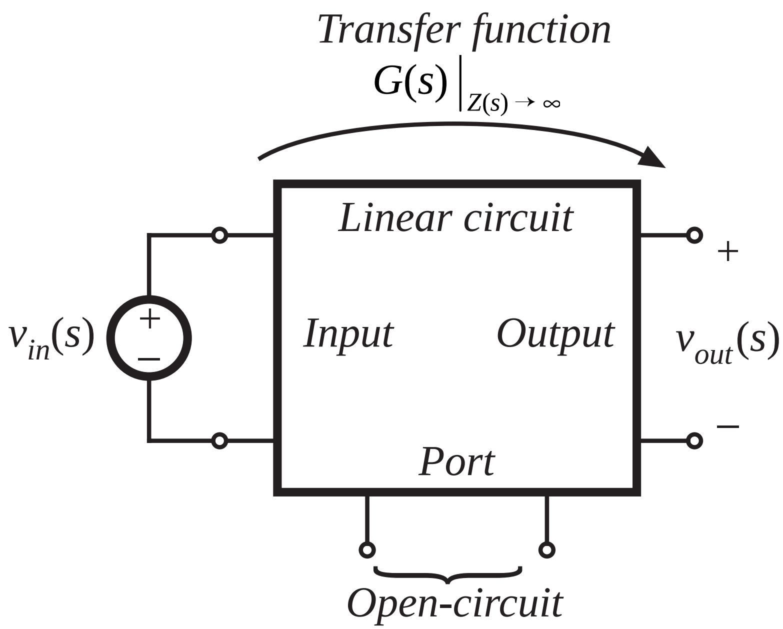
Appendix C
Middlebrook's Extra Element Theorem

C.1 Basic Result

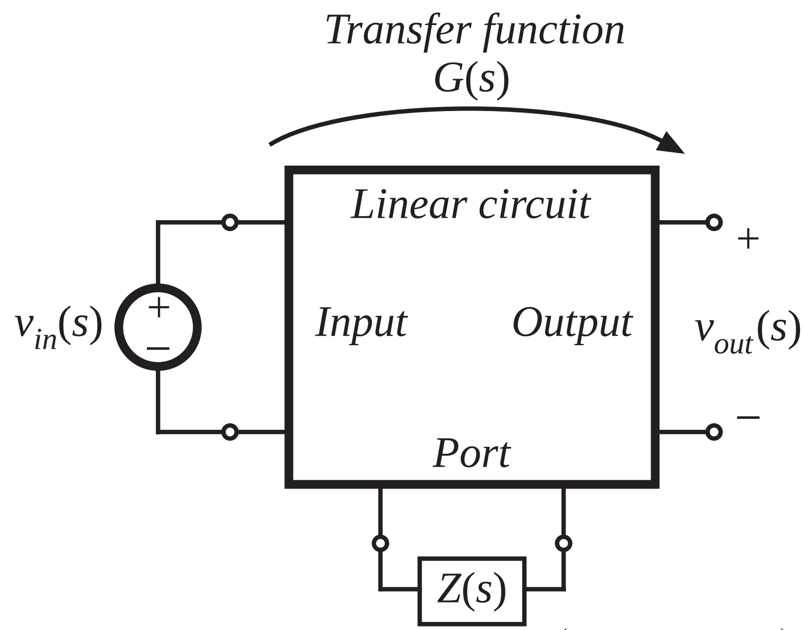
Object: find how addition of an element changes a transfer function $G(s)$

Original conditions:



$$\frac{v_{out}(s)}{v_{in}(s)} = G(s)\Big|_{Z(s)\to\infty}$$

Addition of element $Z(s)$:

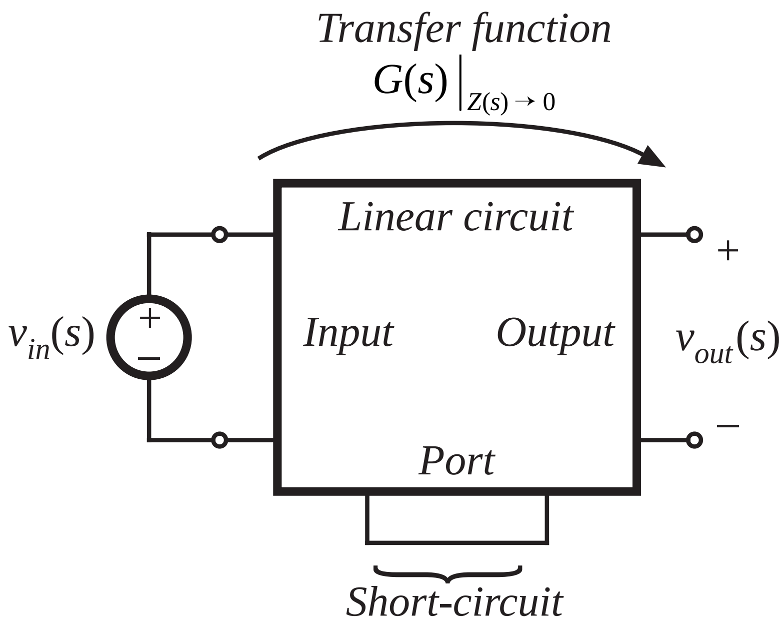


$$\frac{v_{out}(s)}{v_{in}(s)} = \left(G(s)\Big|_{Z(s)\to\infty} \right) \left(\frac{1 + \frac{Z_N(s)}{Z(s)}}{1 + \frac{Z_D(s)}{Z(s)}} \right)$$

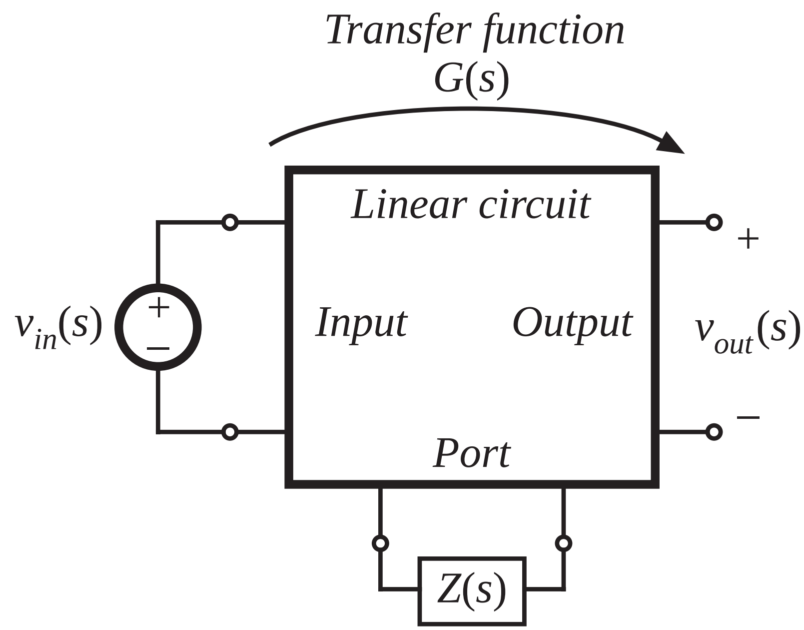
Dual Form of Basic Result

When the added impedance replaces a short circuit:

Original conditions:



Addition of element Z(s):



$$\frac{v_{out}(s)}{v_{in}(s)} = \left(G(s) \Big|_{Z(s) \to 0} \right) \left(\frac{1 + \frac{Z(s)}{Z_N(s)}}{1 + \frac{Z(s)}{Z_D(s)}} \right)$$

Comparison of forms

The two forms of the extra element theorem:

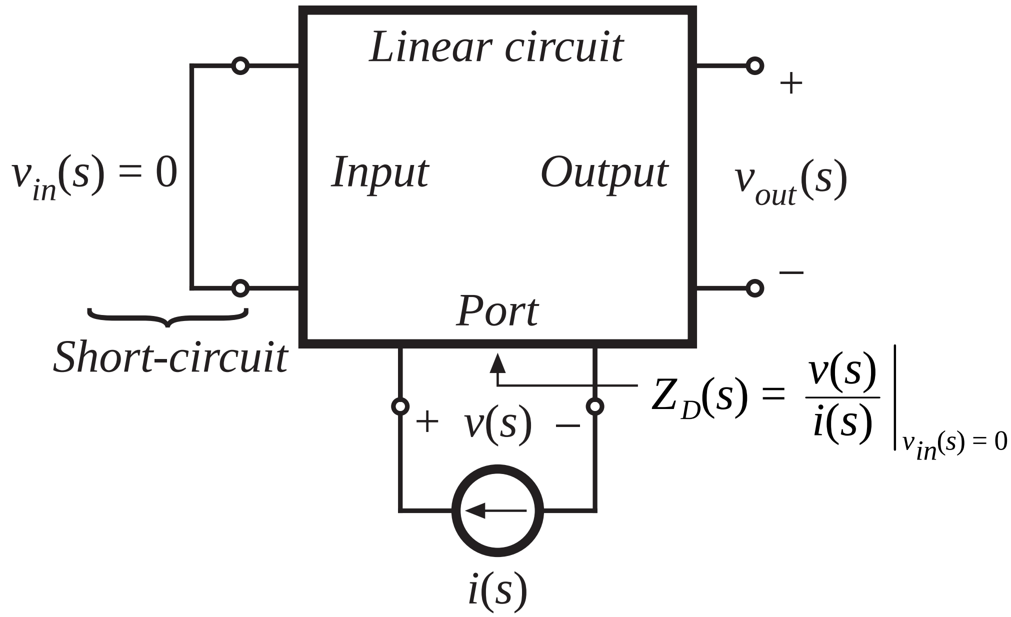
$$\frac{v_{out}(s)}{v_{in}(s)} = \left(G(s) \Big|_{Z(s) \to \infty} \right) \left(\frac{1 + \frac{Z_N(s)}{Z(s)}}{1 + \frac{Z_D(s)}{Z(s)}} \right) \qquad \frac{v_{out}(s)}{v_{in}(s)} = \left(G(s) \Big|_{Z(s) \to 0} \right) \left(\frac{1 + \frac{Z(s)}{Z_N(s)}}{1 + \frac{Z(s)}{Z_D(s)}} \right)$$

These equations describe the same transfer function, referenced to the limiting cases of $Z = 0$ and $Z = \infty$. Upon equating them, one obtains the *reciprocity relationship*:

$$\frac{G(s) \Big|_{Z(s) \to \infty}}{G(s) \Big|_{Z(s) \to 0}} = \frac{Z_D(s)}{Z_N(s)}$$

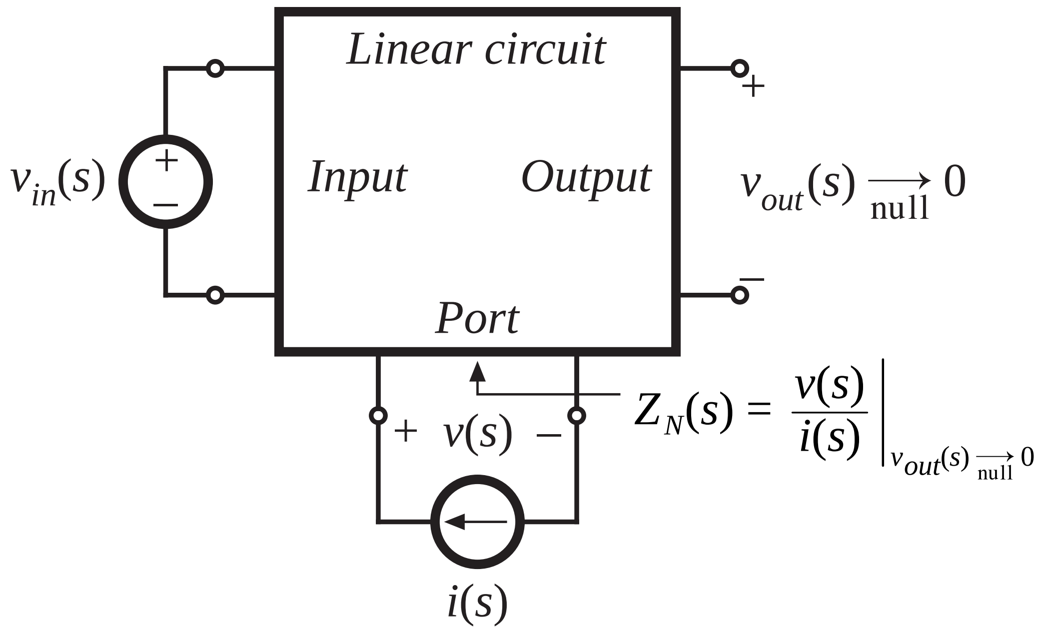
The quantities Z_N and Z_D are the same in both forms.

Finding Z_D



Z_D is the driving-point impedance (i.e., the Thevenin-equivalent impedance) at the port where the new element is connected. Formally, it is found by setting independent sources to zero, and injecting a current $i(s)$ at the port. $Z_D(s)$ is the ratio of $v(s)$ to $i(s)$.

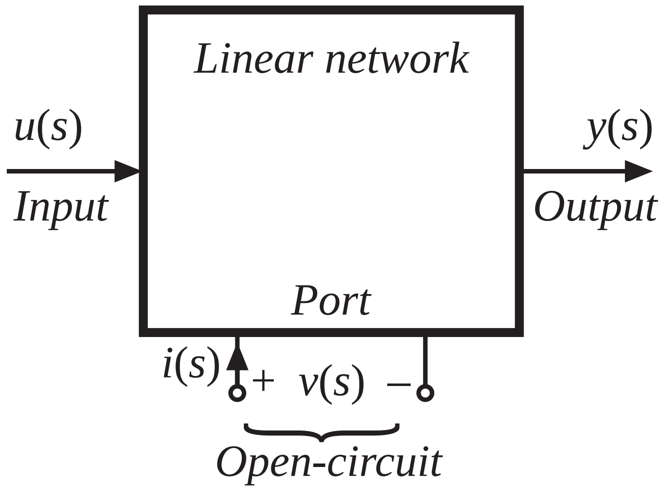
Finding Z_N



Z_N is the impedance seen at the port when the output is *nulled*. In the presence of the input $v_{in}(s)$, a current $i(s)$ is injected at the port. This current is adjusted such that the output $v_{out}(s)$ is nulled to zero. Under these conditions, $Z_N(s)$ is the ratio of $v(s)$ to $i(s)$. **Note:** nulling is not the same as shorting.

C.2 Derivation

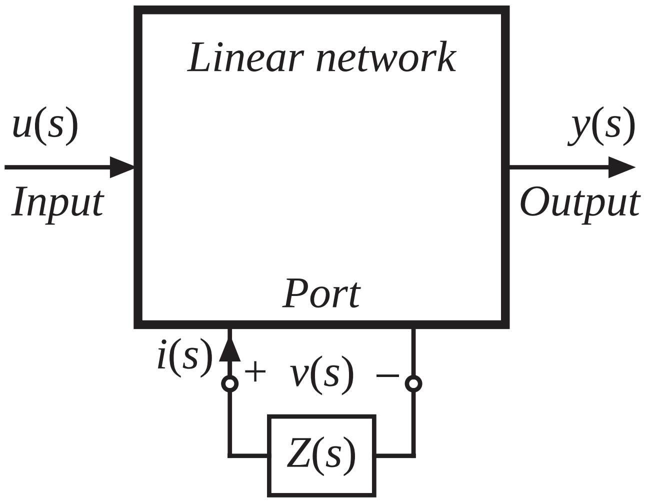
Original system:



$$G_{old}(s) = \left. \frac{y(s)}{u(s)} \right|_{i(s)=0}$$

[The input and output need not be voltages, and are denoted here by the general names *u*(*s*) and *y*(*s*)]

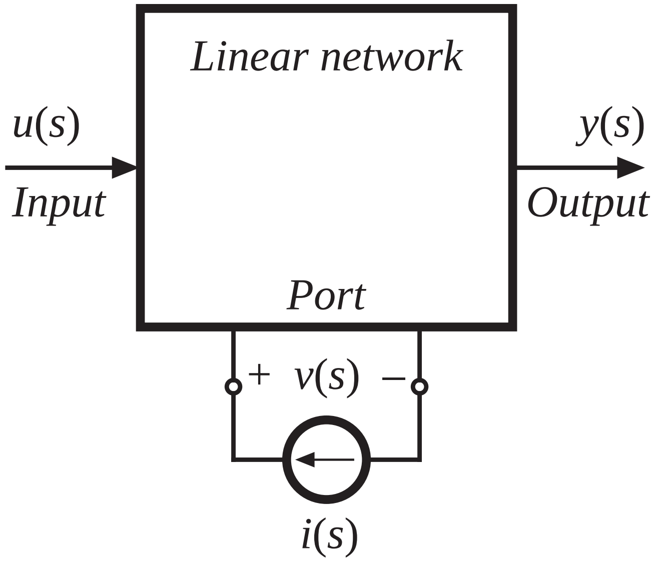
With extra element:



$$G(s) = \frac{y(s)}{u(s)}$$

$$v(s) = -\,i(s)Z(s)$$

Current injection at the port



There are now two independent inputs:

$u(s)$ and $i(s)$

The dependent quantities $y(s)$ and $v(s)$ can be expressed as functions of the independent inputs using superposition:

$$y(s) = G_{old}(s)u(s) + G_i(s)i(s)$$

$$v(s) = G_v(s)u(s) + Z_D(s)i(s)$$

with:

$$G_{old}(s) = \left.\frac{y(s)}{u(s)}\right|_{i(s)=0} \qquad Z_D(s) = \left.\frac{v(s)}{i(s)}\right|_{u(s)=0}$$

$$G_i(s) = \left.\frac{y(s)}{i(s)}\right|_{u(s)=0} \qquad G_v(s) = \left.\frac{v(s)}{u(s)}\right|_{i(s)=0}$$

Solution for $G(s)$

Now eliminate $v(s)$ and $i(s)$ from equations of previous slide, and solve from transfer function $G(s)$:

$$G(s) = \frac{y(s)}{u(s)} = G_{old}(s) - \frac{G_v(s)G_i(s)}{Z(s) + Z_D(s)}$$

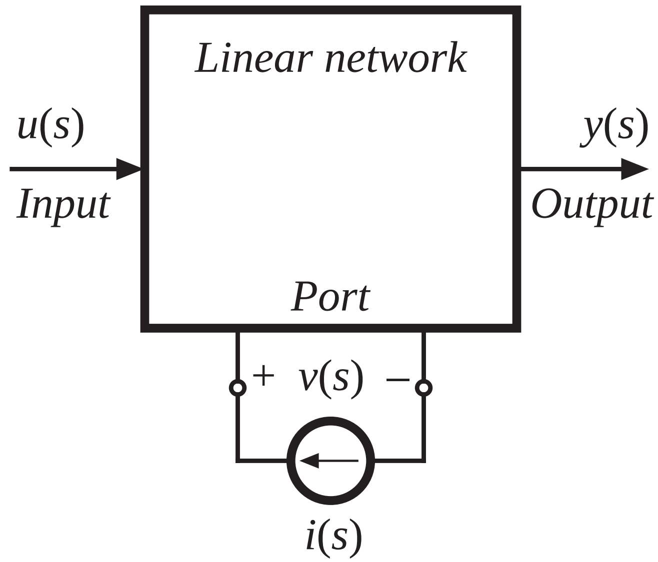
$G_{old}(s)$ and $Z_D(s)$ are found using definitions on previous slide.

We could stop at this point, and use the above equation to evaluate $G(s)$. The quantities $G_i(s)$ and $G_v(s)$ would be evaluated using the definitions on the previous slide. However, it is preferable to eliminate $G_i(s)$ and $G_v(s)$, and instead express $G(s)$ in terms of impedances measured at the given port. This can be accomplished with an alternate thought experiment involving null double injection.

Null double injection

In the presence of the input $u(s)$, inject current $i(s)$ at the port. Adjust $i(s)$ in the special way that causes the output $y(s)$ to be nulled to zero. Under these conditions, the impedance $Z_N(s)$ is defined as:

$$Z_N(s) = \left. \frac{v(s)}{i(s)} \right|_{y(s) \underset{\text{null}}{\longrightarrow} 0}$$



Nulling: Note that

$$y(s) = G_{old}(s)u(s) + G_i(s)i(s)$$

Therefore, the value of $i(s)$ that achieves the null condition $y(s) \underset{\text{null}}{\longrightarrow} 0$ is given by

$$\left[G_{old}(s)u(s) + G_i(s)i(s) \right] \underset{\text{null}}{\longrightarrow} 0$$

So the output is nulled when $i(s)$ is chosen to satisfy

$$u(s) \Big|_{y(s) \underset{\text{null}}{\longrightarrow} 0} = - \frac{G_i(s)}{G_{old}(s)} \; i(s) \Big|_{y(s) \underset{\text{null}}{\longrightarrow} 0}$$

Expression for $Z_N(s)$

Now substitute result from previous slide,

$$u(s)\Big|_{y(s)\underset{\text{null}}{\longrightarrow}0} = -\frac{G_i(s)}{G_{old}(s)}\; i(s)\Big|_{y(s)\underset{\text{null}}{\longrightarrow}0}$$

into previous expression for output voltage

$$v(s) = G_v(s)u(s) + Z_D(s)i(s)$$

The result is:

$$\begin{aligned} v(s)\Big|_{y(s)\underset{\text{null}}{\longrightarrow}0} &= G_v(s)\,u(s)\Big|_{y(s)\underset{\text{null}}{\longrightarrow}0} + Z_D(s)\,i(s)\Big|_{y(s)\underset{\text{null}}{\longrightarrow}0} \\ &= \left(-\frac{G_v(s)G_i(s)}{G_{old}(s)} + Z_D(s)\right) i(s)\Big|_{y(s)\underset{\text{null}}{\longrightarrow}0} \end{aligned}$$

Use definition of $Z_N(s)$:

$$v(s)\Big|_{y(s)\underset{\text{null}}{\longrightarrow}0} = Z_N(s)\,i(s)\Big|_{y(s)\underset{\text{null}}{\longrightarrow}0} = \left(-\frac{G_v(s)G_i(s)}{G_{old}(s)} + Z_D(s)\right) i(s)\Big|_{y(s)\underset{\text{null}}{\longrightarrow}0}$$

Hence:

$$Z_N(s) = Z_D(s) - \frac{G_v(s)G_i(s)}{G_{old}(s)}$$

Expression for $G(s)$

Now, eliminate $G_i(s)$ and $G_v(s)$ from expression for $G(s)$, using Z_N result:

$$G(s) = G_{old}(s) - \frac{Z_D(s) - Z_N(s)}{Z(s) + Z_D(s)} G_{old}(s)$$

Simplify:

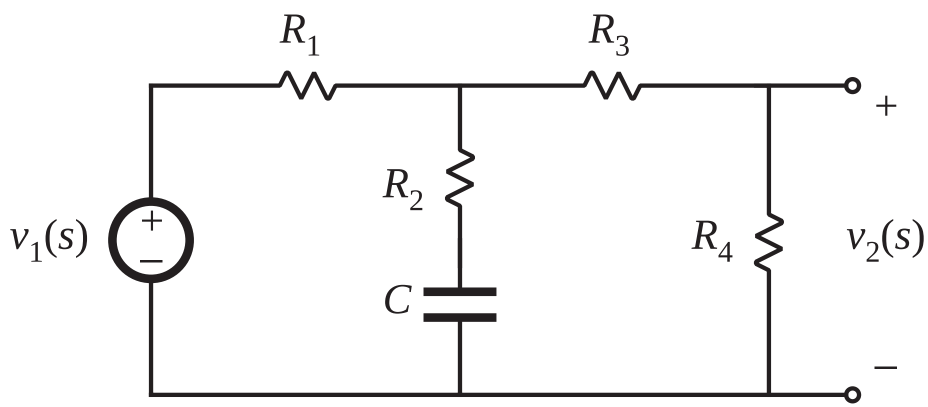
$$G(s) = G_{old}(s) \frac{1 + \frac{Z_N(s)}{Z(s)}}{1 + \frac{Z_D(s)}{Z(s)}}$$

Or,

$$G(s) = \left(G(s) \Big|_{Z(s) \to \infty} \right) \left(\frac{1 + \frac{Z_N(s)}{Z(s)}}{1 + \frac{Z_D(s)}{Z(s)}} \right)$$

(Desired result)

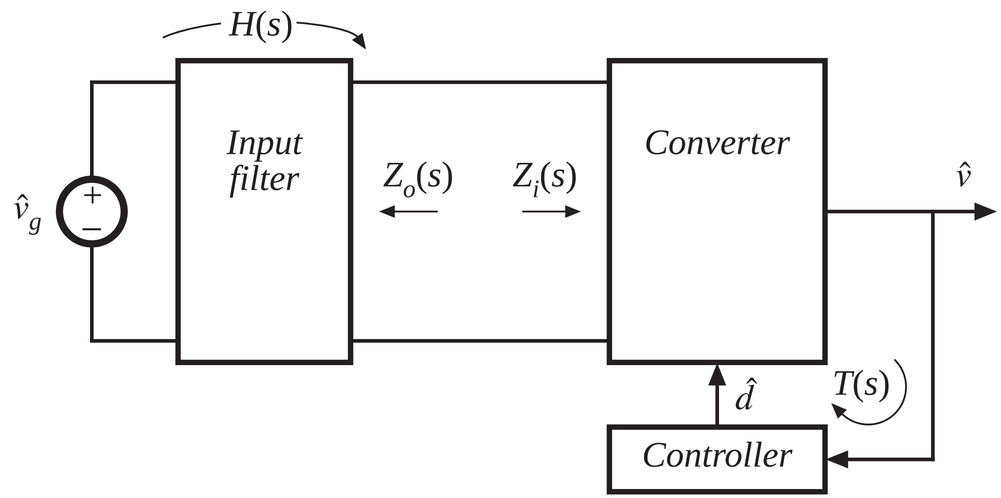
Example:
A simple transfer function



- Find $G(s) = \frac{v_2(s)}{v_1(s)}$
- Express result in factored pole-zero form.

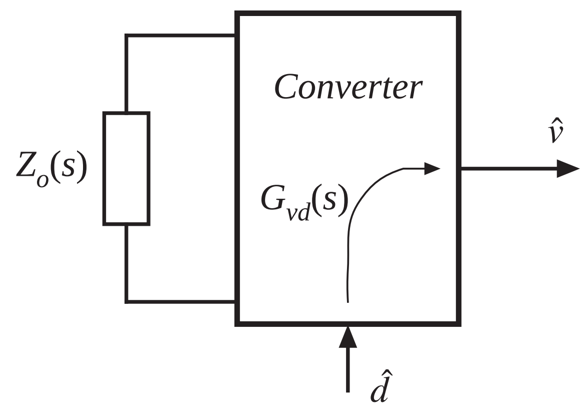
C.4.3 Addition of an Input Filter to a Converter

Addition of an input filter changes the small-signal transfer functions of a converter



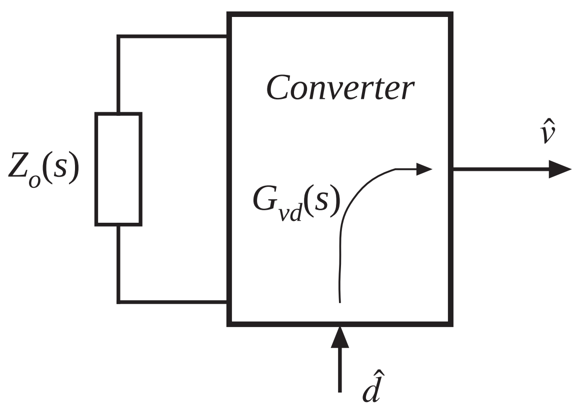
Control-to-output transfer function $G_{vd}(s)$:

$$G_{vd}(s) = \left.\frac{\hat{v}(s)}{\hat{d}(s)}\right|_{\hat{v}_g(s) = 0}$$



Set $\hat{v}_g = 0$. Input filter effectively becomes an impedance $Z_o(s)$, added to the converter power input port.

Application of Extra Element Theorem



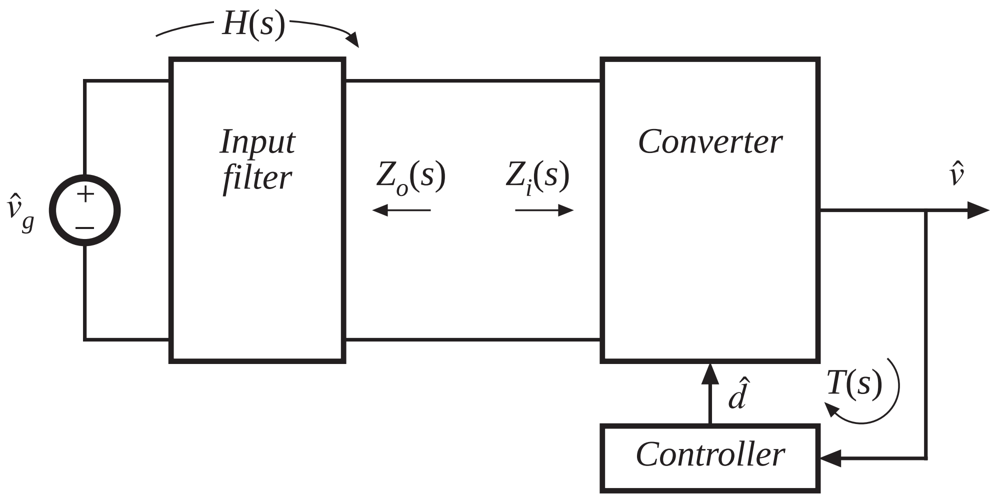
With no input filter, the following "original" transfer function is obtained:

$$G_{vd}(s)\Big|_{Z_o(s)=0}$$

In the presence of the input filter, the control-to-output transfer function can be expressed as:

$$G_{vd}(s) = \left(G_{vd}(s)\Big|_{Z_o(s)=0} \right) \frac{\left(1 + \dfrac{Z_o(s)}{Z_N(s)}\right)}{\left(1 + \dfrac{Z_o(s)}{Z_D(s)}\right)}$$

Z_N and Z_D



$$Z_D(s) = Z_i(s)\Big|_{\hat{d}(s)=0}$$

$$Z_N(s) = Z_i(s)\Big|_{\hat{v}(s) \xrightarrow[\text{null}]{} 0}$$

The input filter does not significantly change the control-to-output transfer function when

$$\| Z_o \| \ll \| Z_N \|, \quad \text{and}$$
$$\| Z_o \| \ll \| Z_D \|$$

Results for basic converters are listed in Table 10.1