2.3 Boost converter example

Boost converter with ideal switch

Realization using power MOSFET and diode
Inductor voltage and capacitor current waveforms

\[ v_L(t) = V_g - V \]

\[ i_C(t) = I - \frac{V}{R} \]

\[ v_L(t) = V_g - V \]

\[ i_C(t) = I - \frac{V}{R} \]
Inductor volt-second balance

Net volt-seconds applied to inductor over one switching period:

\[ \int_{0}^{T_s} v_L(t) \, dt = (V_g) \, DT_s + (V_g - V) \, D'T_s \]

Equate to zero and collect terms:

\[ V_g (D + D') - V D' = 0 \]

Solve for V:

\[ V = \frac{V_g}{D'} \]

The voltage conversion ratio is therefore

\[ M(D) = \frac{V}{V_g} = \frac{1}{D'} = \frac{1}{1 - D} \]
Conversion ratio $M(D)$ of the boost converter

$$M(D) = \frac{1}{D'} = \frac{1}{1-D}$$
Determination of inductor current dc component

Capacitor charge balance:

\[ \int_{0}^{T_s} i_c(t) \, dt = \left( -\frac{V}{R} \right) D T_s + \left( I - \frac{V}{R} \right) D' T_s \]

Collect terms and equate to zero:

\[ -\frac{V}{R} (D + D') + I D' = 0 \]

Solve for \( I \):

\[ I = \frac{V}{D' R} \]

Eliminate \( V \) to express in terms of \( V_g \):

\[ I = \frac{V_g}{D'^2 R} \]
Determination of inductor current ripple

Inductor current slope during subinterval 1:
\[
\frac{di_L(t)}{dt} = \frac{v_L(t)}{L} = \frac{V_g}{L}
\]

Inductor current slope during subinterval 2:
\[
\frac{di_L(t)}{dt} = \frac{v_L(t)}{L} = \frac{V_g - V}{L}
\]

Change in inductor current during subinterval 1 is \((\text{slope}) \times \text{(length of subinterval)}\):
\[
2\Delta i_L = \frac{V_g}{L} DT_s
\]

Solve for peak ripple:
\[
\Delta i_L = \frac{V_g}{2L} DT_s
\]

- Choose \(L\) such that desired ripple magnitude is obtained
Determination of capacitor voltage ripple

Capacitor voltage slope during subinterval 1:
\[ \frac{dv_C(t)}{dt} = \frac{i_c(t)}{C} = -\frac{V}{RC} \]

Capacitor voltage slope during subinterval 2:
\[ \frac{dv_C(t)}{dt} = \frac{i_c(t)}{C} = \frac{I}{C} - \frac{V}{RC} \]

Change in capacitor voltage during subinterval 1 is \((\text{slope}) \times \text{(length of subinterval)}\):
\[ -2\Delta v = -\frac{V}{RC} DT_s \]

Solve for peak ripple:
\[ \Delta v = \frac{V}{2RC} DT_s \]

- Choose \(C\) such that desired voltage ripple magnitude is obtained
- In practice, capacitor \emph{equivalent series resistance (esr)} leads to increased voltage ripple
2.4 Cuk converter example

Cuk converter, with ideal switch

\[ V_g \]

\[ L_1 \]

\[ C_1 \]

\[ L_2 \]

\[ R \]

\[ i_1 \] \quad 1 \quad 2 \quad i_2 \]

\[ v_1 \]

\[ v_2 \]

Cuk converter: practical realization using MOSFET and diode

\[ V_g \]

\[ L_1 \]

\[ C_1 \]

\[ L_2 \]

\[ C_2 \]

\[ R \]

\[ Q_1 \]

\[ D_1 \]

\[ i_1 \] \quad + \quad v_1 \quad - \quad i_2 \]

\[ v_2 \]
Analysis strategy

This converter has two inductor currents and two capacitor voltages, that can be expressed as

\[
i_1(t) = I_1 + i_{1-\text{ripple}}(t)
\]
\[
i_2(t) = I_2 + i_{2-\text{ripple}}(t)
\]
\[
v_1(t) = V_1 + v_{1-\text{ripple}}(t)
\]
\[
v_2(t) = V_2 + v_{2-\text{ripple}}(t)
\]

To solve the converter in steady state, we want to find the dc components \( I_1 \), \( I_2 \), \( V_1 \), and \( V_2 \), when the ripples are small.

**Strategy:**
- Apply volt-second balance to each inductor voltage
- Apply charge balance to each capacitor current
- Simplify using the small ripple approximation
- Solve the resulting four equations for the four unknowns \( I_1, I_2, V_1, \) and \( V_2 \).
Cuk converter circuit
with switch in positions 1 and 2

Switch in position 1:
- MOSFET conducts
- Capacitor $C_1$ releases energy to output

Switch in position 2:
- diode conducts
- Capacitor $C_1$ is charged from input
Waveforms during subinterval 1
MOSFET conduction interval

Inductor voltages and capacitor currents:

\[ v_{L1} = V_g \]
\[ v_{L2} = -v_1 - v_2 \]
\[ i_{C1} = i_2 \]
\[ i_{C2} = i_2 - \frac{v_2}{R} \]

Small ripple approximation for subinterval 1:

\[ v_{L1} = V_g \]
\[ v_{L2} = -V_1 - V_2 \]
\[ i_{C1} = I_2 \]
\[ i_{C2} = I_2 - \frac{V_2}{R} \]
Waveforms during subinterval 2
Diode conduction interval

Inductor voltages and capacitor currents:

\[ v_{L1} = V_g - v_1 \]
\[ v_{L2} = -v_2 \]
\[ i_{C1} = i_1 \]
\[ i_{C2} = i_2 - \frac{v_2}{R} \]

Small ripple approximation for subinterval 2:

\[ v_{L1} = V_g - V_1 \]
\[ v_{L2} = -V_2 \]
\[ i_{C1} = I_1 \]
\[ i_{C2} = I_2 - \frac{V_2}{R} \]
Equate average values to zero

The principles of inductor volt-second and capacitor charge balance state that the average values of the periodic inductor voltage and capacitor current waveforms are zero, when the converter operates in steady state. Hence, to determine the steady-state conditions in the converter, let us sketch the inductor voltage and capacitor current waveforms, and equate their average values to zero.

**Waveforms:**

Inductor voltage $v_{L1}(t)$

Volt-second balance on $L_1$:

$$\langle v_{L1} \rangle = DV_g + D'(V_g - V_1) = 0$$
Equate average values to zero

**Inductor L₂ voltage**

\[ v_{L2}(t) = \begin{cases} -V_2 & \text{for } D'T_s \\ -V_1 - V_2 & \text{for } DT_s \end{cases} \]

Average the waveforms:

\[ \langle v_{L2} \rangle = D(-V_1 - V_2) + D'(-V_2) = 0 \]

**Capacitor C₁ current**

\[ i_{C1}(t) = \begin{cases} I_1 & \text{for } D'T_s \\ I_2 & \text{for } DT_s \end{cases} \]

\[ \langle i_{C1} \rangle = DI_2 + D'I_1 = 0 \]
Equate average values to zero

Capacitor current $i_{c2}(t)$ waveform

Note: during both subintervals, the capacitor current $i_{c2}$ is equal to the difference between the inductor current $i_2$ and the load current $V_2/R$. When ripple is neglected, $i_{c2}$ is constant and equal to zero.

\[
\langle i_{c2} \rangle = I_2 - \frac{V_2}{R} = 0
\]
Solve for steady-state inductor currents and capacitor voltages

The four equations obtained from volt-sec and charge balance:

\[
\begin{align*}
\langle v_{L1} \rangle &= DV_g + D'\left( V_g - V_1 \right) = 0 \\
\langle v_{L2} \rangle &= D\left( -V_1 - V_2 \right) + D'\left( -V_2 \right) = 0 \\
\langle i_{C1} \rangle &= DI_2 + D'I_1 = 0 \\
\langle i_{C2} \rangle &= I_2 - \frac{V_2}{R} = 0
\end{align*}
\]

Solve for the dc capacitor voltages and inductor currents, and express in terms of the known \( V_g, D, \) and \( R \):

\[
\begin{align*}
V_1 &= \frac{V_g}{D} \\
V_2 &= -\frac{D}{D'} V_g \\
I_1 &= -\frac{D}{D'} I_2 = \left( \frac{D}{D'} \right)^2 \frac{V_g}{R} \\
I_2 &= \frac{V_2}{R} = -\frac{D}{D'} \frac{V_g}{R}
\end{align*}
\]
Cuk converter conversion ratio $M = V / V_g$

$$M(D) = \frac{V_2}{V_g} = - \frac{D}{1 - D}$$
Inductor current waveforms

Interval 1 slopes, using small ripple approximation:

\[
\begin{align*}
\frac{di_1(t)}{dt} &= \frac{v_{L_1}(t)}{L_1} = \frac{V_g}{L_1} \\
\frac{di_2(t)}{dt} &= \frac{v_{L_2}(t)}{L_2} = \frac{-V_1 - V_2}{L_2}
\end{align*}
\]

Interval 2 slopes:

\[
\begin{align*}
\frac{di_1(t)}{dt} &= \frac{v_{L_1}(t)}{L_1} = \frac{V_g - V_1}{L_1} \\
\frac{di_2(t)}{dt} &= \frac{v_{L_2}(t)}{L_2} = \frac{-V_2}{L_2}
\end{align*}
\]
Capacitor $C_1$ waveform

Subinterval 1:

\[ \frac{dv_1(t)}{dt} = \frac{i_{C1}(t)}{C_1} = \frac{I_2}{C_1} \]

Subinterval 2:

\[ \frac{dv_1(t)}{dt} = \frac{i_{C1}(t)}{C_1} = \frac{I_1}{C_1} \]
Ripple magnitudes

Analysis results

\[
\Delta i_1 = \frac{V_gDT_s}{2L_1} \\
\Delta i_2 = \frac{V_1 + V_2}{2L_2} DT_s \\
\Delta v_1 = \frac{-I_2 DT_s}{2C_1}
\]

Use dc converter solution to simplify:

\[
\Delta i_1 = \frac{V_gDT_s}{2L_1} \\
\Delta i_2 = \frac{V_gDT_s}{2L_2} \\
\Delta v_1 = \frac{V_gD^2T_s}{2D'RC_1}
\]

Q: How large is the output voltage ripple?
2.5 Estimating ripple in converters containing two-pole low-pass filters

Buck converter example: Determine output voltage ripple

Inductor current waveform.

What is the capacitor current?
Capacitor current and voltage, buck example

Must not neglect inductor current ripple!

If the capacitor voltage ripple is small, then essentially all of the ac component of inductor current flows through the capacitor.
Estimating capacitor voltage ripple $\Delta v$

Current $i_C(t)$ is positive for half of the switching period. This positive current causes the capacitor voltage $v_C(t)$ to increase between its minimum and maximum extrema. During this time, the total charge $q$ is deposited on the capacitor plates, where

$$q = C (2\Delta v)$$

$(\text{change in charge}) = C (\text{change in voltage})$
Estimating capacitor voltage ripple $\Delta v$

The total charge $q$ is the area of the triangle, as shown:

$$q = \frac{1}{2} \Delta i_L \frac{T_s}{2}$$

Eliminate $q$ and solve for $\Delta v$:

$$\Delta v = \frac{\Delta i_L T_s}{8 C}$$

Note: in practice, capacitor equivalent series resistance (esr) further increases $\Delta v$. 
Inductor current ripple in two-pole filters

Example: problem 2.9

can use similar arguments, with

\[ \lambda = L \ (2\Delta i) \]

\[ \lambda = \text{inductor flux linkages} \]

\[ = \text{inductor volt-seconds} \]
2.6 Summary of Key Points

1. The dc component of a converter waveform is given by its average value, or the integral over one switching period, divided by the switching period. Solution of a dc-dc converter to find its dc, or steady-state, voltages and currents therefore involves averaging the waveforms.

2. The linear ripple approximation greatly simplifies the analysis. In a well-designed converter, the switching ripples in the inductor currents and capacitor voltages are small compared to the respective dc components, and can be neglected.

3. The principle of inductor volt-second balance allows determination of the dc voltage components in any switching converter. In steady-state, the average voltage applied to an inductor must be zero.
Summary of Chapter 2

4. The principle of capacitor charge balance allows determination of the dc components of the inductor currents in a switching converter. In steady-state, the average current applied to a capacitor must be zero.

5. By knowledge of the slopes of the inductor current and capacitor voltage waveforms, the ac switching ripple magnitudes may be computed. Inductance and capacitance values can then be chosen to obtain desired ripple magnitudes.

6. In converters containing multiple-pole filters, continuous (nonpulsating) voltages and currents are applied to one or more of the inductors or capacitors. Computation of the ac switching ripple in these elements can be done using capacitor charge and/or inductor flux-linkage arguments, without use of the small-ripple approximation.

7. Converters capable of increasing (boost), decreasing (buck), and inverting the voltage polarity (buck-boost and Cuk) have been described. Converter circuits are explored more fully in a later chapter.
Chapter 3. Steady-State Equivalent Circuit Modeling, Losses, and Efficiency

3.1. The dc transformer model
3.2. Inclusion of inductor copper loss
3.3. Construction of equivalent circuit model
3.4. How to obtain the input port of the model
3.5. Example: inclusion of semiconductor conduction losses in the boost converter model
3.6. Summary of key points
3.1. The dc transformer model

Basic equations of an ideal dc-dc converter:

\[ P_{in} = P_{out} \]  \hspace{1cm} (\eta = 100\%)

\[ V_g I_g = V I \]

\[ V = M(D) V_g \]  \hspace{1cm} (ideal conversion ratio)

\[ I_g = M(D) I \]

These equations are valid in steady-state. During transients, energy storage within filter elements may cause

\[ P_{in} \neq P_{out} \]
Equivalent circuits corresponding to ideal dc-dc converter equations

\[ P_{in} = P_{out} \quad V_g I_g = V I \quad V = M(D) V_g \quad I_g = M(D) I \]

**Dependent sources**

**DC transformer**
The DC transformer model

Models basic properties of ideal dc-dc converter:

- conversion of dc voltages and currents, ideally with 100% efficiency
- conversion ratio $M$ controllable via duty cycle

- Solid line denotes ideal transformer model, capable of passing dc voltages and currents
- Time-invariant model (no switching) which can be solved to find dc components of converter waveforms

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Chapter 3: Steady-state equivalent circuit modeling, ...
Example: use of the DC transformer model

1. Original system

2. Insert dc transformer model

3. Push source through transformer

4. Solve circuit

\[ V = M(D) V_1 \frac{R}{R + M^2(D) R_1} \]
3.2. Inclusion of inductor copper loss

Dc transformer model can be extended, to include converter nonidealities.

Example: inductor copper loss (resistance of winding):

\[
L \quad \frac{R_L}{\text{---}}
\]

Insert this inductor model into boost converter circuit:
Analysis of nonideal boost converter
Circuit equations, switch in position 1

Inductor current and capacitor voltage:

\[ v_L(t) = V_g - i(t) R_L \]
\[ i_C(t) = -v(t) / R \]

Small ripple approximation:

\[ v_L(t) = V_g - I R_L \]
\[ i_C(t) = -V / R \]
Circuit equations, switch in position 2

\[ v_L(t) = V_g - i(t) R_L - v(t) = V_g - I R_L - V \]
\[ i_C(t) = i(t) - v(t) / R = I - V / R \]
Inductor voltage and capacitor current waveforms

Average inductor voltage:
\[
\langle v_L(t) \rangle = \frac{1}{T_s} \int_0^{T_s} v_L(t) \, dt = D(V_g - I R_L) + D'(V_g - I R_L - V)
\]

Inductor volt-second balance:
\[
0 = V_g - I R_L - D'V
\]

Average capacitor current:
\[
\langle i_C(t) \rangle = D \left( -\frac{V}{R} \right) + D' \left( I - \frac{V}{R} \right)
\]

Capacitor charge balance:
\[
0 = D' I - \frac{V}{R}
\]
Solution for output voltage

We now have two equations and two unknowns:

\[ 0 = V_g - I R_L - D V \]
\[ 0 = D I - V / R \]

Eliminate \( I \) and solve for \( V \):

\[
\frac{V}{V_g} = \frac{1}{D} \left( \frac{1}{1 + \frac{R_L}{D^2 R}} \right)
\]
3.3. Construction of equivalent circuit model

Results of previous section (derived via inductor volt-sec balance and capacitor charge balance):

\[ \langle v_i \rangle = 0 = V_g - I R_t - D' V \]
\[ \langle i_c \rangle = 0 = D'I - V / R \]

View these as loop and node equations of the equivalent circuit. Reconstruct an equivalent circuit satisfying these equations.
Inductor voltage equation

\[ \langle v_L \rangle = 0 = V_g - I R_L - D'V \]

- Derived via Kirchhoff’s voltage law, to find the inductor voltage during each subinterval
- Average inductor voltage then set to zero
- This is a loop equation: the dc components of voltage around a loop containing the inductor sum to zero

- IR_L term: voltage across resistor of value R_L having current I
- D’V term: for now, leave as dependent source
Capacitor current equation

\[ \langle i_C \rangle = 0 = D'I - \frac{V}{R} \]

- Derived via Kirchoff’s current law, to find the capacitor current during each subinterval
- Average capacitor current then set to zero
- This is a node equation: the dc components of current flowing into a node connected to the capacitor sum to zero

- \( V/R \) term: current through load resistor of value \( R \) having voltage \( V \)
- \( D'I \) term: for now, leave as dependent source

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Complete equivalent circuit

The two circuits, drawn together:

The dependent sources are equivalent to a \( D' : 1 \) transformer:

**Dependent sources and transformers**

- sources have same coefficient
- reciprocal voltage/current dependence
Solution of equivalent circuit

Converter equivalent circuit

Refer all elements to transformer secondary:

Solution for output voltage using voltage divider formula:

\[
V = \frac{V_g}{D} \left( R \frac{R_i}{D^2} + \frac{1}{1 + \frac{R_i}{D^2 R}} \right)
\]
Solution for input (inductor) current

\[ I = \frac{V_g}{D' R + R_L} = \frac{V_g}{D'} \cdot \frac{1}{\frac{R_l}{D' R} + 1} \]
Solution for converter efficiency

\[ P_{\text{in}} = (V_g) (I) \]
\[ P_{\text{out}} = (V) (D'I) \]
\[ \eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{(V) (D'I)}{(V_g) (I)} = \frac{V}{V_g} D' \]
\[ \eta = \frac{1 + \frac{R_L}{D'} R}{1 + \frac{R_L}{D'^2 R}} \]
Efficiency, for various values of $R_L$

$$\eta = \frac{1}{1 + \frac{R_L}{D^2 R}}$$

![Diagram showing efficiency for various values of $R_L/R$ with $R_L/R = 0.1$]
3.4. How to obtain the input port of the model

Buck converter example — use procedure of previous section to derive equivalent circuit

Average inductor voltage and capacitor current:

\[ \langle v_L \rangle = 0 = D V_g - I_L R_L - V_C \]
\[ \langle i_c \rangle = 0 = I_L - V_C / R \]
Construct equivalent circuit as usual

\[ \langle v_L \rangle = 0 = D V_g - I_L R_L - V_C \]
\[ \langle i_C \rangle = 0 = I_L - V_C / R \]

What happened to the transformer?
• Need another equation
Modeling the converter input port

Input current waveform $i_g(t)$:

Dc component (average value) of $i_g(t)$ is

$$I_g = \frac{1}{T_s} \int_0^{T_s} i_g(t) \, dt = DI_L$$
Input port equivalent circuit

\[ I_g = \frac{1}{T_s} \int_0^{T_s} i_g(t) \, dt = DI_L \]
Complete equivalent circuit, buck converter

Input and output port equivalent circuits, drawn together:

Replace dependent sources with equivalent dc transformer:
3.5. Example: inclusion of semiconductor conduction losses in the boost converter model

Models of on-state semiconductor devices:

- MOSFET: on-resistance $R_{on}$
- Diode: constant forward voltage $V_D$ plus on-resistance $R_D$

Insert these models into subinterval circuits
Boost converter example: circuits during subintervals 1 and 2

\begin{center}
\includegraphics[width=\textwidth]{diagram.png}
\end{center}

\textbf{switch in position 1} \hspace{1cm} \textbf{switch in position 2}
Average inductor voltage and capacitor current

$\langle v_L \rangle = D(V_g - IR_L - IR_{on}) + D'(V_g - IR_L - V_D - IR_D - V) = 0$

$\langle i_C \rangle = D(-V/R) + D'(I - V/R) = 0$
Construction of equivalent circuits

\[ V_g - IR_L - IDR_{on} - D'V_D - ID'R_D - D'V = 0 \]

\[ D'I - V/R = 0 \]
Complete equivalent circuit

Fundamentals of Power Electronics
Solution for output voltage

\[ V = \left( \frac{1}{D'} \right) \left( V_g - D'V_D \right) \left( \frac{D'^2 R}{D'^2 R + R_L + DR_{in} + D'R_D} \right) \]

\[ \frac{V}{V_g} = \left( \frac{1}{D'} \right) \left( 1 - \frac{D'V_D}{V_g} \right) \left( \frac{1}{1 + \frac{R_L + DR_{in} + D'R_D}{D'^2 R}} \right) \]
Solution for converter efficiency

\[ P_{in} = (V_s) \cdot I \]

\[ P_{out} = (V) \cdot (D'I) \]

\[ \eta = D' \frac{V}{V_s} = \frac{1 - \frac{D'V_D}{V_s}}{1 + \left(\frac{R_L + DR_{on} + D'R_D}{D'^2R}\right)} \]

Conditions for high efficiency:

\[ \frac{V_g}{D'} \gg V_D \]

\[ D'^2R \gg R_L + DR_{on} + D'R_D \]
Accuracy of the averaged equivalent circuit in prediction of losses

- Model uses average currents and voltages
- To correctly predict power loss in a resistor, use rms values
- Result is the same, provided ripple is small

\[
\begin{align*}
\text{Inductor current ripple} & \quad \text{MOSFET rms current} & \quad \text{Average power loss in } R_{on} \\
(a) \; \Delta i = 0 & \quad I \sqrt{T_D} & \quad D \hat{I} R_{on} \\
(b) \; \Delta i = 0.1 \; I & \quad (1.00167) I \sqrt{T_D} & \quad (1.0033) D \hat{I} R_{on} \\
(c) \; \Delta i = I & \quad (1.155) I \sqrt{T_D} & \quad (1.3333) D \hat{I} R_{on}
\end{align*}
\]
1. The dc transformer model represents the primary functions of any dc-dc converter: transformation of dc voltage and current levels, ideally with 100% efficiency, and control of the conversion ratio $M$ via the duty cycle $D$. This model can be easily manipulated and solved using familiar techniques of conventional circuit analysis.

2. The model can be refined to account for loss elements such as inductor winding resistance and semiconductor on-resistances and forward voltage drops. The refined model predicts the voltages, currents, and efficiency of practical nonideal converters.

3. In general, the dc equivalent circuit for a converter can be derived from the inductor volt-second balance and capacitor charge balance equations. Equivalent circuits are constructed whose loop and node equations coincide with the volt-second and charge balance equations. In converters having a pulsating input current, an additional equation is needed to model the converter input port; this equation may be obtained by averaging the converter input current.