Announcements

• On-campus students: turn in HW3

• HW4 is due in class on Monday, Feb 20. The grace period for off-campus students expires 5pm (MT) on Monday, Feb.27
2. Effect of the tank transfer function $H(s)$

From previous slide: under small-signal conditions, the switch voltage waveform includes sinusoidal components at the switching (carrier) frequency and at upper and lower sidebands:

$$v_{s1}(t) \approx V_{s1} \left[ \cos \left( \omega_{s0} t \right) + \frac{\beta}{2} \cos \left( (\omega_{s0} + \omega_m) t \right) - \frac{\beta}{2} \cos \left( (\omega_{s0} - \omega_m) t \right) \right]$$

The tank transfer function $H(s)$ operates on each of these frequency components, and we can compute the resulting output voltage $V(t)$ using superposition as illustrated below:

- Frequency modulator
- Switch network
- Tank transfer function
- Output voltage $V(t)$
The control-to-output-envelope transfer function $G_{env}(s)$

For frequency modulation, the control-to-output-envelope transfer function is the small-signal transfer function from variations in the switching frequency $f_s$ to variations in the envelope of the output voltage $\hat{v}$ (or other output quantity):

$$G_{env}(s) = \frac{\|\hat{v}\|}{f_s}$$

The output $v(t)$ is a sinusoid containing both frequency and amplitude modulation. We want to model how the output amplitude varies.

$$v(t) = (V + \hat{v}(t)) \cos(\phi(t) + \theta(t))$$
Phasors and linear systems: a very brief review

\[ v_i = V \cos \omega t \]

\[ v_i = V \cdot \frac{1}{2} (e^{j\omega t} + e^{-j\omega t}) \]

\[ v_0 = V \cdot \frac{1}{2} \left[ H(j\omega) e^{j\omega t} + H^*(j\omega) e^{-j\omega t} \right] \]

\[ v_0 = V \| H(j\omega) \| \cos (\omega t + \angle H(j\omega)) \]
Phasor analysis

Component at the carrier frequency \( \omega_{s0} \): the switch voltage is

\[
V_{s1} \cos (\omega_{s0} t) = \frac{V_{s1}}{2} (e^{j\omega_{s0} t} + e^{-j\omega_{s0} t})
\]

which has a phasor representation of \( V_{s1} \). This component passes through the tank transfer function and leads to a component of the output voltage \( v(t) \) equal to:

\[
\| A_0 \| \cos (\omega_{s0} t + \angle A_0) = \frac{1}{2} \left( A_0 e^{j\omega_{s0} t} + A_0^* e^{-j\omega_{s0} t} \right)
\]

where \( A_0 = V_{s1} H(j\omega_{s0}) \) is the phasor representing the output carrier component, and \( A_0^* \) is the complex conjugate of \( A_0 \).

\[
A_0 = V_{s1} \cdot H(j\omega_{s0})
\]
Phasor analysis, p. 2

Component at the upper sideband frequency \( \omega_s + \omega_m \): the switch voltage is

\[
V_{s1} \cos \left( (\omega_s + \omega_m) t \right) = \frac{\beta V_{s1}}{4} \left( e^{j(\omega_s + \omega_m) t} + e^{-j(\omega_s + \omega_m) t} \right)
\]

Note that the frequency differs from the carrier frequency used in the phasors of the previous slide. Let's nonetheless define a phasor \( V_{s1} \beta/2 \) for this component. Upon passing through the tank transfer function, this component leads to a component of the output voltage \( V(t) \) equal to:

\[
\frac{1}{2} \left( (A_u e^{j\omega_m t}) e^{j\omega_s t} + (A_u e^{j\omega_m t})^* e^{-j\omega_s t} \right)
\]

where \( A_u = V_{s1} (\beta/2) H(j(\omega_s + \omega_m)) \) is the phasor representing the output upper sideband component, at frequency \( \omega_s + \omega_m \). Similar analysis leads to the following output component at the lower sideband frequency:

\[
\frac{1}{2} \left( (A_l e^{-j\omega_m t}) e^{j\omega_s t} + (A_l e^{-j\omega_m t})^* e^{-j\omega_s t} \right)
\]

with \( A_l = V_{s1} (\beta/2) H(j(\omega_s - \omega_m)) \)

\[
e^{j(\omega_s + \omega_m) t} = e^{j\omega_s t} e^{j\omega_m t}
\]
Total output voltage

Upon adding together the expressions for the three components of the output voltage waveform, we obtain:

\[ v(t) = \frac{\left( A_0 + A_u e^{j\omega_m t} + A_l e^{-j\omega_m t}\right) e^{j\omega_0 t} + \left( A_0 + A_u e^{j\omega_m t} + A_l e^{-j\omega_m t}\right)^* e^{-j\omega_0 t}}{2} \]

This expression can also be written in the following form:

\[ v(t) = \| A(t) \| \cos \left( \omega_0 t + \angle A(t) \right) \]

with

\[ A(t) = A_0 + A_u e^{j\omega_m t} + A_l e^{-j\omega_m t} \]

The complex time-varying quantity \( A(t) \) constitutes a phasor representing the modulated output voltage \( v(t) \). The magnitude \( \| A(t) \| \) is the envelope, or amplitude, of the modulated output voltage.
Interpretation: time-varying phasor representation of output voltage

The result of the previous slide is a sinusoid having a time-varying amplitude \( ||A(t)|| \) (the envelope — amplitude modulation), and also a time-varying phase \( \angle A(t) \) (phase or frequency modulation).

Next: separate the perturbation in the envelope and relate to the perturbation in the control input, to find \( G_{env}(s) \).

\[
v(t) = \|A(t)\| \cos(\omega_0 t + \angle A(t))
\]

\[
A(t) = A_0 + A_e e^{j\omega_m t} + A_i e^{-j\omega_m t}
\]
3. Expression for envelope

\[ A(t) = A_0 + A_u e^{j\omega_m t} + A_\ell e^{-j\omega_m t} \]

We can find the amplitude using the identity \( \| A \|^2 = AA^* \)

\[ \| A \|^2 = AA^* = \left( A_0 + A_u e^{j\omega_m t} + A_\ell e^{-j\omega_m t} \right) \left( A_0^* + A_u^* e^{-j\omega_m t} + A_\ell^* e^{j\omega_m t} \right) \]

After some algebra, we obtain

\[ \| A \|^2 = \| A_0 \|^2 + \| A_u \|^2 + \| A_\ell \|^2 + 2 \| A_0^* A_u + A_0 A_\ell^* \| \cos \left( \omega_m t + \angle (A_0 A_u + A_0 A_\ell^*) \right) \]

\[ + 2 \| A_u A_\ell^* \| \cos \left( 2 \omega_m t + \angle A_u A_\ell^* \right) \]
Small-signal linearization

The small-signal assumption $\beta \ll 1$ implies that
\[
\| A_0 \| \gg \| A_u \|
\]
\[
\| A_0 \| \gg \| A_\ell \|
\]

Hence,
\[
\| A_0 \|^2 \gg \| A_u \|^2 + \| A_\ell \|^2
\]
\[
\| A_0^* A_u + A_0 A_\ell^* \| \gg \| A_u A_\ell^* \|
\]

and the dominant terms in the expression for $\| A \|^2$ are:
\[
\| A \|^2 = \| A_0 \|^2 + 2 \| A_0^* A_u + A_0 A_\ell^* \| \cos (\omega_m t + \zeta (A_0^* A_u + A_0 A_\ell^*))
\]

Next: take the square root.
Expression for magnitude

\[ \sqrt{a^2 + b} = a \sqrt{1 + \frac{b^2}{a^2}} \]

Next, we need to take the square root to find \( \| A \| \), and linearize again. We want to linearize a function of the form

\[ \| A \| = \sqrt{a^2 + b(t)} \text{ with } a^2 = \| A_0 \|^2 \text{ and } a^2 \gg |b(t)| \]

Use of a Taylor series for the above function, and linearization by neglect of high-order terms, leads to

\[ \| A \| = a + \frac{b(t)}{2a} + \text{higher order terms} \]

This is the desired result.
Result
Small-signal expression for envelope

\[ \|A\| = \|A_0\| + \frac{\|A_0 A_u + A_0 A_f\|}{\|A_0\|} \cos (\omega_m t + \angle (A_0 A_u + A_0 A_f)) \]

- Envelope \(\|A(t)\|\)
- Steady-state \(\|A_0\|\)
- Small-signal amplitude of envelope
- Envelope phase

\(v(t)\)

\(t\)
The transfer function $G_{env}(s) = \|A_o\| \frac{j\omega}{s}$

From the expression of the envelope from the previous slide, the small-signal perturbation in the output amplitude is

$$\frac{A_o^* A_u + A_o A^*_\ell}{\|A_o\|}$$

We take the switching frequency variation as the transfer function input, having amplitude $\Delta\omega$ and zero phase. Hence the control-to-output-envelope transfer function is

$$G_{env}(j\omega) = \frac{1}{\Delta\omega} \frac{A_o^* A_u + A_o A^*_\ell}{\|A_o\|}$$
If $H(s)$ has 2 poles, $G_{env}(s)$ has 4 poles!

Express $G_{env}(s)$ in terms of $H(s)$

Let the tank transfer function $H(s)$ be expressed as a rational fraction:

$$H(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are the numerator and denominator polynomials, respectively. Simplification of the previous expression for $G_{env}(s)$, with some algebra, leads to the following expression:

$$G_{env}(s) = \frac{jV_{in}}{2s} \left\| \frac{H(-j\omega_0)N(s + j\omega_0)D(s - j\omega_0) - H(j\omega_0)N(s - j\omega_0)D(s + j\omega_0)}{D(s - j\omega_0)D(s + j\omega_0)} \right\|$$

This is the desired result: the small-signal transfer function from the control input (the switching frequency variation in rad/sec) to the output envelope, expressed in terms of the tank transfer function $H(s)$ and the quiescent switching frequency $\omega_{00}$. To evaluate the magnitude and phase of $G_{env}(s)$, we let $s = j\omega_m$. It can be shown that the numerator of the right-side fraction is purely imaginary and has no $s^0$ term; hence the remaining terms cancel the $j/s$ terms of the left-side fraction.

So $s = sp - j\omega_0$ is a pole of $G_{env}(s)$.

$s = sp + j\omega_0$ is a pole of $G_{env}(s)$.
The poles of $G_{env}(s)$

$$G_{env}(s) = \frac{jV_{st}}{2s \|H(j\omega_0)\|} \frac{H(-j\omega_o)N(s + j\omega_o)D(s - j\omega_o) - H(j\omega_o)N(s - j\omega_o)D(s + j\omega_o)}{D(s - j\omega_o)D(s + j\omega_o)}$$

The poles of $G_{env}(s)$ are the roots of $D(s - j\omega_0)D(s + j\omega_0)$:

- $G_{env}$ has twice as many poles as $H$.
- Tank poles are shifted up and down by $j\omega_0$.
- Resulting poles are shifted in frequency and Q-factor.

Example:
- Poles of $H(s)$
- Poles of $G_{env}(s)$
- $\omega_0$, $\omega_1$, $\omega_2$, $\omega_3$, $\omega_4$, $\omega_5$, $\omega_6$
Key features of $G_{env}(s)$

DC gain of $G_{env}(s)$ is the slope of the $V$ vs. $F$ curve at the quiescent operating point

Interpretation of resonant frequency $\omega_0$ and $Q$ factor of dominant poles of $G_{env}(s)$