Solution

Homework Assignment 4
Resonant and Soft Switching Techniques in Power Electronics
ECEN 5817
Spring 2006

Dynamic Analysis of a Parallel Resonant Inverter

A certain parallel resonant inverter circuit includes a full bridge switch network as illustrated in Fig. 19.1 of the text, with the parallel tank network of Fig. 19.1(c). The element values are: \( L = 627 \, \mu\text{H} \), \( C = 7.9 \, \text{nF} \), \( R = 400 \, \Omega \), and \( V_c = 150 \, \text{V} \).

The controller includes a frequency modulator circuit whose control input is voltage \( v_{in}(t) \). This frequency modulator generates the gate drive signals for the switch network; its switching frequency \( f_s \) depends on \( v_{in}(t) \) according to the formula \( f_s = K_m v_{in} \), with \( K_m = 50 \, \text{kHz per volt} \). Variations in \( v_{in}(t) \) therefore cause the switching frequency to vary proportionally, leading to variations in the amplitude (envelope) of the ac output voltage \( v(t) \).

The objective of this assignment is to work out analytical expressions for the control-to-output small-signal transfer function \( G_{cm}(s) \), i.e., the transfer function from perturbations in the control voltage \( v_{in}(t) \) to variations in the envelope of the ac output voltage waveform \( v(t) \). You should accomplish this by application of the methods developed in lectures 12 to 14.

1. \textit{Quiescent operating point analysis.} At what switching frequency does the converter operate with an output power of 50 W? For this quiescent switching frequency, determine the quiescent value of the peak ac output voltage. Sketch the elliptical output characteristic and label the values of the open-circuit voltage, short-circuit current, and tank output impedance.

2. \textit{Tank transfer function.} Determine the numerical values (real and imaginary parts) of the poles and any zeroes of the tank transfer function \( H(s) \). Sketch the pole locations in the complex \( s \)-plane.

3. \textit{Control-to-output transfer function.} Apply the analysis derived in class to find an analytical expression for the control-to-output transfer function \( G_{cm}(s) \). \textit{Hint:} it is possible to derive relatively simple expressions for the zeroes and dc gain in this example. The result contains a dc gain, one zero, and four poles.

4. \textit{Root locus.} Determine the numerical values of the poles and zero of \( G_{cm}(s) \). Sketch the pole and zero locations in the complex \( s \)-plane. Which corner frequencies are lower than half of the switching frequency?

5. \textit{Frequency response.} Construct a Bode diagram of the magnitude and phase of \( G_{cm}(s) \). Label salient features.
1. Quiescent operating point analysis

The steady-state solution of the parallel resonant inverter is given by

\[ M = \frac{V}{V_g} = \frac{4}{\pi} \left| \left( |H|\right) = \frac{4}{\pi} \frac{1}{\sqrt{(1-F^2)^2 + \left(\frac{F}{Q}\right)^2}} \right. \]

with \( \frac{f_g}{f_o} = F \), \( f_o = \frac{1}{2\pi \sqrt{L C}} \), \( R_o = \sqrt{\frac{L}{C}} \), \( Q = \frac{R}{R_o} \)

The converter drives the 400Ω load resistor at 50W when the peak voltage \( V \) is

\[ \frac{(V/\sqrt{2})^2}{R} = P \Rightarrow V = \sqrt{2RP} = 200V \]

Hence \( M = \frac{V}{V_g} = \frac{200}{150} = \frac{4}{3} \)

Also, \( R_o = \sqrt{\frac{L}{C}} = 281.7 \Omega \) and \( Q = \frac{R}{R_o} = 1.42 \)

Now solve the above equation \( \theta \) for \( F \). The result is

\[ F = 1 - \frac{1}{2Q^2} \pm \sqrt{\left(1 - \frac{1}{2Q^2}\right)^2 - 1 + \left(\frac{4}{\pi M}\right)^2} \]

The plus sign gives the solution above resonance:

\[ F = 1.20 \quad \text{with} \quad f_o = \frac{1}{2\pi \sqrt{L C}} = 71.5kHz \]

this leads to \( f_s = F f_o = 86kHz \).
The open-circuit output voltage is given by

\[ V_{oc} = \frac{4}{\pi} V_d \cdot \| H_{oo}(j\omega_0) \| = 431 \text{ V} \]

with \[ \| H_{oo}(j\omega_0) \| = \frac{1}{\| 1 - F \|} \]

The short-circuit output current is given by

\[ I_{sc} = \frac{V_{oc}}{\| 2 \omega_0 (j\omega_0) \|} = 0.56 \text{A} \]

with \[ \| 2 \omega_0 (j\omega_0) \| = \frac{\omega_0 L}{\| 1 - F \|} = 764 \Omega \]
2. Tank transfer function

\[ H(s) = \frac{1}{1 + \frac{sL}{R} + s^2LC} = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \]

with \(\omega_0 = \frac{1}{\sqrt{LC}}\) \(\Rightarrow\) \(Q = R\sqrt{\frac{L}{C}}\)

The denominator polynomial has roots at

\[ s = -\frac{\omega_0}{2Q} \pm j\frac{\omega_0}{2Q} \sqrt{4Q^2 - 1} \quad \text{for} \quad Q > \frac{1}{2} \]

\[ = -\omega_r \pm j\omega_i \quad \text{with} \quad \omega_r = \frac{\omega_0}{2Q}, \quad \omega_i = \frac{\omega_0}{2Q} \sqrt{4Q^2 - 1} \]

For the given component values, we obtain

\(\omega_r = 158 \text{ krad/sec} \Rightarrow 25.2 \text{ kHz}\)

\(\omega_i = 421 \text{ krad/sec} \Rightarrow 67 \text{ kHz}\)

Compare with switching frequency

\(\omega_s = 2\pi(86 \text{ kHz}) = 540 \text{ krad/sec}\)
3. Control-to-output transfer function

From the course notes Eq. (38), we have

\[
G_{ou}(s) = \frac{V_{s1}}{2\omega_m \|H(j\omega_0)\|} \frac{H(-j\omega_0)N(s+j\omega_0)D(s-j\omega_0) - H(j\omega_0)N(s-j\omega_0)D(s+j\omega_0)}{D(s+j\omega_0)D(s-j\omega_0)}
\]

\(H(s)\) is given at the top of the previous page, and

\[
H(s) = \frac{N(s)}{D(s)}
\]

so \(N(s) = 1\) and \(D(s) = 1 + \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2\)

the denominator \(D(s+j\omega_0)D(s-j\omega_0)\)

On the previous page we found the roots of \(D(s)\) as \(s = -\omega_r \pm j \omega_i\). Hence, \(D(s)\) can be written as

\[
D(s) = \left(1 + \frac{s}{\omega_r + j \omega_i}\right)\left(1 + \frac{s}{\omega_r - j \omega_i}\right)
\]

and

\[
D(s+j\omega_0) = \left(1 + \frac{s+j\omega_0}{\omega_r + j \omega_i}\right)\left(1 + \frac{s+j\omega_0}{\omega_r - j \omega_i}\right)
\]

\[
\text{after a little more algebra, we get}
\]

\[
D(s+j\omega_0) = \frac{[\omega_r + j(\omega_i + \omega_0)][\omega_r + j(\omega_i - \omega_0)]}{(\omega_r^2 + \omega_i^2)}\left(1 + \frac{s}{\omega_r + j(\omega_i + \omega_0)}\right)\left(1 + \frac{s}{\omega_r + j(\omega_i - \omega_0)}\right)
\]
in a similar manner,

\[ D(s-j\omega_o) = \frac{[\omega_r + j(\omega_i - \omega_o)] \omega_r + j(-\omega_o - \omega_i)}{(\omega_r^2 + \omega_i^2)} \left(1 + \frac{s}{\omega_r + j(\omega_i - \omega_o)}\right) \left(1 + \frac{s}{\omega_r - j(-\omega_o - \omega_i)}\right) \]

The denominator is therefore

\[ D(s-j\omega_o) D(s+j\omega_o) = \frac{(\omega_r^2 + (\omega_i + \omega_o)^2)(\omega_r^2 + (\omega_o - \omega_i)^2)}{(\omega_r^2 + \omega_i^2)^2} \left(1 + \frac{s}{\omega_r + j(\omega_o - \omega_i)}\right) \left(1 + \frac{s}{\omega_r - j(\omega_o + \omega_i)}\right) \left(1 + \frac{s}{\omega_r + j(\omega_o + \omega_i)}\right) \]

we can further write the complex conjugate poles in the standard quadratic form:

\[ \left(1 + \frac{s}{\omega_r + j(\omega_o - \omega_i)}\right) \left(1 + \frac{s}{\omega_r - j(\omega_o - \omega_i)}\right) = 1 + \frac{2\omega_r s}{\omega_r^2 + (\omega_o - \omega_i)^2} + \frac{s^2}{\omega_r^2 + (\omega_o - \omega_i)^2} \]

\[ = 1 + \frac{s}{Q_r \omega_r} + \left(\frac{s}{\omega_r}\right)^2 \quad \text{with} \quad \omega_r = \sqrt{\omega_r^2 + (\omega_o - \omega_i)^2} \]

and \( Q_r = \frac{\omega_r}{2\omega_r} \)

likewise,

\[ \left(1 + \frac{s}{\omega_r + j(\omega_o + \omega_i)}\right) \left(1 + \frac{s}{\omega_r - j(\omega_o + \omega_i)}\right) = 1 + \frac{2\omega_r s}{\omega_r^2 + (\omega_o + \omega_i)^2} + \frac{s^2}{\omega_r^2 + (\omega_o + \omega_i)^2} \]

\[ = 1 + \frac{s}{Q_r \omega_r} + \left(\frac{s}{\omega_r}\right)^2 \quad \text{with} \quad \omega_r = \sqrt{\omega_r^2 + (\omega_o + \omega_i)^2} \]

and \( Q_r = \frac{\omega_r}{2\omega_r} \)
The denominator is therefore

\[
D(s+j\omega_0)D(s-j\omega_0) = \frac{w_0^2 w_0^2}{\omega_0^4} \left[ 1 + \frac{s}{Q \omega L} + \left(\frac{s}{\omega L}\right)^2 \right] \left[ 1 + \frac{s}{Q \omega L} + \left(\frac{s}{\omega L}\right)^2 \right]
\]

The "big numerator" or BN

\[
N(s) = 1 \text{ for this converter, so }
BN = H(-j\omega_0)D(s-j\omega_0) - H(j\omega_0)D(s+j\omega_0)
\]

Now, \( H(j\omega_0) = \frac{1}{D(j\omega_0)} = \frac{1}{\left(1+\frac{j\omega_0}{\omega_0+j\omega_i}\right)\left(1+\frac{j\omega_0}{\omega_0-j\omega_i}\right)} \) which can be simplified to

\[
H(j\omega_0) = \frac{\omega_0^2}{(\omega_0+j(\omega_0+\omega_i))(\omega_0+j(\omega_0-\omega_i))}
\]
In a similar manner,

\[ H(-j\omega_0) = \frac{\omega_0^2}{(\omega_r+j(\omega_0-w_i))(\omega_r+j(-\omega_0+w_i))} \]

and from page 6,

\[ D(s+j\omega_0) = \frac{(\omega_r+j(\omega_i+w_0))(\omega_r+j(\omega_0-w_i))}{\omega_0^2} \cdot \left(1 + \frac{s}{\omega_r+j(\omega_0+w_i)}\right) \]

\[ \cdot \left(1 + \frac{s}{\omega_r+j(\omega_0-w_i)}\right)
\]

\[ D(s-j\omega_0) = \frac{(\omega_r+j(\omega_i-w_0))(\omega_r+j(-\omega_i-w_0))}{\omega_0^2} \cdot \left(1 + \frac{s}{\omega_r+j(-\omega_0+w_i)}\right) \]

\[ \cdot \left(1 + \frac{s}{\omega_r+j(-\omega_0-w_i)}\right) \]

Note that the constant coefficients at the beginning of the \(D(s+j\omega_0)\) and \(D(s-j\omega_0)\) expressions are the reciprocals of \(H(j\omega_0)\) and \(H(-j\omega_0)\), respectively. Hence, these terms cancel out in the expression for the big numerator, and we are left with

\[ BN = \left(1 + \frac{s}{\omega_r+j(-\omega_0+w_i)}\right) \left(1 + \frac{s}{\omega_r+j(-\omega_0-w_i)}\right) \]

\[ - \left(1 + \frac{s}{\omega_r+j(\omega_0+w_i)}\right) \left(1 + \frac{s}{\omega_r+j(\omega_0-w_i)}\right) \]
Multiply out:

$$BN = 1 + S \left[ \frac{1}{\omega_R + j(-\omega_s + \omega_i)} + \frac{1}{\omega_R + j(-\omega_s - \omega_i)} \right] + \frac{S^2}{\omega_R^2 - 2\omega_R \omega_s - \omega_s^2 + \omega_i^2}$$

$$- 1 - S \left[ \frac{1}{\omega_R + j(\omega_s - \omega_i)} + \frac{1}{\omega_R + j(\omega_s + \omega_i)} \right] - \frac{S^2}{\omega_R^2 - 2\omega_R \omega_s - \omega_s^2 + \omega_i^2}$$

$$= S \left[ \frac{1}{\omega_R + j(-\omega_s + \omega_i)} + \frac{1}{\omega_R + j(-\omega_s - \omega_i)} - \frac{1}{\omega_R + j(\omega_s - \omega_i)} - \frac{1}{\omega_R + j(\omega_s + \omega_i)} \right]$$

$$+ S^2 \left[ \frac{1}{\omega_R^2 - 2\omega_R \omega_s - \omega_s^2 + \omega_i^2} - \frac{1}{\omega_R^2 + 2\omega_R \omega_s - \omega_s^2 + \omega_i^2} \right]$$

$$= S \left[ \frac{2j(\omega_s - \omega_i)}{\omega_R^2 + (\omega_s - \omega_i)^2} + \frac{2j(\omega_s + \omega_i)}{\omega_R^2 + (\omega_s + \omega_i)^2} \right] + S \left[ \frac{4j\omega_R \omega_s}{\omega_R^2 \omega_i} \right]$$

$$= S \left[ \frac{2j(\omega_s - \omega_i)(\omega_R^2 + (\omega_s + \omega_i)^2) + (\omega_s + \omega_i)(\omega_R^2 + (\omega_s - \omega_i)^2)}{\omega_R^2 \omega_i} \right] + S \left[ \frac{4j\omega_R \omega_s}{\omega_R^2 \omega_i} \right]$$

$$= S \cdot \frac{4j\omega_s}{\omega_R^2 \omega_i} \left( \omega_R^2 + \omega_s^2 - \omega_i^2 \right) + S \cdot \frac{4j\omega_R \omega_s}{\omega_R^2 \omega_i}$$

$$BN = \frac{4j\omega_s}{\omega_R^2 \omega_i} \left( \omega_R^2 + \omega_s^2 - \omega_i^2 \right) \left( 1 + S \frac{\omega_R}{\omega_R^2 + \omega_s^2 - \omega_i^2} \right)$$
The coefficient of \( G_{em}(s) \)

\[
\text{coeff} = \frac{V_{s1}}{2\omega_m \| H(j\omega_0) \|} \quad \text{Note that } \omega_m \text{ is the frequency of the control variation,}
\]

i.e., of the input to this transfer function.

we can view \( s = j\omega_m \) and hence \( \omega_m = s/j \).

\[ V_{s1} = \frac{4}{\pi} V_g \] (amplitude of fundamental component of switch network output voltage)

\( H(j\omega_0) \) is given at the bottom of page 7;
its magnitude is

\[
\| H(j\omega_0) \| = \frac{\omega_0^2}{\sqrt{\omega_r^2 + (\omega_0 + \omega_i)^2} \sqrt{\omega_r^2 + (\omega_0 - \omega_i)^2}}
\]

\[
= \frac{1}{\sqrt{(1-F^2)^2 + (F\frac{\pi}{Q})^2}} = \frac{\omega_0^2}{\omega_0 \omega_Q}
\]

so

\[
\text{coeff} = \frac{2V_g j\omega_0 \omega_Q}{\pi s \omega_0^2}
\]

- must also multiply by modulator gain

\[ K_m = 50 \text{kHz/V} = 2\pi \times 50 \text{ krad/sec per volt} \]
Finally,

\[ G_{eq}(s) = \frac{2j V_0 \omega l \omega H K_m}{\pi s \omega_0^2} - \frac{4j \omega_0 \left( \omega_r^2 + \omega_0^2 - \omega_i^2 \right)}{\omega_l^2 \omega_H^2} \leq \left( 1 + \frac{s \omega_r}{\omega_r^2 + \omega_0^2 - \omega_i^2} \right) \]

\[ \frac{\omega_0}{\omega_l^2 \omega_H^2} \left( 1 + \frac{s}{\omega_l^2} + \left( \frac{s}{\omega_0^2} \right)^2 \right) \left( 1 + \frac{s}{\omega_H^2} + \left( \frac{s}{\omega_0^2} \right)^2 \right) \]

\[ = -\frac{8V_0 K_m}{\pi} \frac{\omega_0}{\omega_l^2 \omega_H^2} \frac{\omega_r^2 + \omega_0^2 - \omega_i^2}{\omega_r^2 \omega_0^2 \omega_H^2} \left( 1 + \frac{s \omega_r}{\omega_r^2 + \omega_0^2 - \omega_i^2} \right) \]

\[ \frac{\left( 1 + \frac{s}{\omega_l^2} + \left( \frac{s}{\omega_0^2} \right)^2 \right)}{\left( 1 + \frac{s}{\omega_H^2} + \left( \frac{s}{\omega_0^2} \right)^2 \right)} \]

For the given values: salient features are tabulated on next page.

2 low-frequency poles at 31.5 kHz, \( Q = 0.63 \)

2 high-frequency poles at 155 kHz, \( Q = 3.1 \) (ignore above switching frequency)

zero at 140 kHz (ignore above switching frequency)

dc gain of -2.55 \( \Rightarrow \) 48.1 dB

4) For root locus diagram, see p. 7
Dynamic analysis of the parallel resonant converter

\[
\begin{align*}
M & = 1.33333333 \\
Vg & = 150 \\
V & = 200 \\
L & = 0.000627506 \quad 627.5056511 \mu\text{H} \\
C & = 7.91186E-09 \\
R & = 400 \\
f_s & = 75000 \\
K_m & = 50000 \text{ Hz/V} \\
F & = 1.05 \\
Q_e & = 1.752266974 \\
f_0 & = 71428.57143 \\
P & = 50 \\
Re & = 493.4802201 \\
R_0 & = 281.6238777
\end{align*}
\]

Gen  salient features

\[
\begin{align*}
w_L & = 198094.5235 \quad 31527.72261 = \frac{f}{f_L} \\
Q_L & = 6.26E-01 \\
w_H & = 973204.8861 \quad 154890.3683 = \frac{f}{f_H} \\
Q_H & = 3.08E+00
\end{align*}
\]

Problem solutions

stated element values and part 1 calcs

\[
\begin{align*}
L & = 6.27E-04 \\
R & = 400 \\
C & = 7.90E-09 \\
P & = 50 \\
V_g & = 1.50E+02 \\
M & = 1.33E+00 \\
V & = 200 \\
R_0 & = 281.7216509 \\
Q & = 1.419841175 \\
F & = 1.20E+00 \\
f_0 & = 7.15E+04 \\
f_s & = 8.59E+04 \\
5.40E+05 & = \omega_{sD}
\end{align*}
\]

\[
\begin{align*}
||Z_{00}|| & = 7.64E+02 \\
V_{oc} & = 4.31E+02 \\
I_{sc} & = 5.64E-01 \\
w_0 & = 4.49E+05 \\
w_r & = 1.58E+05 \\
w_i & = 4.21E+05 \\
69.38099844 & = \omega_{sD}
\end{align*}
\]
5. GENV transfer function: Bode plot

\[ \text{dc gain} = -255 \Rightarrow 4.8 \text{ dB} \]

\( f_L \quad 31.5 \text{ kHz} \)

\( Q_L = 0.63 \)

\( -40 \text{ dB/dec} \)

\( f_s = 86 \text{ kHz} \)

\( -60 \text{ dB/dec} \)

\( f_H \quad 140 \text{ kHz} \quad Q_H = 3.1 \Rightarrow 10 \text{ dB} \)

\( \text{Switching frequency} \)

\( f_s \quad 86 \text{ kHz} \)

\( \text{Dynamics invalid} \)

\( \text{near and above} \ f_{so} \)

\( (\text{GENV} \quad \text{not including the minus sign in the dc gain term}) \)

\( f_{so} = 10^{1/2} e \)

\( = 5 \text{ kHz} \)

\( 180 \angle 0 \text{/dec} = -13\text{/dec} \)

\( 140 \text{ kHz} = f_s/10 \)

\( f_{so} = 107 \text{ kHz} \)

\( -23 \text{/dec} \)

\( f_{so} = 86 \text{ kHz} \)

\( \text{etc} \)

The frequency modulator samples the control signal \( v(t) \) at a rate of \( f_s \).

The model is valid for frequencies sufficiently less than half of the switching frequency.