Modeling and Control of Power Electronics Systems
ECEN 5807

Lecture 39
April 20, 2015

R. W. Erickson

Department of Electrical, Computer, and Energy Engineering
University of Colorado, Boulder
A complete 1ø system containing three feedback loops

**Diagram:**
- **Boost converter**
- **DC–DC Converter**
- **Wide-bandwidth input current controller**
- **Wide-bandwidth output voltage controller**
- **Low-bandwidth energy-storage capacitor voltage controller**

**Equations:**
- $v_{g}(t) = k_{x} v_{g}(t) v_{\text{control}}(t)$
- $v_{\text{ref}}(t) = k_{x} v_{g}(t) v_{\text{control}}(t)$

**Text:**
- **Fundamentals of Power Electronics** 62  Chapter 18: PWM Rectifiers
Bandwidth of capacitor voltage loop

- The energy-storage-capacitor voltage feedback loop causes the dc component of $v_c(t)$ to be equal to some reference value.
- Average rectifier power is controlled by variation of $R_e$.
- $R_e$ must not vary too quickly; otherwise, ac line current harmonics are generated.
- Extreme limit: loop has infinite bandwidth, and $v_c(t)$ is perfectly regulated to be equal to a constant reference value.
  - Energy storage capacitor voltage then does not change, and this capacitor does not store or release energy.
  - Instantaneous load and ac line powers are then equal.
- Input current becomes

$$i_{ac}(t) = \frac{P_{ac}(t)}{v_{ac}(t)} = \frac{P_{load}(t)}{v_{ac}(t)} = \frac{P_{load}}{V_M \sin(\omega t)}$$
Input current waveform, extreme limit

\[ i_{ac}(t) = \frac{P_{ac}(t)}{v_{ac}(t)} = \frac{P_{load}(t)}{v_{ac}(t)} = \frac{P_{load}}{V_M \sin(\omega t)} \]

So bandwidth of capacitor voltage loop must be limited, and THD increases rapidly with increasing bandwidth.
18.4.2  Modeling the outer low-bandwidth control system

This loop maintains power balance, stabilizing the rectifier output voltage against variations in load power, ac line voltage, and component values.

The loop must be slow, to avoid introducing variations in $R_e$ at the harmonics of the ac line frequency.

Objective of our modeling efforts: low-frequency small-signal model that predicts transfer functions at frequencies below the ac line frequency.
Large signal model
averaged over switching period $T_s$

Ideal rectifier model, assuming that inner wide-bandwidth loop operates ideally

High-frequency switching harmonics are removed via averaging

Ac line-frequency harmonics are included in model

Nonlinear and time-varying
Predictions of large-signal model

If the input voltage is

$$v_g(t) = \sqrt{2} \, v_{g,rms} \sin(\omega t)$$

Then the instantaneous power is:

$$\langle p(t) \rangle_{T_s} = \frac{\langle v_g(t) \rangle_{T_s}^2}{R_e(v_{control}(t))} = \frac{v_{g,rms}^2}{R_e(v_{control}(t))} \left( 1 - \cos(2\omega t) \right)$$

which contains a constant term plus a second-harmonic term.
Separation of power source into its constant and time-varying components

The second-harmonic variation in power leads to second-harmonic variations in the output voltage and current.
Removal of even harmonics via averaging

\[
\langle v(t) \rangle_{T_s} = \frac{1}{2} \frac{2\pi}{\omega} = \frac{\pi}{\omega}
\]

\[
T_{2L} = \frac{1}{2} \frac{2\pi}{\omega} = \frac{\pi}{\omega}
\]
Resulting averaged model

Time invariant model
Power source is nonlinear
Perturbation and linearization

The averaged model predicts that the rectifier output current is

\[
\left\langle i_2(t) \right\rangle_{T_{2L}} = \frac{\left\langle p(t) \right\rangle_{T_{2L}}}{\left\langle v(t) \right\rangle_{T_{2L}}} = \frac{v_{g,\text{rms}}^2(t)}{R_{e}(v_{\text{control}}(t)) \left\langle v(t) \right\rangle_{T_{2L}}}
= f \left( v_{g,\text{rms}}(t), \left\langle v(t) \right\rangle_{T_{2L}}, v_{\text{control}}(t) \right)
\]

Let

\[
\left\langle v(t) \right\rangle_{T_{2L}} = V + \hat{v}(t)
\]
\[
\left\langle i_2(t) \right\rangle_{T_{2L}} = I_2 + \hat{i}_2(t)
\]
\[
v_{g,\text{rms}} = V_{g,\text{rms}} + \hat{v}_{g,\text{rms}}(t)
\]
\[
v_{\text{control}}(t) = V_{\text{control}} + \hat{v}_{\text{control}}(t)
\]

with

\[
V >> |\hat{v}(t)|
\]
\[
I_2 >> |\hat{i}_2(t)|
\]
\[
V_{g,\text{rms}} >> |\hat{v}_{g,\text{rms}}(t)|
\]
\[
V_{\text{control}} >> |\hat{v}_{\text{control}}(t)|
\]
Linearized result

\[ I_2 + \hat{i}_2(t) = g_2 \hat{v}_{g,rms}(t) + j_2 \hat{v}(t) - \frac{\hat{v}_{control}(t)}{r_2} \]

where

\[ g_2 = \frac{df\left(v_{g,rms}, V, V_{control}\right)}{dv_{g,rms}} \bigg|_{v_{g,rms} = v_{g,rms}} = \frac{2}{R_e(V_{control})} \frac{V_{g,rms}}{V} \]

\[ \left( -\frac{1}{r_2} \right) = \frac{df\left(V_{g,rms}, \langle v \rangle_{T_2L}, V_{control}\right)}{d\langle v \rangle_{T_2L}} \bigg|_{\langle v \rangle_{T_2L} = V} = - \frac{I_2}{V} \]

\[ j_2 = \frac{df\left(V_{g,rms}, V, V_{control}\right)}{dv_{control}} \bigg|_{v_{control} = V_{control}} = \frac{V^2_{g,rms}}{VR_e^2(V_{control})} \frac{dR_e(v_{control})}{dv_{control}} \bigg|_{v_{control} = V_{control}} \]
Small-signal equivalent circuit

Predicted transfer functions

**Control-to-output**

\[
\frac{\hat{v}(s)}{\hat{v}_{\text{control}}(s)} = j_2 \frac{R || r_2}{1 + sC \frac{R || r_2}{1}}
\]

**Line-to-output**

\[
\frac{\hat{v}(s)}{\hat{v}_{g,\text{rms}}(s)} = g_2 \frac{R || r_2}{1 + sC \frac{R || r_2}{1}}
\]
# Model parameters

## Table 18.1  Small-signal model parameters for several types of rectifier control schemes

<table>
<thead>
<tr>
<th>Controller type</th>
<th>$g_2$</th>
<th>$j_2$</th>
<th>$r_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average current control with feedforward, Fig. 18.14</td>
<td>0</td>
<td>$\frac{P_{av}}{V_{V_{control}}}$</td>
<td>$\frac{V^2}{P_{av}}$</td>
</tr>
<tr>
<td>Current-programmed control, Fig. 18.16</td>
<td>$\frac{2P_{av}}{V_{g,rms}}$</td>
<td>$\frac{P_{av}}{V_{V_{control}}}$</td>
<td>$\frac{V^2}{P_{av}}$</td>
</tr>
<tr>
<td>Nonlinear-carrier charge control of boost rectifier, Fig. 18.21</td>
<td>$\frac{2P_{av}}{V_{g,rms}}$</td>
<td>$\frac{P_{av}}{V_{V_{control}}}$</td>
<td>$\frac{V^2}{2P_{av}}$</td>
</tr>
<tr>
<td>Boost with critical conduction mode control, Fig. 18.20</td>
<td>$\frac{2P_{av}}{V_{g,rms}}$</td>
<td>$\frac{P_{av}}{V_{V_{control}}}$</td>
<td>$\frac{V^2}{P_{av}}$</td>
</tr>
<tr>
<td>DCM buck-boost, flyback, SEPIC, or Čuk converters</td>
<td>$\frac{2P_{av}}{V_{g,rms}}$</td>
<td>$\frac{2P_{av}}{VD}$</td>
<td>$\frac{V^2}{P_{av}}$</td>
</tr>
</tbody>
</table>
Constant power load

Rectifier and dc-dc converter operate with same average power

Incremental resistance $R$ of constant power load is negative, and is

$$R = - \frac{V^2}{P_{av}}$$

which is equal in magnitude and opposite in polarity to rectifier incremental output resistance $r_z$ for all controllers except NLC
Transfer functions with constant power load

When \( r_2 = -R \), the parallel combination \( r_2 \parallel R \) becomes equal to zero. The small-signal transfer functions then reduce to

\[
\frac{\hat{v}(s)}{\hat{v}_{\text{cont}}(s)} = \frac{j_2}{sC}
\]

\[
\frac{\hat{v}(s)}{\hat{v}_{g,\text{rms}}(s)} = \frac{g_2}{sC}
\]
18.5 RMS values of rectifier waveforms

Doubly-modulated transistor current waveform, boost rectifier:

Computation of rms value of this waveform is complex and tedious

Approximate here using double integral

Generate tables of component rms and average currents for various rectifier converter topologies, and compare
RMS transistor current

RMS transistor current is

\[ I_{Q_{\text{rms}}} = \sqrt{\frac{1}{T_{\text{ac}}} \int_0^{T_{\text{ac}}} i_Q^2(t) \, dt} \]

Express as sum of integrals over all switching periods contained in one ac line period:

\[ I_{Q_{\text{rms}}} = \sqrt{\frac{1}{T_{\text{ac}}} T_s \sum_{n=1}^{\frac{T_{\text{ac}}}{T_s}} \left( \frac{1}{T_s} \int_{(n-1)T_s}^{nT_s} i_Q^2(t) \, dt \right)} \]

Quantity in parentheses is the value of \( i_Q^2 \), averaged over the \( n^{\text{th}} \) switching period.
Approximation of RMS expression

\[ I_{Q_{rms}} = \sqrt{\frac{1}{T_{ac}} T_s \sum_{n=1}^{T_{ac}/T_s} \left( \frac{1}{T_s} \int_{(n-1)T_s}^{nT_s} i_Q^2(t)\,dt \right)} \]

When \( T_s \ll T_{ac} \), then the summation can be approximated by an integral, which leads to the double-average:

\[ I_{Q_{rms}} \approx \sqrt{\frac{1}{T_{ac}} \lim_{T_s \to 0} \left[ T_s \sum_{n=1}^{T_{ac}/T_s} \left( \frac{1}{T_s} \int_{(n-1)T_s}^{nT_s} i_Q^2(\tau)\,d\tau \right) \right]} \]

\[ = \sqrt{\frac{1}{T_{ac}} \int_0^{T_{ac}} \frac{1}{T_s} \int_0^{T_s} i_Q^2(\tau)\,d\tau\,dt} \]

\[ = \sqrt{\left\langle i_Q^2(t) \right\rangle_{T_s} / T_{ac}} \]
18.5.1 Boost rectifier example

For the boost converter, the transistor current \( i_Q(t) \) is equal to the input current when the transistor conducts, and is zero when the transistor is off. The average over one switching period of \( i_Q^2(t) \) is therefore

\[
\langle i_Q^2 \rangle_{T_s} = \frac{1}{T_s} \int_{t}^{t+T_s} i_Q^2(t) \, dt = d(t) i_{ac}^2(t)
\]

If the input voltage is

\[ v_{ac}(t) = V_M \left| \sin \omega t \right| \]

then the input current will be given by

\[ i_{ac}(t) = \frac{V_M}{R_e} \left| \sin \omega t \right| \]

and the duty cycle will ideally be

\[ \frac{V}{v_{ac}(t)} = \frac{1}{1 - d(t)} \quad \text{(this neglects converter dynamics)} \]
Boost rectifier example

Duty cycle is therefore

\[ d(t) = 1 - \frac{V_M}{V} \left| \sin \omega t \right| \]

Evaluate the first integral:

\[ \langle i_Q^2 \rangle_{T_s} = \frac{V_M^2}{R_e^2} \left( 1 - \frac{V_M}{V} \left| \sin \omega t \right| \right) \sin^2 (\omega t) \]

Now plug this into the RMS formula:

\[
I_{Q_{\text{rms}}} = \sqrt{\frac{1}{T_{ac}} \int_0^{T_{ac}} \langle i_Q^2 \rangle_{T_s} dt} = \sqrt{\frac{1}{T_{ac}} \int_0^{T_{ac}} \frac{V_M^2}{R_e^2} \left( 1 - \frac{V_M}{V} \left| \sin \omega t \right| \right) \sin^2 (\omega t) dt}
\]

\[
I_{Q_{\text{rms}}} = \sqrt{\frac{2}{T_{ac}} \frac{V_M^2}{R_e^2} \int_0^{T_{ac}/2} \left( \sin^2 (\omega t) - \frac{V_M}{V} \sin^3 (\omega t) \right) dt}
\]
Integration of powers of \( \sin \theta \) over complete half-cycle

\[
\frac{1}{\pi} \int_0^\pi \sin^n(\theta) d\theta = \begin{cases} 
\frac{2 \cdot 2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n} & \text{if } n \text{ is odd} \\
\frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} & \text{if } n \text{ is even}
\end{cases}
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \frac{1}{\pi} \int_0^\pi \sin^n(\theta) d\theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{2}{\pi} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{\pi} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{4}{3\pi} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{3}{8} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{16}{15\pi} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{15}{48} )</td>
</tr>
</tbody>
</table>
Boost example: transistor RMS current

\[ I_{Q_{rms}} = \frac{V_M}{\sqrt{2} R_e} \sqrt{1 - \frac{8}{3\pi} \frac{V_M}{V}} = I_{ac\,rms} \sqrt{1 - \frac{8}{3\pi} \frac{V_M}{V}} \]

Transistor RMS current is minimized by choosing \( V \) as small as possible: \( V = V_M \). This leads to

\[ I_{Q_{rms}} = 0.39 I_{ac\,rms} \]

When the dc output voltage is not too much greater than the peak ac input voltage, the boost rectifier exhibits very low transistor current. Efficiency of the boost rectifier is then quite high, and 95% is typical in a 1kW application.
Table of rectifier current stresses for various topologies

<table>
<thead>
<tr>
<th>Topology</th>
<th>Current Stress</th>
<th>Average</th>
<th>Peak</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCM boost</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Transistor</td>
<td>$I_{ac,rms} \sqrt{1 - \frac{8}{3\pi} \frac{V_M}{V}}$</td>
<td>$I_{ac,rms} \frac{2\sqrt{2}}{\pi} \left(1 - \frac{\pi}{8} \frac{V_M}{V}\right)$</td>
<td>$I_{ac,rms} \sqrt{2}$</td>
</tr>
<tr>
<td>Diode</td>
<td>$I_{dc} \sqrt{\frac{16}{3\pi} \frac{V}{V_M}}$</td>
<td>$I_{dc}$</td>
<td>$2I_{dc} \frac{V}{V_M}$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$I_{ac,rms}$</td>
<td>$I_{ac,rms} \frac{2\sqrt{2}}{\pi}$</td>
<td>$I_{ac,rms} \sqrt{2}$</td>
</tr>
<tr>
<td>CCM flyback, with n:1 isolation transformer and input filter</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Transistor, xfmr primary</td>
<td>$I_{ac,rms} \sqrt{1 + \frac{8}{3\pi} \frac{V_M}{V}}$</td>
<td>$I_{ac,rms} \frac{2\sqrt{2}}{\pi}$</td>
<td>$I_{ac,rms} \sqrt{2} \left(1 + \frac{V}{n}\right)$</td>
</tr>
<tr>
<td>$L_1$</td>
<td>$I_{ac,rms}$</td>
<td>$I_{ac,rms} \frac{2\sqrt{2}}{\pi}$</td>
<td>$I_{ac,rms} \sqrt{2}$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$I_{ac,rms} \sqrt{\frac{8}{3\pi} \frac{V_M}{nV}}$</td>
<td>0</td>
<td>$I_{ac,rms} \sqrt{2} \max \left(1, \frac{V_M}{nV}\right)$</td>
</tr>
<tr>
<td>Diode, xfmr secondary</td>
<td>$I_{dc} \sqrt{\frac{3}{2} + \frac{16}{3\pi} \frac{nV}{V_M}}$</td>
<td>$I_{dc}$</td>
<td>$2I_{dc} \left(1 + \frac{nV}{V_M}\right)$</td>
</tr>
</tbody>
</table>

Table 18.3 Summary of rectifier current stresses for several converter topologies

Fundamentals of Power Electronics
Table of rectifier current stresses

<table>
<thead>
<tr>
<th>Component</th>
<th>CCM SEPIC, nonisolated</th>
<th>CCM SEPIC, with n:1 isolation transformer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transistor</td>
<td>$I_{\text{ac rms}} \sqrt{1 + \frac{8}{3\pi} \frac{V_M}{V}}$</td>
<td>$I_{\text{ac rms}} \sqrt{1 + \frac{8}{3\pi} \frac{V_M}{nV}}$</td>
</tr>
<tr>
<td>$L_1$</td>
<td>$I_{\text{ac rms}}$</td>
<td>$I_{\text{ac rms}}$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$I_{\text{ac rms}} \sqrt{\frac{8}{3\pi} \frac{V_M}{V}}$</td>
<td>$I_{\text{ac rms}} \sqrt{\frac{8}{3\pi} \frac{V_M}{nV}}$</td>
</tr>
<tr>
<td>$L_2$</td>
<td>$I_{\text{ac rms}} \frac{V_M}{\sqrt{2}}$</td>
<td>$I_{\text{ac rms}} \frac{V_M}{\sqrt{2}}$</td>
</tr>
<tr>
<td>Diode</td>
<td>$I_{\text{dc}} \sqrt{\frac{3}{2} + \frac{16}{3\pi} \frac{V_M}{V}}$</td>
<td>$I_{\text{dc}} \sqrt{\frac{3}{2} + \frac{16}{3\pi} \frac{nV}{V}}$</td>
</tr>
</tbody>
</table>

Diode, xfmr secondary

with, in all cases, $\frac{I_{\text{ac rms}}}{I_{\text{dc}}} = \sqrt{\frac{\pi}{2}} \frac{V}{V_M}$, ac input voltage = $V_M \sin(\omega t)$

dc output voltage = $V$

Fundamentals of Power Electronics 85 Chapter 18: PWM Rectifiers
Comparison of rectifier topologies

Boost converter
- Lowest transistor rms current, highest efficiency
- Isolated topologies are possible, with higher transistor stress
- No limiting of inrush current
- Output voltage must be greater than peak input voltage

Buck-boost, SEPIC, and Cuk converters
- Higher transistor rms current, lower efficiency
- Isolated topologies are possible, without increased transistor stress
- Inrush current limiting is possible
- Output voltage can be greater than or less than peak input voltage
Comparison of rectifier topologies

1kW, 240Vrms example. Output voltage: 380Vdc. Input current: 4.2Arms

<table>
<thead>
<tr>
<th>Converter</th>
<th>Transistor rms current</th>
<th>Transistor voltage</th>
<th>Diode rms current</th>
<th>Transistor rms current, 120V</th>
<th>Diode rms current, 120V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boost</td>
<td>2 A</td>
<td>380 V</td>
<td>3.6 A</td>
<td>6.6 A</td>
<td>5.1 A</td>
</tr>
<tr>
<td>Nonisolated SEPIC</td>
<td>5.5 A</td>
<td>719 V</td>
<td>4.85 A</td>
<td>9.8 A</td>
<td>6.1 A</td>
</tr>
<tr>
<td>Isolated SEPIC</td>
<td>5.5 A</td>
<td>719 V</td>
<td>36.4 A</td>
<td>11.4 A</td>
<td>42.5 A</td>
</tr>
</tbody>
</table>

Isolated SEPIC example has 4:1 turns ratio, with 42V 23.8A dc load
18.6 Modeling losses and efficiency in CCM high-quality rectifiers

**Objective:** extend procedure of Chapter 3, to predict the output voltage, duty cycle variations, and efficiency, of PWM CCM low harmonic rectifiers.

**Approach:** Use the models developed in Chapter 3. Integrate over one ac line cycle to determine steady-state waveforms and average power.

**Boost example**

---

**Dc-dc boost converter circuit**

---

**Averaged dc model**

---
Modeling the ac-dc boost rectifier
Boost rectifier waveforms

Typical waveforms
(low frequency components)

\[ i_g(t) = \frac{v_g(t)}{R_e} \]
Example: boost rectifier with MOSFET on-resistance

Averaged model

Inductor dynamics are neglected, a good approximation when the ac line variations are slow compared to the converter natural frequencies.
18.6.1 Expression for controller duty cycle $d(t)$

Solve input side of model:

$$i_g(t)d(t)R_{on} = v_g(t) - d'(t)v$$

with

$$i_g(t) = \frac{v_g(t)}{R_e}$$

$$v_g(t) = V_m |\sin \omega t|$$

eliminate $i_g(t)$:

$$\frac{v_g(t)}{R_e} \cdot d(t)R_{on} = v_g(t) - d'(t)v$$

solve for $d(t)$:

$$d(t) = \frac{v - v_g(t)}{v - v_g(t) \frac{R_{on}}{R_e}}$$

Again, these expressions neglect converter dynamics, and assume that the converter always operates in CCM.
18.6.2 Expression for the dc load current

Solve output side of model, using charge balance on capacitor C:

\[ I = \langle i_d \rangle_{T_{ac}} \]

\[ i_d(t) = d'(t) i_g(t) = d''(t) \frac{v_g(t)}{R_e} \]

But \( d'(t) \) is:

\[ d'(t) = \frac{v_g(t) \left(1 - \frac{R_{on}}{R_e}\right)}{v - v_g(t) \frac{R_{on}}{R_e}} \]

hence \( i_d(t) \) can be expressed as

\[ i_d(t) = \frac{v_g^2(t)}{R_e} \left(1 - \frac{R_{on}}{R_e}\right) \frac{1}{v - v_g(t) \frac{R_{on}}{R_e}} \]

Next, average \( i_d(t) \) over an ac line period, to find the dc load current \( I \).
Dc load current $I$

Now substitute $v_g(t) = V_M \sin \omega t$, and integrate to find $\langle i_d(t) \rangle_{T_{ac}}$:

$$I = \langle i_d \rangle_{T_{ac}} = \frac{2}{T_{ac}} \int_0^{T_{ac}/2} \left( \frac{V_M^2}{R_e} \right) \left( 1 - \frac{R_{on}}{R_e} \right) \sin^2 (\omega t) \left( v - \frac{V_M R_{on}}{R_e} \sin (\omega t) \right) dt$$

This can be written in the normalized form

$$I = \frac{2}{T_{ac}} \frac{V_M^2}{VR_e} \left( 1 - \frac{R_{on}}{R_e} \right) \int_0^{T_{ac}/2} \frac{\sin^2 (\omega t)}{1 - a \sin (\omega t)} dt$$

with

$$a = \left( \frac{V_M}{V} \right) \left( \frac{R_{on}}{R_e} \right)$$
Integration

By waveform symmetry, we need only integrate from 0 to \( T_{ac}/4 \). Also, make the substitution \( \theta = \omega t \):

\[
I = \frac{V_M^2}{VR_e} \left( 1 - \frac{R_{on}}{R_e} \right) \frac{2}{\pi} \int_0^{\pi/2} \frac{\sin^2(\theta)}{1 - a \sin(\theta)} \, d\theta
\]

This integral is obtained not only in the boost rectifier, but also in the buck-boost and other rectifier topologies. The solution is

\[
\frac{4}{\pi} \int_0^{\pi/2} \frac{\sin^2(\theta)}{1 - a \sin(\theta)} \, d\theta = F(a) = \frac{2}{a^2 \pi} \left( -2a - \pi + \frac{4 \sin^{-1}(a) + 2 \cos^{-1}(a)}{\sqrt{1 - a^2}} \right)
\]

• Result is in closed form
• \( a \) is a measure of the loss resistance relative to \( R_e \)
• \( a \) is typically much smaller than unity
The integral $F(a)$

$$
\frac{4}{\pi} \int_0^{\pi/2} \frac{\sin^2(\theta)}{1 - a \sin(\theta)} d\theta = F(a) = \frac{2}{a^2 \pi} \left( -2a - \pi + \frac{4 \sin^{-1}(a) + 2 \cos^{-1}(a)}{\sqrt{1 - a^2}} \right)
$$

Approximation via polynomial:

$$
F(a) \approx 1 + 0.862a + 0.78a^2
$$

For $|a| \leq 0.15$, this approximate expression is within 0.1% of the exact value. If the $a^2$ term is omitted, then the accuracy drops to ±2% for $|a| \leq 0.15$. The accuracy of $F(a)$ coincides with the accuracy of the rectifier efficiency $\eta$. 

Fundamentals of Power Electronics 96 Chapter 18: PWM Rectifiers
18.6.3 Solution for converter efficiency $\eta$

Converter average input power is

$$P_{in} = \left\langle p_{in}(t) \right\rangle_{T_{ac}} = \frac{V_M^2}{2R_e}$$

Average load power is

$$P_{out} = VI = \left(V\right) \left(\frac{V_M}{VR_e} \left(1 - \frac{R_{on}}{R_e}\right) \frac{F(a)}{2}\right)$$

with

$$a = \left(\frac{V_M}{V}\right) \left(\frac{R_{on}}{R_e}\right)$$

So the efficiency is

$$\eta = \frac{P_{out}}{P_{in}} = \left(1 - \frac{R_{on}}{R_e}\right) F(a)$$

Polynomial approximation:

$$\eta \approx \left(1 - \frac{R_{on}}{R_e}\right) \left(1 + 0.862 \frac{V_M}{V} \frac{R_{on}}{R_e} + 0.78 \left(\frac{V_M}{V} \frac{R_{on}}{R_e}\right)^2\right)$$
Boost rectifier efficiency

\[ \eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \left( 1 - \frac{R_{\text{on}}}{R_e} \right) F(a) \]

- To obtain high efficiency, choose \( V \) slightly larger than \( V_M \)
- Efficiencies in the range 90% to 95% can then be obtained, even with \( R_{\text{on}} \) as high as \( 0.2R_e \)
- Losses other than MOSFET on-resistance are not included here
18.6.4 Design example

Let us design for a given efficiency. Consider the following specifications:

- Output voltage: 390 V
- Output power: 500 W
- RMS input voltage: 120 V
- Efficiency: 95%

Assume that losses other than the MOSFET conduction loss are negligible.

Average input power is

$$ P_{in} = \frac{P_{out}}{\eta} = \frac{500 \text{ W}}{0.95} = 526 \text{ W} $$

Then the emulated resistance is

$$ R_e = \frac{V_{g,rms}^2}{P_{in}} = \frac{(120 \text{ V})^2}{526 \text{ W}} = 27.4 \Omega $$
Design example

Also, \[
\frac{V_M}{V} = \frac{120\sqrt{2}}{390} V = 0.435
\]

95% efficiency with \( V_M/V = 0.435 \) occurs with \( R_{on}/R_e \approx 0.075 \).

So we require a MOSFET with on resistance of

\[
R_{on} \leq (0.075) R_e
\]

\[
= (0.075) (27.4 \Omega) = 2 \Omega
\]