ECEN 4797/5797

Introduction to Power Electronics

Lecture #22

Wednesday, October 14, 2009

Part II: Converter Dynamics & Control

State-Space Averaging

Sections 7.3

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System of Equations: State-space matrix form

$v_{2_1} = L \frac{di_1}{dt} = f(i_x's, v_c's, \ldots, v_g)$

v_{2_2}

v_{2_3}

i_{c_1}

i_{c_2}

i_{c_3}

$A \cdot X$

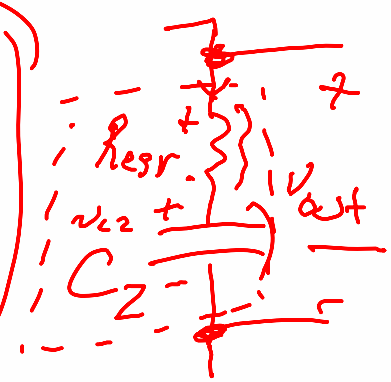
State equations of a linear system, in matrix form

A canonical matrix form:

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{E}\,\mathbf{u}(t)$$

State vector $\mathbf{x}(t)$ contains inductor currents, capacitor voltages, etc.:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix}, \qquad \frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \vdots \end{bmatrix}$$



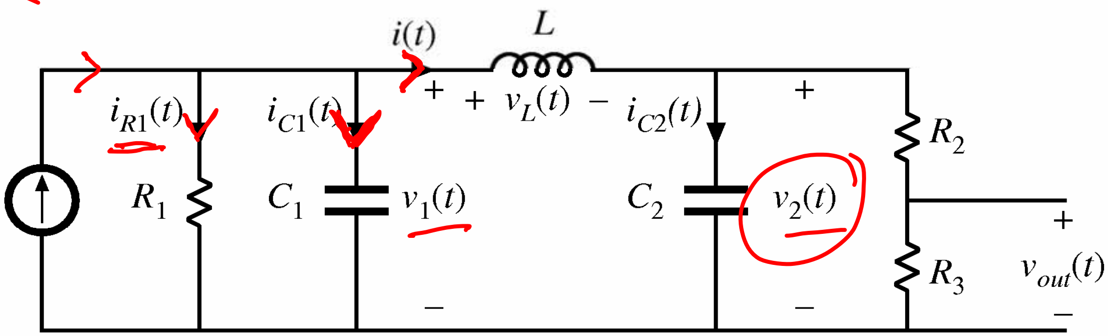
Input vector $\mathbf{u}(t)$ contains independent sources such as $v_g(t)$

Output vector $\mathbf{y}(t)$ contains other dependent quantities to be computed, such as $i_g(t)$

Matrix $\mathbf{K}$ contains values of capacitance, inductance, and mutual inductance, so that $\mathbf{K}\ d\mathbf{x}/dt$ is a vector containing capacitor currents and inductor winding voltages. These quantities are expressed as linear combinations of the independent inputs and state variables. The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{E}$ contain the constants of proportionality.

$$K\cdot\frac{dx}{dt} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix} = \begin{bmatrix} C_1 \frac{dv_1}{dt} + 0 + 0 \\ C_2\, dv_2/dt \\ L\frac{di}{dt} \end{bmatrix} = \begin{bmatrix} C\frac{dv_1}{dt} \\ \\ \end{bmatrix}$$

Example



State vector

$$\mathbf{x}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix}$$

Matrix **K**

$$\mathbf{K} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix} \cdot \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}$$

$x = \ldots$

Input vector

$$\mathbf{u}(t) = \begin{bmatrix} i_{in}(t) \end{bmatrix}$$

Choose output vector as

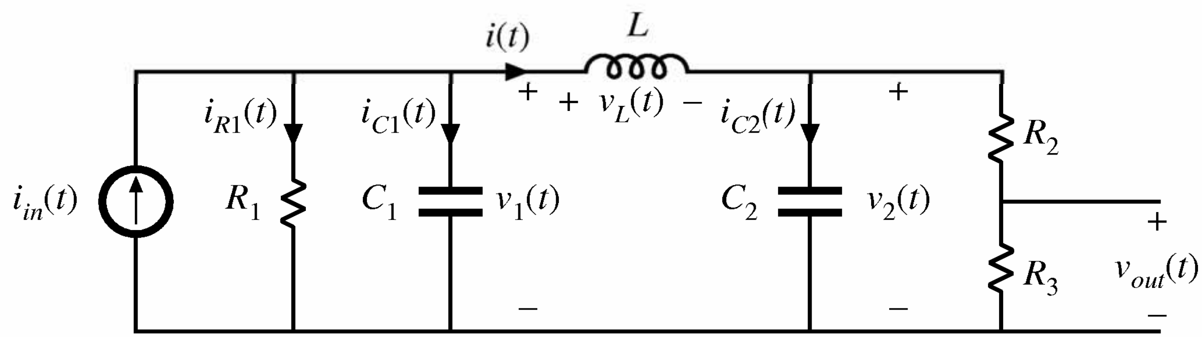
$$\mathbf{y}(t) = \begin{bmatrix} v_{out}(t) \\ i_{R1}(t) \end{bmatrix}$$

$$K\cdot\frac{dx}{dt} = Ax + Bu$$

$$\begin{bmatrix} -\frac{1}{R_1} & 0 & -1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

To write the state equations of this circuit, we must express the inductor voltages and capacitor currents as linear combinations of the elements of the $\mathbf{x}(t)$ and $\mathbf{u}(t)$ vectors.

Circuit equations



Find i_{C1} via node equation: $$i_{C1}(t) = C_1 \frac{dv_1(t)}{dt} = i_{in}(t) - \frac{v_1(t)}{R} - i(t)$$

Find i_{C2} via node equation: $$i_{C2}(t) = C_2 \frac{dv_2(t)}{dt} = i(t) - \frac{v_2(t)}{R_2 + R_3}$$

Find v_L via loop equation: $$v_L(t) = L \frac{di(t)}{dt} = v_1(t) - v_2(t)$$

Equations in matrix form

The same equations:

$$i_{C1}(t) = C_1 \frac{dv_1(t)}{dt} = i_{in}(t) - \frac{v_1(t)}{R} - i(t)$$

$$i_{C2}(t) = C_2 \frac{dv_2(t)}{dt} = i(t) - \frac{v_2(t)}{R_2 + R_3}$$

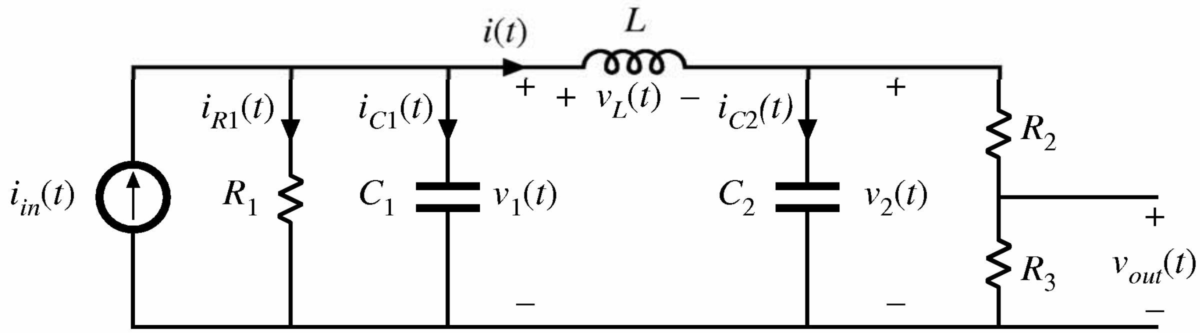
$$v_L(t) = L \frac{di(t)}{dt} = v_1(t) - v_2(t)$$

Express in matrix form:

$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} \frac{dv_1(t)}{dt} \\ \frac{dv_2(t)}{dt} \\ \frac{di(t)}{dt} \end{bmatrix}}_{\frac{d\mathbf{x}(t)}{dt}} = \underbrace{\begin{bmatrix} -\frac{1}{R_1} & 0 & -1 \\ 0 & -\frac{1}{R_2 + R_3} & 1 \\ 1 & -1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} i_{in}(t) \end{bmatrix}}_{\mathbf{u}(t)}$$

Output (dependent signal) equations

$$\mathbf{y}(t) = \begin{bmatrix} v_{out}(t) \\ i_{R1}(t) \end{bmatrix}$$



Express elements of the vector $\mathbf{y}$ as linear combinations of elements of $\mathbf{x}$ and $\mathbf{u}$:

$$v_{out}(t) = v_2(t)\,\frac{R_3}{R_2 + R_3}$$

$$i_{R1}(t) = \frac{v_1(t)}{R_1}$$

Express in matrix form

The same equations:

$$v_{out}(t) = v_2(t)\,\frac{R_3}{R_2 + R_3}$$

$$i_{R1}(t) = \frac{v_1(t)}{R_1}$$

Express in matrix form:

$$\underbrace{\begin{bmatrix} v_{out}(t) \\ i_{R1}(t) \end{bmatrix}}_{\mathbf{y}(t)} = \underbrace{\begin{bmatrix} 0 & \dfrac{R_3}{R_2 + R_3} & 0 \\ \dfrac{1}{R_1} & 0 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} i_{in}(t) \end{bmatrix}}_{\mathbf{u}(t)}$$

7.3.2 The basic state-space averaged model

Given: a PWM converter, operating in continuous conduction mode, with two subintervals during each switching period.

During subinterval 1, when the switches are in position 1, the converter reduces to a linear circuit that can be described by the following state equations:

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_1\,\mathbf{x}(t) + \mathbf{B}_1\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_1\,\mathbf{x}(t) + \mathbf{E}_1\,\mathbf{u}(t)$$

1

- Avg.
- DC
- Linearize

AC

During subinterval 2, when the switches are in position 2, the converter reduces to another linear circuit, that can be described by the following state equations:

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_2\,\mathbf{x}(t) + \mathbf{B}_2\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_2\,\mathbf{x}(t) + \mathbf{E}_2\,\mathbf{u}(t)$$

2

1)

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_1\,\mathbf{x}(t) + \mathbf{B}_1\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_1\,\mathbf{x}(t) + \mathbf{E}_1\,\mathbf{u}(t)$$

2)

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_2\,\mathbf{x}(t) + \mathbf{B}_2\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_2\,\mathbf{x}(t) + \mathbf{E}_2\,\mathbf{u}(t)$$

Avg- nonlinear:

$$\begin{cases} K\dfrac{d\langle x\rangle}{dt} = A(t)\cdot\langle x\rangle + B(t)\langle u\rangle \\ \langle y\rangle = C(t)\langle x\rangle + E(t)\langle u\rangle \end{cases}$$

$$A(t) = d(t)\cdot A_1 + d'(t)A_2$$

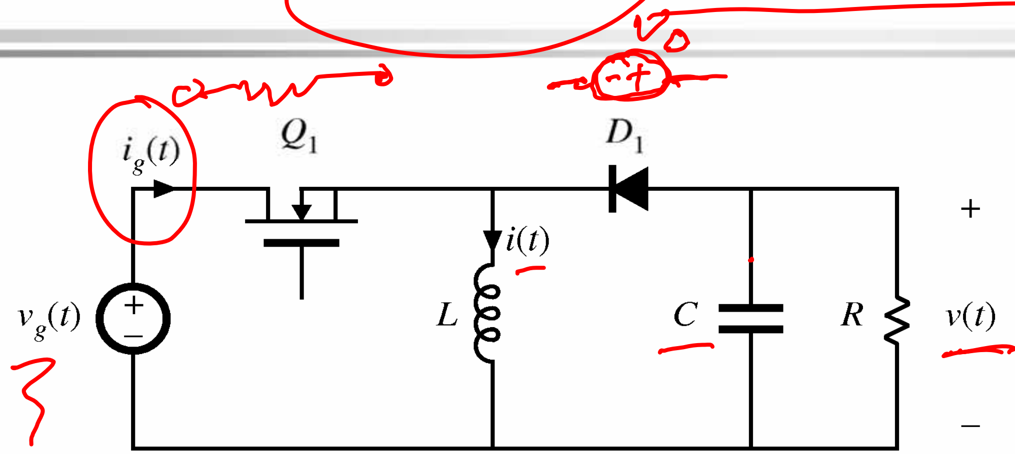
DC:

$$0 = \overbrace{AX + BU}^{d(t)=D} \rightarrow X = -A^{-1}BU$$
$$Y = CX + EU \qquad Y = (-CA^{-1}B + E)U$$

AC:

$$* \boxed{\begin{aligned} K\frac{d\hat{x}}{dt} &= A\hat{x} + B\hat{u} + \big((A_1 - A_2)X + (B_1 - B_2)U\big)\hat{d} \\ \hat{y} &= C\hat{x} + E\hat{u} + \big((C_1 - C_2)X + (E_1 - E_2)U\big)\hat{d} \end{aligned}}$$

7.3.4 Example: State-space averaging of a nonideal buck-boost converter



Model nonidealities:

- MOSFET on-resistance R_{on}
- Diode forward voltage drop V_D

state vector

$$\mathbf{x}(t) = \begin{bmatrix} i(t) \\ v(t) \end{bmatrix}$$

input vector

$$\mathbf{u}(t) = \begin{bmatrix} v_g(t) \\ V_D \end{bmatrix}$$

output vector

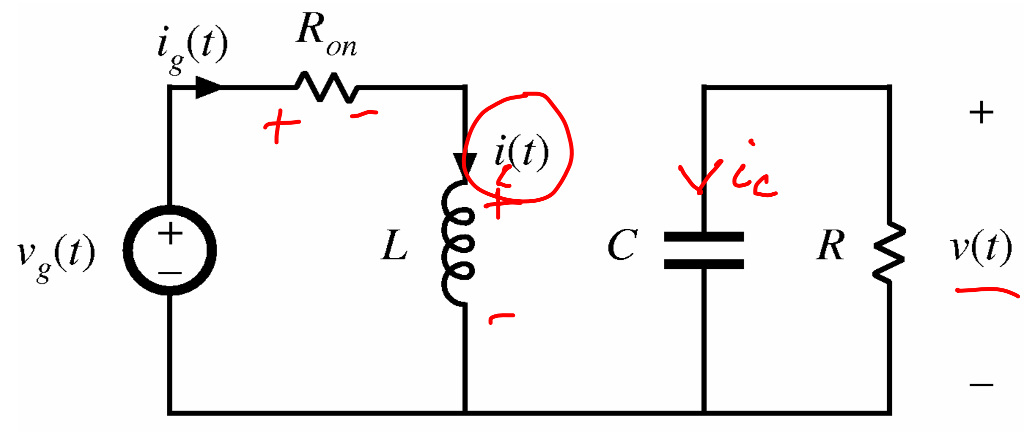
$$\mathbf{y}(t) = \begin{bmatrix} i_g(t) \end{bmatrix}$$

Interval #1

$\rightarrow v_L = v_g - i_L R_{on}$

$i_C = -v/R$

$i_g = i_L$



$$\begin{bmatrix} v_L \\ i_C \end{bmatrix} = \begin{bmatrix} L\frac{di_L}{dt} \\ C\frac{dv}{dt} \end{bmatrix} = \overbrace{\begin{bmatrix} -R_{on} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}}^{A_1} \begin{bmatrix} i_L \\ v \end{bmatrix} + \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}^{B_1} \begin{bmatrix} v_g \\ V_D \end{bmatrix}$$

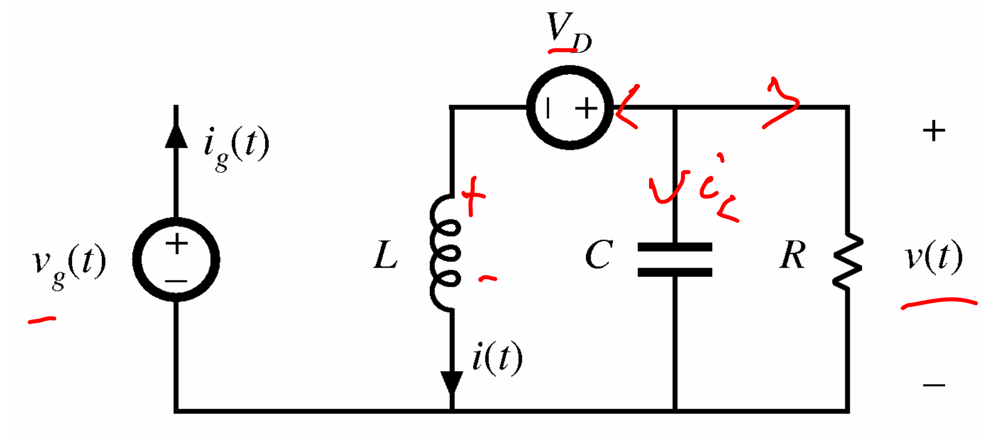
$$[i_g] = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{C_1} \begin{bmatrix} i_L \\ v \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}^{E_1} \begin{bmatrix} v_g \\ V_D \end{bmatrix}$$

#2

$v_L = v - V_D$

$i_c = -i_L - v/R$

$i_g = 0$



$$A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -1/R \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\cancel{E_2 = 0}$$

$$\left.\underbrace{\begin{bmatrix} -R_{on} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}}_{\mathbf{A}_1} \quad \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{R} \end{bmatrix}}_{\mathbf{A}_2}\right\} \quad A = \begin{bmatrix} -DR_{on} & D' \\ -D' & -\frac{1}{R} \end{bmatrix}, \quad A_1 - A_2 = \begin{bmatrix} -R_{on} & -1 \\ 1 & 0 \end{bmatrix}$$

$$\left.\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{B}_1} \quad \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{B}_2}\right) \quad B = \begin{bmatrix} D & -D' \\ 0 & 0 \end{bmatrix} \ldots$$

$$\left.\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}_1} \quad \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{\mathbf{C}_2}\right) \quad C = \begin{bmatrix} D & 0 \end{bmatrix} \ldots$$

DC state equations

$$\mathbf{0} = \mathbf{A}\,\mathbf{X} + \mathbf{B}\,\mathbf{U}$$
$$\mathbf{Y} = \mathbf{C}\,\mathbf{X} + \mathbf{E}\,\mathbf{U}$$

or,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -DR_{on} & D' \\ -D' & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} D & -D' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

$$\begin{bmatrix} I_g \end{bmatrix} = \begin{bmatrix} D & 0 \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

DC solution:

$$\begin{bmatrix} I \\ V \end{bmatrix} = \left(\frac{1}{1 + \frac{D}{D'^2}\frac{R_{on}}{R}} \right) \begin{bmatrix} \frac{D}{D'^2 R} & \frac{1}{D'\,R} \\ -\frac{D}{D'} & 1 \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

$$\begin{bmatrix} I_g \end{bmatrix} = \left(\frac{1}{1 + \frac{D}{D'^2}\frac{R_{on}}{R}} \right) \begin{bmatrix} \frac{D^2}{D'^2 R} & \frac{D}{D'R} \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

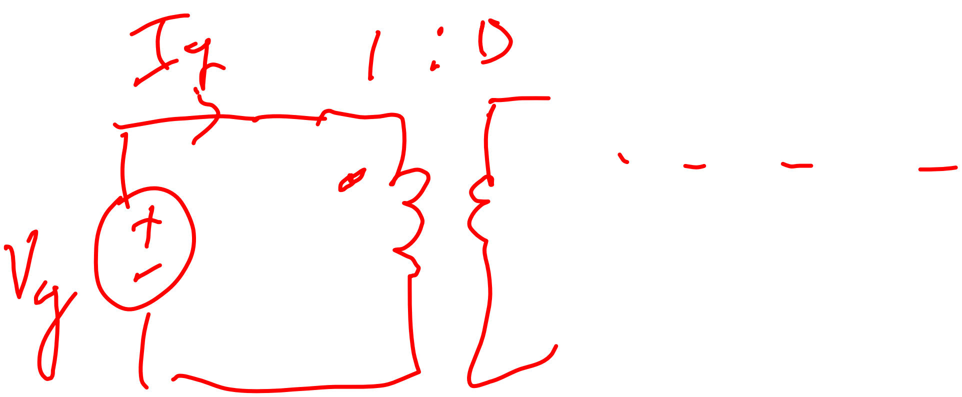
Steady-state equivalent circuit

DC state equations:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -DR_{on} & D' \\ -D' & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} D & -D' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

$$\begin{bmatrix} I_g \end{bmatrix} = \begin{bmatrix} D & 0 \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

Corresponding equivalent circuit:



$$\mathbf{A} = \begin{bmatrix} -DR_{on} & -D' \\ -D' & -1/R \end{bmatrix}, \mathbf{B} = \begin{bmatrix} D & -D' \\ 0 & 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} D & 0 \end{bmatrix}$$

$$\mathbf{A_1} - \mathbf{A_2} = \begin{bmatrix} -R_{on} & -1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{B_1} - \mathbf{B_2} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ \mathbf{C_1} - \mathbf{C_2} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\mathbf{K}\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{A}\,\hat{\mathbf{x}}(t) + \mathbf{B}\,\hat{\mathbf{u}}(t) + \left\{ \left(\mathbf{A}_1 - \mathbf{A}_2\right)\mathbf{X} + \left(\mathbf{B}_1 - \mathbf{B}_2\right)\mathbf{U} \right\} \hat{d}(t)$$

$$\hat{\mathbf{y}}(t) = \mathbf{C}\,\hat{\mathbf{x}}(t) + \mathbf{E}\,\hat{\mathbf{u}}(t) + \left\{ \left(\mathbf{C}_1 - \mathbf{C}_2\right)\mathbf{X} + \left(\mathbf{E}_1 - \mathbf{E}_2\right)\mathbf{U} \right\} \hat{d}(t)$$

$$\begin{bmatrix} L\frac{d\hat{i}_L}{dt} \\ C\frac{d\hat{v}}{dt} \end{bmatrix} = \begin{bmatrix} -DR_{on} & -D' \\ -D' & -1/R \end{bmatrix} \begin{bmatrix} \hat{i}_L \\ \hat{v} \end{bmatrix} + \cdots$$

Small-signal ac equivalent circuit

$$\begin{bmatrix} L\dfrac{d\hat{i}_l}{dt} \\ C\dfrac{d\hat{v}}{dt}\end{bmatrix} = \begin{bmatrix} -DR_{on} & D' \\ -D' & -\dfrac{1}{R}\end{bmatrix}\begin{bmatrix}\hat{i}_L\\ \hat{v}\end{bmatrix} + \begin{bmatrix} D & -\cancel{D'} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{v}_g \\ \hat{v}_D\end{bmatrix} + \begin{bmatrix} V_g + V_D - I_L R_{on} - V \\ I_L\end{bmatrix}\hat{d}$$

$$\hat{i}_g = \begin{bmatrix} D & 0\end{bmatrix}\begin{bmatrix}\hat{i}_L\\ \hat{v}\end{bmatrix} + I_L\hat{d}$$