Current Programmed Control
(i.e. Peak Current-Mode Control)

Lecture slides part 2
More Accurate Models

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Simple First-Order CPM Model: Summary

- Assumption: CPM controller operates ideally, $\langle i_L \rangle = i_c$
- Useful results at low frequencies, well suited for conservative (relatively low bandwidth) design of voltage feedback loop around CPM controlled converter
- Limitations:
  - The simple model does not indicate possible instability of the current controller, or the need for a compensation ("artificial") ramp
  - Does not include high-frequency dynamics, which is relevant for wide-bandwidth voltage loop designs
  - Does not correctly model line-to-output responses in CPM buck or buck-derived converters even at low frequencies (it incorrectly predicts complete rejection of line disturbances)
More Accurate CPM Models: Outline

- Sampled-data modeling of inductor dynamics in current programmed mode
  - Instability of the current loop and the need for compensation ("artificial") ramp
  - Improved modeling of high-frequency dynamics to enable design of wide-bandwidth voltage loops

- More accurate averaged model
  - Large-signal and small-signal averaged modulator model
  - Accurate averaged small-signal models, including high-frequency dynamics
  - Accurate modeling of line-to-output responses

- Discussion of results for basic converters
- CPM model for simulations
- Design examples
Inductor current waveform, CCM

Inductor current slopes $m_1$ and $-m_2$

buck converter

$$m_1 = \frac{v_g - v}{L} \quad -m_2 = -\frac{v}{L}$$

boost converter

$$m_1 = \frac{v_g}{L} \quad -m_2 = \frac{v_g - v}{L}$$

buck–boost converter

$$m_1 = \frac{v_g}{L} \quad -m_2 = \frac{v}{L}$$
Steady-state inductor current waveform, CPM

First interval:

\[ i_{L}(dT_s) = i_c = i_{L}(0) + m_1 dT_s \]

Solve for \( d \):

\[ d = \frac{i_c - i_{L}(0)}{m_1 T_s} \]

Second interval:

\[ i_{L}(T_s) = i_{L}(dT_s) - m_2 dT_s \]
\[ = i_{L}(0) + m_1 dT_s - m_2 dT_s \]

in steady state:

\[ 0 = M_1 DT_s - M_2 D'T_s \]

\[ \frac{M_2}{M_1} = \frac{D}{D'} \]
Inductor current in transient

\[ i_L(t) \]

\[ m_1(t) \]

\[ i_c[n] \]

\[ -m_2(t) \]

\[ i_c[n+1] \]

\[ d[n]T_s \]

\[ d'[n]T_s \]

\[ d[n+1]T_s \]

\[ d'[n+1]T_s \]
High-frequency small-signal inductor-current dynamics

• Assume that voltage perturbations are negligibly small at high frequencies: the slopes $m_1$ and $m_2$ can be considered constant

• Apply sampled-data modeling:

$$
\hat{i}_c(t) \rightarrow \hat{i}_c^{\ast} \rightarrow \hat{i}_c[n] \rightarrow \hat{i}_L[n] \rightarrow \hat{i}_L(t)
$$

$$
\hat{i}_c(s) \rightarrow \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \hat{i}_c(s - jk\omega_s) \rightarrow \hat{i}_c(z) \rightarrow \hat{i}_L(z) \rightarrow \hat{i}_L(s)
$$
Small-signal perturbation

\[ \hat{i}_c[n] - \hat{i}_c'[n] + i_{L}[n] d[n]T_s - m_2(t) \]
Discrete-time dynamics \( \hat{i}_c[n] \rightarrow \hat{i}_L[n] \)

\[
\hat{i}_L[n] = \hat{i}_L[n-1] + (m_1 + m_2) \hat{d}[n]T_s
\]

\[
\hat{i}_c[n] = \hat{i}_L[n-1] + m_1 \hat{d}[n]T_s
\]
Discrete-time dynamics \( \hat{i}_c[n] \rightarrow \hat{i}_L[n] \)

\[
\hat{i}_L[n] = \hat{i}_L[n-1] + (m_1 + m_2) \hat{d}[n] T_s
\]

\[
\hat{i}_c[n] = \hat{i}_L[n-1] + m_1 \hat{d}[n] T_s
\]

\[
\hat{i}_L[n] = -\frac{m_2}{m_1} \hat{i}_L[n-1] + \frac{m_1 + m_2}{m_1} \hat{i}_c[n] = \alpha \hat{i}_L[n-1] + (1-\alpha)\hat{i}_c[n]
\]

\[
\alpha = -\frac{m_2}{m_1} = -\frac{M_2}{M_1} = -\frac{D}{D'}
\]
Instability for $D > 0.5$

\[ \hat{i}_L[n] = \alpha \hat{i}_L[n-1] + (1-\alpha)\hat{i}_c[n] \]

\[ \alpha = -\frac{m_2}{m_1} = -\frac{M_2}{M_1} = -\frac{D}{D'} \]
Change in inductor current perturbation over many switching periods

\[
\dot{i}_L(T_s) = \dot{i}_L(0) \left(- \frac{D}{D'} \right)
\]

\[
\dot{i}_L(2T_s) = \dot{i}_L(T_s) \left(- \frac{D}{D'} \right) = \dot{i}_L(0) \left(- \frac{D}{D'} \right)^2
\]

\[
\dot{i}_L(nT_s) = \dot{i}_L((n-1)T_s) \left(- \frac{D}{D'} \right) = \dot{i}_L(0) \left(- \frac{D}{D'} \right)^n
\]

\[
\left| \dot{i}_L(nT_s) \right| \rightarrow \begin{cases} 
0 & \text{when } \left| - \frac{D}{D'} \right| < 1 \\
\infty & \text{when } \left| - \frac{D}{D'} \right| > 1
\end{cases}
\]

For stability: \( D < 0.5 \)
Stabilization via addition of an artificial ramp to the measured switch current waveform

Now, transistor switches off when

\[ i_a(dT_s) + i_L(dT_s) = i_c \]

or,

\[ i_L(dT_s) = i_c - i_a(dT_s) \]
Steady state waveforms with artificial ramp

\[ i_L(dT_s) = i_c - i_a(dT_s) \]
small-signal perturbation with compensation ramp

\[ i_c(t) + \hat{i}_c[n] \]

\[ -m_a(t) \]

\[ \hat{i}_c[n] \]

\[ \hat{i}_L[n] \]

\[ m_1(t) \]

\[ \hat{i}_L[n-1] \]

\[ d[n]T_s \]

\[ -m_2(t) \]
Discrete-time dynamics with compensation ramp:

\[ \hat{i}_c[n] \rightarrow \hat{i}_L[n] \]

\[ \hat{i}_L[n] = \hat{i}_L[n-1] + (m_1 + m_2)\hat{d}[n]T_s \]

\[ \hat{i}_c[n] = \hat{i}_L[n-1] + (m_1 + m_a)\hat{d}[n]T_s \]
Discrete-time dynamics with compensation ramp: \( \hat{i}_c[n] \rightarrow \hat{i}_L[n] \)

\[
\hat{i}_L[n] = \hat{i}_L[n-1] + (m_1 + m_2)\hat{d}[n]T_s
\]

\[
\hat{i}_c[n] = \hat{i}_L[n-1] + (m_1 + m_a)\hat{d}[n]T_s
\]

\[
\hat{i}_L[n] = -\frac{m_2 - m_a}{m_1 + m_a} \hat{i}_L[n-1] + \frac{m_1 + m_2}{m_1 + m_a} \hat{i}_c[n] = \alpha \hat{i}_L[n-1] + (1-\alpha)\hat{i}_c[n]
\]

\[
\alpha = -\frac{m_2 - m_a}{m_1 + m_a} = \frac{1 - \frac{m_a}{m_2}}{D' + \frac{m_a}{m_2}}
\]
\[ \hat{i}_L[n] = \alpha \hat{i}_L[n-1] + (1-\alpha)\hat{i}_c[n] \]

The characteristic value \( \alpha \)

\[ \alpha = -\frac{1 - \frac{m_a}{m_2}}{\frac{D'}{D} + \frac{m_a}{m_2}} \]

- For stability, require \( |\alpha| < 1 \)
- Buck and buck-boost converters: \( m_2 = -\frac{v}{L} \)
  So if \( v \) is well-regulated, then \( m_2 \) is also well-regulated
- A common choice: \( m_a = 0.5 m_2 \)
  This leads to \( \alpha = -1 \) at \( D = 1 \), and \( |\alpha| < 1 \) for \( 0 \leq D < 1 \).
  The minimum \( \alpha \) that leads to stability for all \( D \).
- Another common choice: \( m_a = m_2 \)
  This leads to \( \alpha = 0 \) for \( 0 \leq D < 1 \).
  Deadbeat control, finite settling time
Sensitivity to noise

With small ripple: a small amount of noise in the control current $\hat{i}_c$ leads to a large perturbation in the duty cycle.
Artificial ramp reduces sensitivity to noise

The same amount of noise in the control current $i_c$ leads to a smaller perturbation in the duty cycle, because the gain has been reduced.

Fundamentals of Power Electronics

Chapter 12: Current Programmed Control
Discrete-time dynamics: \( \hat{i}_c(z) \rightarrow \hat{i}_L(z) \)

Difference equation: \( \hat{i}_L[n] = \alpha \hat{i}_L[n-1] + (1-\alpha)\hat{i}_c[n] \)

Z-transform: \( \hat{i}_L(z) = \alpha \hat{i}_L(z)z^{-1} + (1-\alpha)\hat{i}_c(z) \)

Discrete-time (z-domain) control-to-inductor current transfer function:

\[
\frac{\hat{i}_L(z)}{\hat{i}_c(z)} = \frac{1-\alpha}{1-\alpha z^{-1}}
\]

• Pole at \( z = \alpha \)
• Stability condition: pole inside the unit circle, \( |\alpha| < 1 \)
• Frequency response (note that \( z^{-1} \) corresponds to a delay of \( T_s \) in time domain):

\[
\frac{1-\alpha}{1-\alpha e^{-sT_s}} \rightarrow \frac{1-\alpha}{1-\alpha e^{-j\omega T_s}}
\]
Equivalent hold: $\hat{i}_L[n] \rightarrow \hat{i}_L(t)$, $\hat{i}_L(z) \rightarrow \hat{i}_L(s)$
Equivalent hold

- The response from the samples $i_L[n]$ of the inductor current to the inductor current perturbation $i_L(t)$ is a pulse of amplitude $i_L[n]$ and length $T_s$
- Hence, in frequency domain, the equivalent hold has the transfer function previously derived for the zero-order hold:

$$1 - e^{-sT_s} \over s$$
Complete sampled-data “transfer function”

Control-to-inductor current small-signal response:

\[
\frac{\hat{i}_L(s)}{\hat{i}_c(s)} = \frac{1 - \alpha}{1 - \alpha e^{-sT_s}} \frac{1 - e^{-sT_s}}{sT_s}
\]

\[
\alpha = -\frac{m_2 - m_a}{m_1 + m_a} = -\frac{1 - \frac{m_a}{m_2}}{\frac{D'}{D} + \frac{m_a}{m_2}}
\]
Example

- CPM buck converter:
  
  \[ V_g = 10\text{V}, \ L = 5 \ \mu\text{H}, \ C = 75 \ \mu\text{F}, \ D = 0.5, \ V = 5 \ \text{V}, \]
  
  \[ I = 20 \ \text{A}, \ R = V/I = 0.25 \ \Omega, \ f_s = 100 \ \text{kHz} \]

- Inductor current slopes:

  \[ m_1 = (V_g - V)/L = 1 \ \text{A}/\mu\text{s} \]
  
  \[ m_2 = V/L = 1 \ \text{A}/\mu\text{s} \]

\[
\frac{\hat{i}_L(s)}{\hat{i}_c(s)} = \frac{1 - \alpha}{1 - \alpha e^{-sT_s}} \frac{1 - e^{-sT_s}}{sT_s} \]

\[ \alpha = -\frac{m_2 - m_a}{m_1 + m_a} = -\frac{1 - \frac{m_a}{m_2}}{\frac{D'}{D} + \frac{m_a}{m_2}} = \frac{1 - \frac{m_a}{m_2}}{1 + \frac{m_a}{m_2}} \]

\[ D = 0.5: \text{CPM controller is stable for any compensation ramp, } m_a/m_2 > 0 \]
Control-to-inductor current responses for several compensation ramps ($m_d/m_2$ is a parameter)

MATLAB file: CPMfr.m
First-order approximation

\[ e^{-sT_s} \approx \frac{1 - \frac{s}{(\omega_s / \pi)}}{1 + \frac{s}{(\omega_s / \pi)}} \]

\[ \frac{\hat{i}_L(s)}{\hat{i}_c(s)} = \frac{1 - \alpha}{1 - \alpha e^{-sT_s}} \frac{1 - e^{-sT_s}}{sT_s} \approx \frac{1}{1 + \frac{1 + \alpha}{1 - \alpha (\omega_s / \pi)} \frac{s}{\omega_{hf}}} = \frac{1}{1 + \frac{s}{\omega_{hf}}} \]

Control-to-inductor current response behaves approximately as a single-pole transfer function with a high-frequency pole at

\[ f_{hf} = \frac{1 - \alpha}{1 + \alpha} \frac{f_s}{\pi} = \frac{1}{1 - 2D + 2D \frac{m_a}{m_2}} \frac{f_s}{\pi} \]
Control-to-inductor current responses for several compensation ramps ($m_d/m_2 = 0.1, 0.5, 1, 5$)
Second-order approximation

\[
e^{-sT_s} \approx \frac{1 - \frac{\pi}{2} \frac{s}{(\omega_s / 2)} + \left( \frac{s}{\omega_s / 2} \right)^2}{1 + \frac{\pi}{2} \frac{s}{(\omega_s / 2)} + \left( \frac{s}{\omega_s / 2} \right)^2}
\]

\[
\frac{\hat{i}_{L}(s)}{\hat{i}_{c}(s)} = \frac{1 - \alpha}{1 - \alpha e^{-sT_s}} \frac{1 - e^{-sT_s}}{sT_s} \approx \frac{1}{1 + \frac{\pi}{2} \frac{(1 + \alpha)}{(1 - \alpha) (\omega_s / 2)} \frac{s}{\omega_s / 2} + \left( \frac{s}{\omega_s / 2} \right)^2}
\]

Control-to-inductor current response behaves approximately as a second-order transfer function with corner frequency \(f_s/2\) and Q-factor given by

\[
Q = \frac{2}{\pi} \frac{1 - \alpha}{1 + \alpha} = \frac{2}{\pi} \frac{1}{1 - 2D + 2D \frac{m_a}{m_2}}
\]
Control-to-inductor current responses for several compensation ramps ($m_d/m_2 = 0.1, 0.5, 1, 5$)

2$^{nd}$-order transfer-function approximation
Conclusions

• In CPM converters, high-frequency inductor dynamics depend strongly on the compensation (“artificial”) ramp slope $m_a$
• Without compensation ramp ($m_a = 0$), CPM controller is unstable for $D > 0.5$, resulting in period-doubling or other sub-harmonic (or even chaotic) oscillations
• For $m_a = 0.5m_2$, CPM controller is stable for all $D$
• Relatively large compensation ramp ($m_a > 0.5m_2$) is a practical choice not just to ensure stability of the CPM controller, but also to reduce sensitivity to noise
• For relatively large values of $m_a$, high-frequency inductor current dynamics can be well approximated by a single high-frequency pole
• Second-order approximation is very accurate for any $m_a$
• Next: more accurate averaged model, including high-frequency dynamics