1-D}$
Inductor current waveforms

Interval 1 slopes, using small ripple approximation:

\[ \frac{di_1(t)}{dt} = \frac{v_{L1}(t)}{L_1} = \frac{V_g}{L_1} \]
\[ \frac{di_2(t)}{dt} = \frac{v_{L2}(t)}{L_2} = \frac{-V_1 - V_2}{L_2} \]

Interval 2 slopes:

\[ \frac{di_1(t)}{dt} = \frac{v_{L1}(t)}{L_1} = \frac{V_g - V_1}{L_1} \]
\[ \frac{di_2(t)}{dt} = \frac{v_{L2}(t)}{L_2} = \frac{-V_2}{L_2} \]
Capacitor $C_1$ waveform

Subinterval 1:
\[
\frac{dv_1(t)}{dt} = \frac{i_{C1}(t)}{C_1} = \frac{I_2}{C_1}
\]

Subinterval 2:
\[
\frac{dv_1(t)}{dt} = \frac{i_{C1}(t)}{C_1} = \frac{I_1}{C_1}
\]
Ripple magnitudes

Analysis results

\[ \Delta i_1 = \frac{V_s DT_s}{2L_1} \]
\[ \Delta i_2 = \frac{V_1 + V_2}{2L_2} DT_s \]
\[ \Delta v_1 = -\frac{I_2 DT_s}{2C_1} \]

Use dc converter solution to simplify:

\[ \Delta i_1 = \frac{V_s DT_s}{2L_1} \]
\[ \Delta i_2 = \frac{V_s DT_s}{2L_2} \]
\[ \Delta v_1 = \frac{V_s D^2 T_s}{2D'R C_1} \]

Q: How large is the output voltage ripple?
2.5 Estimating ripple in converters containing two-pole low-pass filters

Buck converter example: Determine output voltage ripple

Inductor current waveform.
What is the capacitor current?
Capacitor current and voltage, buck example

Must not neglect inductor current ripple!

If the capacitor voltage ripple is small, then essentially all of the ac component of inductor current flows through the capacitor.
Estimating capacitor voltage ripple $\Delta v$

Current $i_C(t)$ is positive for half of the switching period. This positive current causes the capacitor voltage $v_C(t)$ to increase between its minimum and maximum extrema. During this time, the total charge $q$ is deposited on the capacitor plates, where

$$q = C (2\Delta v)$$

$$\text{(change in charge)} = C \text{(change in voltage)}$$
Estimating capacitor voltage ripple $\Delta v$

The total charge $q$ is the area of the triangle, as shown:

$$q = \frac{1}{2} \Delta i_L \frac{T_s}{2}$$

Eliminate $q$ and solve for $\Delta v$:

$$\Delta v = \frac{\Delta i_L T_s}{8 C}$$

Note: in practice, capacitor equivalent series resistance (esr) further increases $\Delta v$. 
Inductor current ripple in two-pole filters

Example: problem 2.9

\[ \lambda = L(2\Delta i) \]

\( \lambda \) = inductor flux linkages

\( \lambda \) = inductor volt-seconds
Chapter 3. Steady-State Equivalent Circuit Modeling, Losses, and Efficiency

3.1. The dc transformer model
3.2. Inclusion of inductor copper loss
3.3. Construction of equivalent circuit model
3.4. How to obtain the input port of the model
3.5. Example: inclusion of semiconductor conduction losses in the boost converter model
3.6. Summary of key points
3.1. The dc transformer model

Basic equations of an ideal dc-dc converter:

\[ P_{in} = P_{out} \]
\[ V_g I_g = V I \]  \( (\eta = 100\%) \)

\[ V = M(D) V_g \]  \( \text{(ideal conversion ratio)} \)
\[ I_g = M(D) I \]

These equations are valid in steady-state. During transients, energy storage within filter elements may cause 
\[ P_{in} \neq P_{out} \]
Equivalent circuits corresponding to ideal dc-dc converter equations

\[ P_{in} = P_{out} \quad V_g I_g = V I \quad V = M(D) V_g \quad I_g = M(D) I \]
The DC transformer model

Models basic properties of ideal dc-dc converter:
- conversion of dc voltages and currents, ideally with 100% efficiency
- conversion ratio $M$ controllable via duty cycle

- Solid line denotes ideal transformer model, capable of passing dc voltages and currents
- Time-invariant model (no switching) which can be solved to find dc components of converter waveforms
Example: use of the DC transformer model

1. Original system

2. Insert dc transformer model

3. Push source through transformer

4. Solve circuit

\[ V = M(D) V_1 \frac{R}{R + M^2(D) R_1} \]
3.2. Inclusion of inductor copper loss

Dc transformer model can be extended, to include converter nonidealities.

Example: inductor copper loss (resistance of winding):

\[ L \quad R_L \]

Insert this inductor model into boost converter circuit:

[Diagram of boost converter circuit with inductor and resistor added]
Analysis of nonideal boost converter

![Diagram of boost converter with switch in position 1 and switch in position 2]
Circuit equations, switch in position 1

Inductor current and capacitor voltage:

\[ v_L(t) = V_g - i(t) R_L \]
\[ i_C(t) = -v(t) / R \]

Small ripple approximation:

\[ v_L(t) = V_g - I R_L \]
\[ i_C(t) = -V / R \]
Circuit equations, switch in position 2

\[ v_L(t) = V_g - i(t) R_L - v(t) = V_g - I R_L - V \]

\[ i_C(t) = i(t) - v(t) / R \approx I - V / R \]
Inductor voltage and capacitor current waveforms

Average inductor voltage:
\[
\langle v_L(t) \rangle = \frac{1}{T_s} \int_0^{T_s} v_L(t) dt = D(V_g - I R_L) + D'(V_g - I R_L - V)
\]

Inductor volt-second balance:
\[
0 = V_g - I R_L - D'V
\]

Average capacitor current:
\[
\langle i_C(t) \rangle = D(-V/R) + D'(I-V/R)
\]

Capacitor charge balance:
\[
0 = D'I - V/R
\]
Solution for output voltage

We now have two equations and two unknowns:

\[ 0 = V_g - I R_L - D V \]
\[ 0 = D I - V / R \]

Eliminate \( I \) and solve for \( V \):

\[
V = \frac{1}{D} \frac{1}{1 + R_L / D^2 R}
\]
3.3. Construction of equivalent circuit model

Results of previous section (derived via inductor volt-sec balance and capacitor charge balance):

\[
\begin{align*}
\langle v_L \rangle &= 0 = V_g - I R_t - D V \\
\langle i_c \rangle &= 0 = D I - V / R
\end{align*}
\]

View these as loop and node equations of the equivalent circuit.
Reconstruct an equivalent circuit satisfying these equations
Inductor voltage equation

\[ \langle v_L \rangle = 0 = V_g - I R_L - D'V \]

- Derived via Kirchhoff’s voltage law, to find the inductor voltage during each subinterval
- Average inductor voltage then set to zero
- This is a loop equation: the dc components of voltage around a loop containing the inductor sum to zero

\[ V_g + \langle v_L \rangle - I R_L - D'V = 0 \]

- \( IR_L \) term: voltage across resistor of value \( R_L \) having current \( I \)
- \( D'V \) term: for now, leave as dependent source
Capacitor current equation

\[ \langle i_c \rangle = 0 = D'I - \frac{V}{R} \]

- Derived via Kirchoff’s current law, to find the capacitor current during each subinterval
- Average capacitor current then set to zero
- This is a node equation: the dc components of current flowing into a node connected to the capacitor sum to zero

- \( \frac{V}{R} \) term: current through load resistor of value \( R \) having voltage \( V \)
- \( D'I \) term: for now, leave as dependent source
Complete equivalent circuit

The two circuits, drawn together:

![Complete equivalent circuit diagram]

The dependent sources are equivalent to a $D' : 1$ transformer:

- sources have same coefficient
- reciprocal voltage/current dependence

Dependent sources and transformers:

- $I_i$
- $nV_2$
- $nI_1$
- $V_2$
- $I_f$
- $n : 1$

Fundamentals of Power Electronics
Solution of equivalent circuit

Converter equivalent circuit

Refer all elements to transformer secondary:

Solution for output voltage using voltage divider formula:

$$V = \frac{V_g}{D} \left( \frac{R}{R + \frac{R_L}{D^2}} \right) = \frac{V_g}{D} \left( 1 + \frac{R_L}{D^2 R} \right)$$
Solution for input (inductor) current

\[ I = \frac{V_g}{D^2 R + R_L} = \frac{V_g}{D^2} \left( \frac{1}{1 + \frac{R_L}{D^2 R}} \right) \]
Solution for converter efficiency

\[ P_{\text{in}} = (V_g)(I) \]

\[ P_{\text{out}} = (V)(D'I) \]

\[ \eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{(V)(D'I)}{(V_g)(I)} = \frac{V}{V_g} D' \]

\[ \eta = \frac{1}{1 + \frac{R_L}{D'I^2 R}} \]
Efficiency, for various values of $R_L$

$$\eta = \frac{1}{1 + \frac{R_L}{D^2 R}}$$

For $R_L/R = 0.1$
3.4. How to obtain the input port of the model

Buck converter example — use procedure of previous section to derive equivalent circuit

\[
\langle v_L \rangle = 0 = D V_g - I_L R_L - V_C \\
\langle i_c \rangle = 0 = I_L - V_c / R
\]
Construct equivalent circuit as usual

\[ \langle v_L \rangle = 0 = D V_g - I_L R_L - V_C \]

\[ \langle i_C \rangle = 0 = I_L - V_C / R \]

What happened to the transformer?

\* Need another equation
Modeling the converter input port

Input current waveform $i_g(t)$:

Dc component (average value) of $i_g(t)$ is

$$I_g = \frac{1}{T_s} \int_0^{T_s} i_g(t) \, dt = DI_L$$
Input port equivalent circuit

\[ I_g = \frac{1}{T_s} \int_0^{T_s} i_g(t) \, dt = DI_L \]
Complete equivalent circuit, buck converter

Input and output port equivalent circuits, drawn together:

Replace dependent sources with equivalent dc transformer:

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Fundamentals of Power Electronics
3.5. Example: inclusion of semiconductor conduction losses in the boost converter model

Boost converter example

Models of on-state semiconductor devices:

MOSFET: on-resistance \( R_{on} \)

Diode: constant forward voltage \( V_D \) plus on-resistance \( R_D \)

Insert these models into subinterval circuits
Boost converter example: circuits during subintervals 1 and 2

\[ V_g \]

\[ \begin{align*}
  i &+ L \\
  DT_s &+ Ts \\
  i_L &+ v_L - \\
  v &+ v &- \\
  i_C &+ v &- \\
  CR &+ v &- \\
  i &+ v &- \\
  v &+ v &- \\
  \text{switch in position 1} & \text{switch in position 2} \\
  \end{align*} \]
Average inductor voltage and capacitor current

\[
\langle v_L \rangle = D(V_g - I_R L - I_{on}) + D'(V_g - I_R L - V_D - I_R D - V) = 0
\]

\[
\langle i_C \rangle = D(-V/R) + D'(I - V/R) = 0
\]
Construction of equivalent circuits

\[ V_g - IR_L - IDR_{on} - D'V_D - ID'R_D - D'V = 0 \]

\[ D'I - V/R = 0 \]
Complete equivalent circuit
Solution for output voltage

\[ V = \left( \frac{1}{D'} \right) \left( V_g - D'V_D \right) \left( \frac{D'^2R}{D'^2R + R_L + DR_{\text{on}} + D'R_D} \right) \]

\[ \frac{V}{V_g} = \left( \frac{1}{D'} \right) \left( 1 - \frac{D'V_D}{V_g} \right) \left( \frac{1}{1 + \frac{R_L + DR_{\text{on}} + D'R_D}{D'^2R}} \right) \]
Solution for converter efficiency

\[ P_{in} = (V_g) (I) \]

\[ P_{out} = (V) (D'I) \]

\[ \eta = D' \frac{V}{V_g} = \frac{1 - \frac{D'V_D}{V_g}}{1 + \frac{R_L + DR_{on} + D'R_D}{D'^2R}} \]

Conditions for high efficiency:

\[ \frac{V_g}{D'} \gg V_D \]

\[ D'^2R \gg R_L + DR_{on} + D'R_D \]
Accuracy of the averaged equivalent circuit in prediction of losses

- Model uses average currents and voltages
- To correctly predict power loss in a resistor, use rms values
- Result is the same, provided ripple is small

**MOSFET current waveforms, for various ripple magnitudes:**

<table>
<thead>
<tr>
<th>Inductor current ripple</th>
<th>MOSFET rms current</th>
<th>Average power loss in $R_{on}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\Delta i = 0$</td>
<td>$I \sqrt{D}$</td>
<td>$D \bar{I} R_{on}$</td>
</tr>
<tr>
<td>(b) $\Delta i = 0.1 \ I$</td>
<td>$(1.00167) I \sqrt{D}$</td>
<td>$(1.0033) D \bar{I} R_{on}$</td>
</tr>
<tr>
<td>(c) $\Delta i = I$</td>
<td>$(1.155) I \sqrt{D}$</td>
<td>$(1.3333) D \bar{I} R_{on}$</td>
</tr>
</tbody>
</table>
Summary of chapter 3

1. The dc transformer model represents the primary functions of any dc-dc converter: transformation of dc voltage and current levels, ideally with 100% efficiency, and control of the conversion ratio $M$ via the duty cycle $D$. This model can be easily manipulated and solved using familiar techniques of conventional circuit analysis.

2. The model can be refined to account for loss elements such as inductor winding resistance and semiconductor on-resistances and forward voltage drops. The refined model predicts the voltages, currents, and efficiency of practical nonideal converters.

3. In general, the dc equivalent circuit for a converter can be derived from the inductor volt-second balance and capacitor charge balance equations. Equivalent circuits are constructed whose loop and node equations coincide with the volt-second and charge balance equations. In converters having a pulsating input current, an additional equation is needed to model the converter input port; this equation may be obtained by averaging the converter input current.