Extension of State Space Averaging to Model Some Other Types of Switches

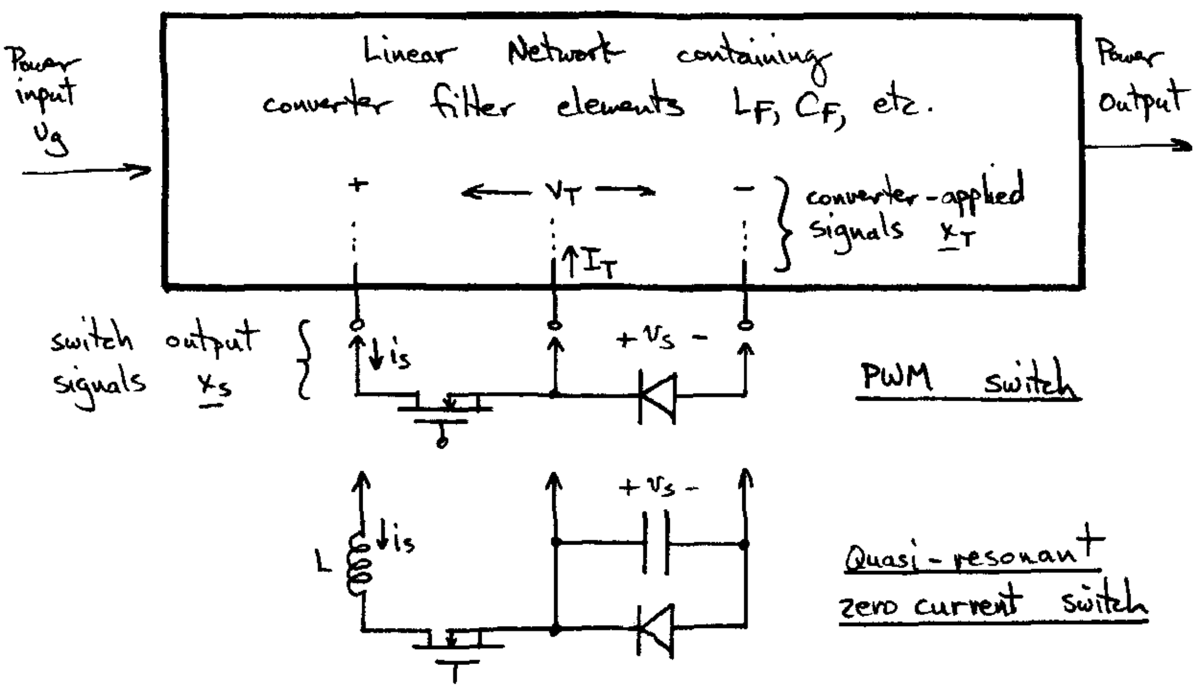
We know how to model the properties and dynamics of PWM converters whose dynamic elements contain small switching ripple. This problem was solved in a physical circuit-oriented way using circuit averaging, and in general using state space averaging.

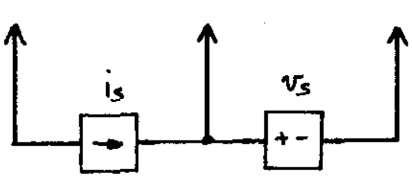
We also now know how to model resonant switch cells, thereby determining the switch average output waveforms in terms of the converter-applied signals (V_T and I_T) and the control inputs (f_s, etc.).

The state-space averaging method is generalized here to model systems incorporating most types of switches (resonant or otherwise) embedded inside converters. The result is that, provided the switch contains no internal dynamics and can directly replace an equivalent PWM switch (these conditions are defined more rigorously later), then state space averaging can be generalized, with the switch conversion ratio μ replacing the duty cycle D. However, μ may itself depend on V_T and I_T, so "built-in" feedback loops may occur.

Equivalent circuit models, the canonical model, and the input filter problem can all be generalized to cover other types of switches.

A Converter System Containing a Filter Network and a Switch Network

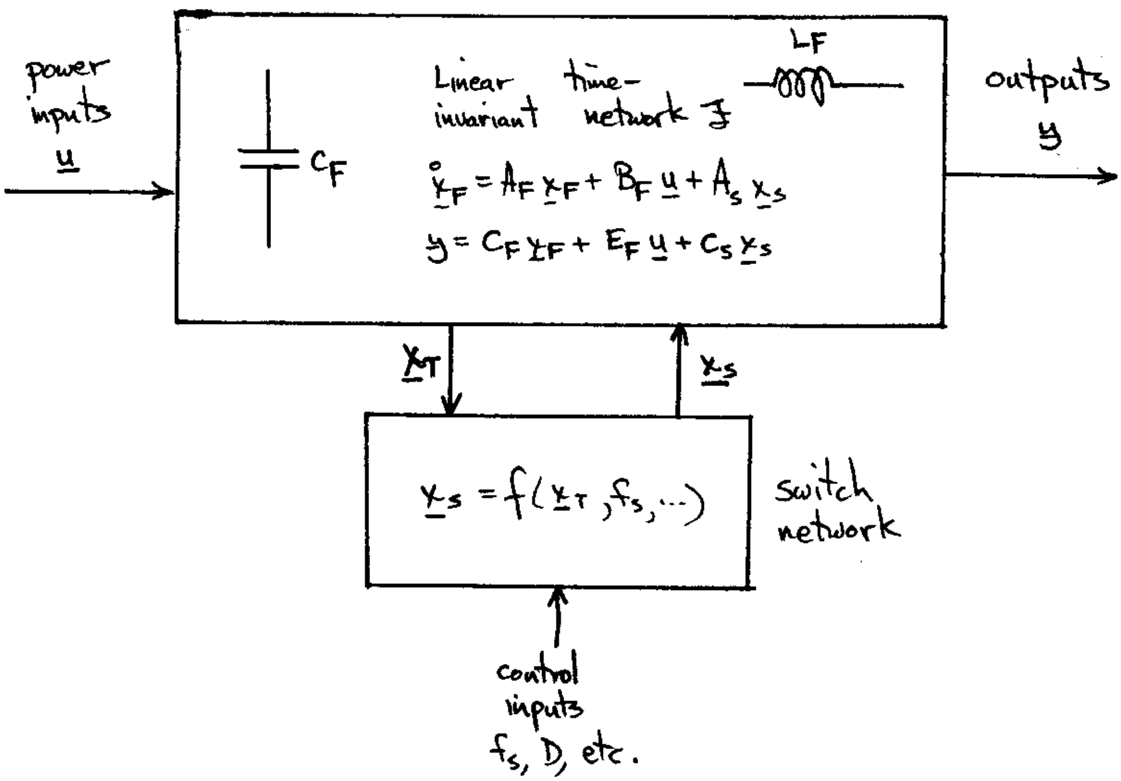




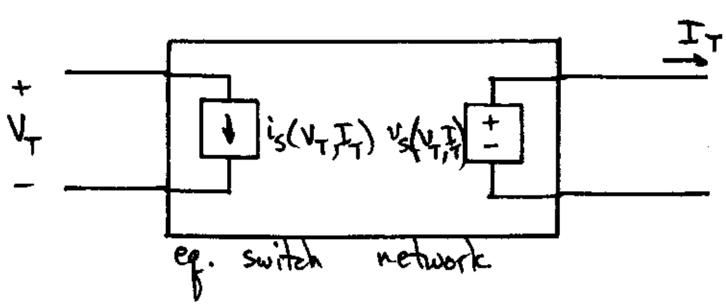
Equivalent model

- Sources i_s, v_s are defined to have waveforms identical to those of actual switch
- i_s, v_s depend on I_T, V_T.

In general, the system can be written as a linear time invariant network of filter elements, power sources, and loads, connected to a switching network:



The switch therefore becomes a two-port device, with x_T the independent inputs and x_s the dependent outputs:



In general, $\dot{\underline{x}}_F$ and $\underline{y}$ are linear combinations of the independent quantities $\underline{x}_F$, $\underline{u}$, and $\underline{x}_S$:

$$\begin{aligned} \dot{\underline{x}}_F &= A_F \underline{x}_F + B_F \underline{u} + A_S \underline{x}_S \\ \underline{y} &= C_F \underline{x}_F + E_F \underline{u} + C_S \underline{x}_S \end{aligned} \tag{1}$$

Since this part of the system is, by definition, time-invariant, A_F, B_F, A_S, C_F, E_F, and C_S are constant matrices.

We will assume that all of the switch tank components operate in discontinuous conduction mode, so that the switch network introduces no additional poles into the system dynamics. Then the low frequency components of $\underline{x}_S(t)$ depend only on $\underline{x}_T$ and the control inputs.

Circuit Averaging

As in PWM converter models, we can average the system signals, over a period which is short compared with the system natural response times, without significantly changing the waveforms. This effectively removes the switching and ringing harmonics without modifying the desired low frequency response.

In particular, replace the switch network outputs $\underline{x}_s$ by their average values $\langle \underline{x}_s \rangle$.

$$\langle \underline{x}_s \rangle = \begin{bmatrix} \langle v_c \rangle \\ \langle i_L \rangle \end{bmatrix}$$ for the zero current switch; we have already solved for these.

Hence, we must solve the switch using our phase plane techniques, and express the average outputs $\langle \underline{x}_s \rangle$ in terms of $\underline{x}_T$. We can then plug the result into Eq. (1) to find the system average dynamics. We could then perturb, linearize, and construct a small-signal equivalent circuit.

Before doing so, it is desirable to perform an additional step:

<u>Expressing Eq. (1) in Canonical State-Space Averaging Form</u>

Regardless of the type of switch used, a buck converter is still a buck converter, and its equations should not change significantly. Since we have a large body of knowledge pertaining to PWM switches, let's manipulate Eq. (1) into the same form as the previous PWM result. Doing so requires that we be able to write $\langle \underline{x}_s \rangle$ in the form

$$\langle \underline{x}_s \rangle = \mu \, \underline{X}_{s1} + \mu' \, \underline{X}_{s2} \qquad (2)$$

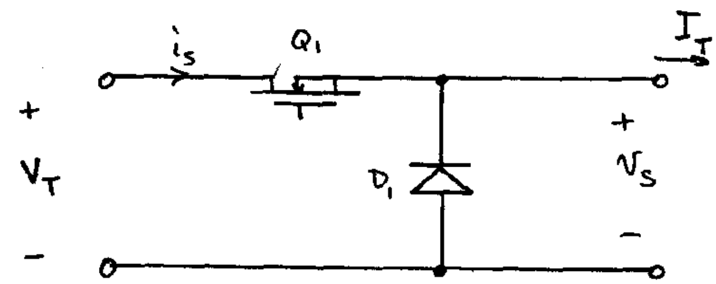
where $\mu =$ switch conversion ratio

$\mu' = 1-\mu$

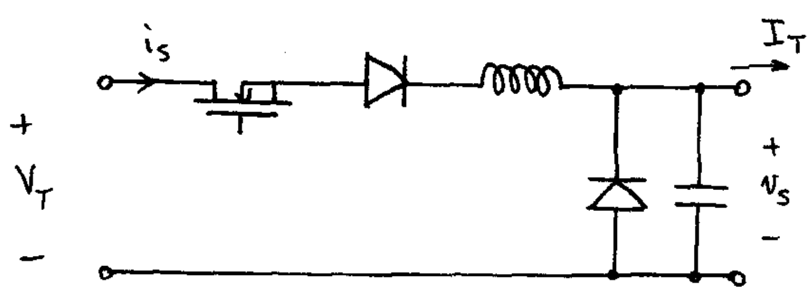
$\underline{x}_{s1} =$ The value of $\underline{x}_s$ during subinterval 1 (switch in position 1)

$\underline{x}_{s2} =$ The value of $\underline{x}_s$ during subinterval 2 (switch in position 2)

Many types of switches can be written in the form of Eq. (2). For example, for the linear zero current switch (type a), we have;



PWM



ZCS

<u>PWM</u>

position 1:
Q_1 on
D_1 off

so $v_s = V_T$
$i_s = I_T$

$$\Rightarrow \underline{x}_{s1} = \begin{bmatrix} V_T \\ I_T \end{bmatrix}$$

position 2:
Q_1 off
D_1 on

so $v_s = 0$
$i_s = 0$

$$\Rightarrow \underline{x}_{s2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We have previously shown that, for the ZCS,

$$v_s = \mu V_T$$
$$i_s = \mu I_T$$

which can be written in the form

$$\underline{x}_S = \mu \underline{X}_{S1} + \mu' \underline{X}_{S2}$$

Insertion of Eq. (2) into Eq. (1) yields

$$\overset{\circ}{\underline{x}}_F = A_F \underline{x}_F + B_F \underline{u} + \mu A_s \underline{X}_{S1} + \mu' A_s \underline{X}_{S2}$$
$$\underline{y} = C_F \underline{x}_F + E_F \underline{u} + \mu C_s \underline{X}_{S1} + \mu' C_s \underline{X}_{S2} \qquad (3)$$

Consider next the linear time invariant network $\mathcal{F}$. Suppose that, in place of the switch outputs $\underline{x}_s$ we connected sources of value $\underline{X}_{S1}$, the same as during subinterval 1 of the equivalent PWM converter. By definition, the system state equations should then be

$$\overset{\circ}{\underline{x}}_F = A_1 \underline{x}_F + B_1 \underline{u}$$
$$\underline{y} = C_1 \underline{x}_F + E_1 \underline{u} \qquad (4)$$

However, Eq. (1) predicts that the state equations would be

$$\mathring{\underline{x}}_F = A_F \underline{x}_F + B_F \underline{u} + A_S \underline{X}_{S1}$$
$$y = C_F \underline{x}_F + E_F \underline{u} + C_S \underline{X}_{S1} \tag{5}$$

Hence, Eqs. (4) and (5) must be equal:

$$\mathring{\underline{x}}_F = A_F \underline{x}_F + B_F \underline{u} + A_S \underline{X}_{S1} = A_1 \underline{x}_F + B_1 \underline{u}$$
$$y = C_F \underline{x}_F + E_F \underline{u} + C_S \underline{X}_{S1} = C_1 \underline{x}_F + E_1 \underline{u} \tag{6}$$

Solve for $A_S \underline{X}_{S1}$ and $C_S \underline{X}_{S1}$:

$$A_S \underline{X}_{S1} = (A_1 - A_F)\,\underline{x}_F + (B_1 - B_F)\,\underline{u}$$
$$C_S \underline{X}_{S1} = (C_1 - C_F)\,\underline{x}_F + (E_1 - E_F)\,\underline{u} \tag{7}$$

Similar arguments apply if sources $\underline{X}_{S2}$ are applied to network $\mathcal{F}$ whose values are identical to the actual value of $\underline{x}_S$ during subinterval 2 of the equivalent PWM converter. Then,

$$\mathring{\underline{x}}_F = A_2 \underline{x}_F + B_2 \underline{u} = A_F \underline{x}_F + B_F \underline{u} + A_S \underline{X}_{S2}$$
$$y = C_2 \underline{x}_F + E_2 \underline{u} = C_F \underline{x}_F + E_F \underline{u} + C_S \underline{X}_{S2} \tag{8}$$

Solve for $A_s \underline{X}_{s2}$ and $C_s \underline{X}_{s2}$:

$$
\begin{aligned}
A_s \underline{X}_{s2} &= (A_2 - A_F)\,\underline{x}_F + (B_2 - B_F)\,\underline{u} \\
C_s \underline{X}_{s2} &= (C_2 - C_F)\,\underline{x}_F + (E_2 - E_F)\,\underline{u}
\end{aligned}
\qquad (9)
$$

Now let's eliminate $\underline{X}_{s1}$ and $\underline{X}_{s2}$ from our model. Insertion of Eqs. (7) and (9) into the averaged equations (3) yields

$$
\begin{aligned}
\mathring{\underline{x}}_F &= A_F \underline{x}_F + B_F \underline{u} + \mu\Big((A_1 - A_F)\underline{x}_F + (B_1 - B_F)\underline{u}\Big) \\
&\qquad + \mu'\Big((A_2 - A_F)\underline{x}_F + (B_2 - B_F)\underline{u}\Big) \\
\underline{y} &= C_F \underline{x}_F + E_F \underline{u} + \mu\Big((C_1 - C_F)\underline{x}_F + (E_1 - E_F)\underline{u}\Big) \\
&\qquad + \mu'\Big((C_2 - C_F)\underline{x}_F + (E_2 - E_F)\underline{u}\Big)
\end{aligned}
\qquad (10)
$$

Collect terms, and use the identity $\mu + \mu' = 1$:

$$
\begin{aligned}
\mathring{\underline{x}}_F &= (\mu A_1 + \mu' A_2)\,\underline{x}_F + (\mu B_1 + \mu' B_2)\,\underline{u} \\
\underline{y} &= (\mu C_1 + \mu' C_2)\,\underline{x}_F + (\mu E_1 + \mu' E_2)\,\underline{u}
\end{aligned}
\qquad (11)
$$

This is the desired result. It is identical to the state space averaging result, with D replaced by μ.

Hence, the results of PWM state space averaging can be applied to other types of switches, using μ_0 instead of D_0, $\hat{\mu}$ instead of $\hat{d}$, etc.

Perturbation and Linearization

To construct a small signal model, let

$$\mu = \mu_0 + \hat{\mu} \qquad \underline{u} = \underline{U}_0 + \hat{\underline{u}}$$

$$\underline{x}_F = \underline{X}_{F0} + \hat{\underline{x}}_F \qquad \underline{y} = \underline{Y}_0 + \hat{\underline{y}} \tag{12}$$

$$\underline{x}_T = \underline{X}_{T0} + \hat{\underline{x}}_T \qquad \underline{u}_c = \underline{U}_{c0} + \hat{\underline{u}}_c$$

Note that μ may depend on $\underline{x}_T$ any control inputs $\underline{u}_c$. Hence, the perturbation in μ may depend on $\hat{\underline{x}}_T$ and $\hat{\underline{u}}_c$, as follows:

$$\hat{\mu} = \left.\frac{d\mu}{d\underline{x}_T}\right|_{\underline{x}_T = \underline{X}_{T0}} \cdot \hat{\underline{x}}_T + \left.\frac{d\mu}{d\underline{u}_c}\right|_{\underline{u}_c = \underline{U}_{c0}} \cdot \hat{\underline{u}}_c \tag{13}$$

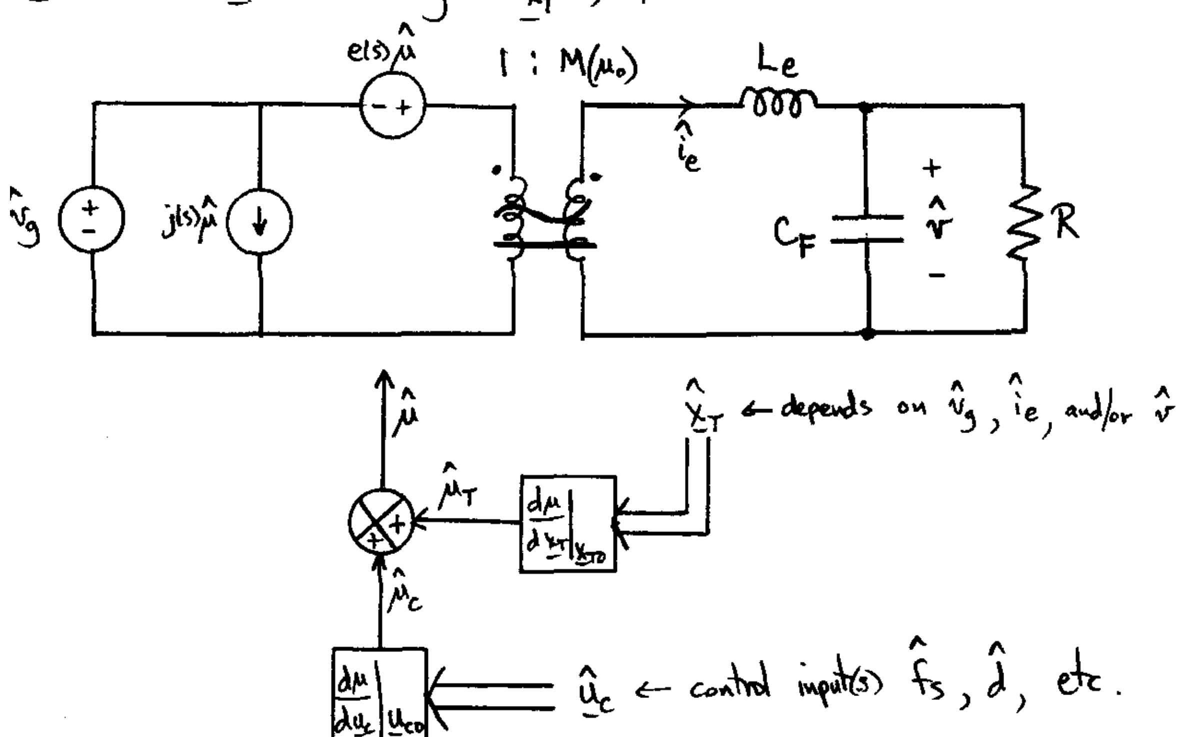
Since the elements of $\underline{x}_T$ are in fact converter states and/or power inputs, Eq. (13) constitutes feedback and/or feedforward.

Substitute Eq. (12) into Eq. (11), and eliminate nonlinear terms:

$$\frac{d\,\hat{\underline{x}}_F}{dt} = (\mu_0 A_1 + \mu_0' A_2)\,\hat{\underline{x}}_F + (\mu_0 B_1 + \mu_0' B_2)\,\hat{\underline{u}} + \left((A_1 - A_2)\underline{X}_{F0} + (B_1 - B_2)\underline{U}_0\right)\hat{\mu}$$

$$\hat{y} = (\mu_0 C_1 + \mu_0' C_2)\,\hat{\underline{x}}_F + (\mu_0 E_1 + \mu_0' E_2)\,\hat{\underline{u}} + \left((C_1 - C_2)\underline{X}_{F0} + (E_1 - E_2)\underline{U}_0\right)\hat{\mu} \qquad (14)$$

Since Eq. (14) is of the same form as the PWM small signal model, the same canonical circuit model applies. However, $\hat{\mu}$ can depend not only on the control inputs $\hat{\underline{u}}_c$, but also on $\hat{\underline{x}}_F$ and $\hat{\underline{u}}$ (through $\hat{\underline{x}}_T$):



	Buck	Boost	Buck-Boost
V_{T0}	V_{g0}	V_0	$V_{g0} - V_0$
G_{0g}	$\frac{\mu_0+V_{T0}K_v}{1+V_{T0}K_i/R}$	$\frac{\mu_0'(1+\frac{V_{T0}K_i}{\mu_0'^2 R})}{\mu_0'^2-V_{T0}K_v\mu_0'+2V_{T0}K_i/R}$	$\frac{-\mu_0' V_{T0}K_v-\mu_0'\mu_0-\frac{\mu_0'^2 V_{T0}K_i}{\mu_0' R}}{\mu_0'^2-V_{T0}K_v\mu_0'+V_{T0}K_i(1+\mu_0)/R}$
G_{0c}	$\frac{V_{T0}K_c}{1+V_{T0}K_i/R}$	$\frac{\mu_0' V_{T0}K_c}{\mu_0'^2-V_{T0}K_v\mu_0'+2V_{T0}K_i/R}$	$\frac{-\mu_0' V_{T0}K_c}{\mu_0'^2-V_{T0}K_v\mu_0'+V_{T0}K_i(1+\mu_0)/R}$
ω_p	$\omega_F\sqrt{1+V_{T0}K_i/R}$	$\omega_F\sqrt{\mu_0'^2-V_{T0}K_v\mu_0'+2V_{T0}K_i/R}$	$\omega_F\sqrt{\mu_0'^2-V_{T0}K_v\mu_0'+V_{T0}K_i(1+\mu_0)/R}$
Q	$\frac{\sqrt{1+V_{T0}K_i/R}}{\frac{R_F}{R}+\frac{V_{T0}K_i}{R_F}}$	$\frac{\sqrt{\mu_0'^2-V_{T0}K_v\mu_0'+2V_{T0}K_i/R}}{\frac{R_F}{R}(1-\frac{V_{T0}K_v}{\mu_0'})+\frac{V_{T0}K_i}{R_F}}$	$\frac{\sqrt{\mu_0'^2-V_{T0}K_v\mu_0'+V_{T0}K_i(1+\mu_0)/R}}{\frac{R_F}{R}(1-\frac{V_{T0}K_v\mu_0}{\mu_0'})+\frac{V_{T0}K_i}{R_F}}$
ω_{zc}	∞	$\frac{\mu_0'^2 R}{L_F}$	$\frac{\mu_0'^2 R}{L_F\mu_0}$
ω_{zg}	∞	∞	$\frac{\mu_0' R(\mu_0' V_{T0}K_v+\mu_0'\mu_0+\frac{\mu_0'^2 V_{T0}K_i}{R})}{\mu_0 V_{T0}K_v L_F}$

$$G_{vg}(s) = G_{0g}\frac{1-\frac{s}{\omega_{zg}}}{1+\frac{1}{Q}\frac{s}{\omega_p}+\left(\frac{s}{\omega_p}\right)^2}$$

$$G_{vc}(s) = G_{0c}\frac{1-\frac{s}{\omega_{zc}}}{1+\frac{1}{Q}\frac{s}{\omega_p}+\left(\frac{s}{\omega_p}\right)^2}$$

$$\omega_F = 1/\sqrt{L_F C_F}$$
$$R_F = \sqrt{L_F/C_F}$$

Switch	μ_0	$\partial\mu/\partial J_x$	K_i	K_v	K_c
PWM	D	0	0	0	$K_d = 1$
LZCS HW	$\frac{F}{2\pi}(\frac{J_T}{2}+\pi+\sin^{-1}(J_T)+\frac{1}{J_T}(1+\sqrt{1-J_T^2}))$	$\frac{\partial\mu}{\partial J_T}=\frac{F}{2\pi}(\frac{1}{2}-\frac{(1+\sqrt{1-J_T^2})}{J_T^2})$	$-\frac{\partial\mu}{\partial J_T}\frac{R_0}{V_{T0}}$	$-\frac{\partial\mu}{\partial J_T}\frac{R_0 I_{T0}}{V_{T0}^2}$	$K_f = \frac{\mu_0}{F_{s0}}$
LZCS FW	$\frac{F}{2\pi}(\frac{J_T}{2}+2\pi-\sin^{-1}(J_T)+\frac{1}{J_T}(1-\sqrt{1-J_T^2}))$	$\frac{\partial\mu}{\partial J_T}=\frac{F}{2\pi}(\frac{1}{2}-\frac{(1-\sqrt{1-J_T^2})}{J_T^2})$	0	0	$K_f = \frac{\mu_0}{F_{s0}}$
LZVS HW	$1-\frac{F}{2\pi}(\frac{\frac{1}{2}}{J_T}+\pi+\sin^{-1}(\frac{1}{J_T})+J_T(1+\sqrt{1-\frac{1}{J_T^2}}))$	$\frac{\partial\mu}{\partial J_T}=\frac{1}{J_T^2}\frac{F}{2\pi}(\frac{1}{2}-\frac{(1+\sqrt{1-J_T^2})}{J_T^2})$	$-\frac{\partial\mu}{\partial J_T}\frac{R_0}{V_{T0}}$	$-\frac{\partial\mu}{\partial J_T}\frac{R_0 I_{T0}}{V_{T0}^2}$	$K_f=\frac{\mu_0}{F_{s0}}$
LZVS FW	$1-\frac{F}{2\pi}(\frac{\frac{1}{2}}{J_T}+2\pi-\sin^{-1}(\frac{1}{J_T})+J_T(1-\sqrt{1-\frac{1}{J_T^2}}))$	$\frac{\partial\mu}{\partial J_T}=\frac{1}{J_T^2}\frac{F}{2\pi}(\frac{1}{2}-\frac{(1-\sqrt{1-J_T^2})}{J_T^2})$	0	0	$K_f=\frac{\mu_0}{F_{s0}}$
NRS HW $N_T>1$ $N_B=0$	$\frac{F}{2\pi}(\frac{J_T}{2}+(1+\sqrt{1-J_T^2})/J_T+2\sin^{-1}(J_{crit})$ $+\sin^{-1}(J_T)+\frac{2}{k}\tan^{-1}(\frac{k\sqrt{1-J_{crit}^2}}{J_{crit}}))$ $J_{crit}=\frac{R_0(N_T-1)i_T}{V_{T0}}$	$\frac{\partial\mu}{\partial J_T}=$ $\frac{F}{2\pi}[\frac{1}{2}+\frac{2(N_T-1)}{\sqrt{1-J_{crit}^2}}(1-\frac{1}{(k^2+(1-k^2)J_{crit}^2})$ $-\frac{1}{J_T^2}+\sqrt{\frac{1-J_T}{1+J_T}}]$	$-\frac{\partial\mu}{\partial J_T}\frac{R_0}{V_{T0}}$	$-\frac{\partial\mu}{\partial J_T}\frac{R_0 I_{T0}}{V_{T0}^2}$	$K_f = \frac{\mu_0}{F_{s0}}$
NRS FW $N_T=1$ $N_B\neq0$	$\frac{F}{2\pi}(\frac{J_T}{2}+(1-\sqrt{1-J_T^2})/J_T+2\sin^{-1}(J_{crit})$ $+\pi-\sin^{-1}(J_T)+\frac{2}{k}\tan^{-1}(\frac{k\sqrt{1-J_{crit}^2}}{J_{crit}}))$ $J_{crit}=J_B=\frac{R_0 N_B i_{bias}}{V_{T0}}$	$\frac{\partial\mu}{\partial J_B}=\frac{F}{\pi}[\frac{1}{\sqrt{1-J_B^2}}-\frac{J_B^2}{(J_B^2+k^2(1-J_B^2))}\cdot$ $(\frac{1}{\sqrt{1-J_B^2}}+\frac{\sqrt{1-J_B^2}}{J_B^2})]$	0	$\frac{-\partial\mu}{\partial J_B}\frac{R_0 N_B I_{bias}}{V_{T0}^2}$	$K_f = \frac{\mu_0}{F_{s0}}$ $K_B=\frac{R_0 N_B}{V_{T0}^2}$

For the half-wave ZCS

$$\mu = F\, P(j_T)$$

with $$j_T = \frac{i_T R_o}{v_T}$$

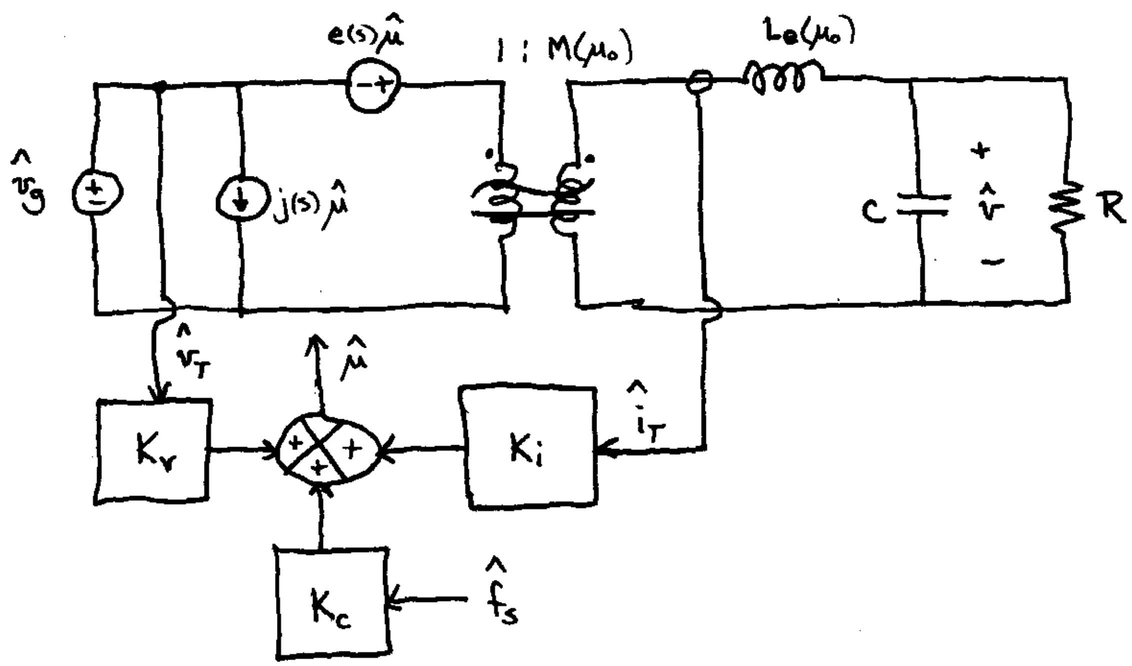
so $$\mu = \frac{f_s}{f_o}\, P\left(\frac{i_T R_o}{v_T}\right)$$

what i_T and v_T are depends on the converter topology. For the buck, $v_T = v_g$ and $i_T = i$ (inductor current). For the boost, $v_T = v$, etc. This affects where the $\hat{i}_T$ and $\hat{v}_T$ feedback connections are taken in the model below.

Perturb and linearize:

$$\hat{\mu} = K_c \hat{f}_s + K_i \hat{i}_T + K_v \hat{v}_T$$

Buck equivalent circuit:



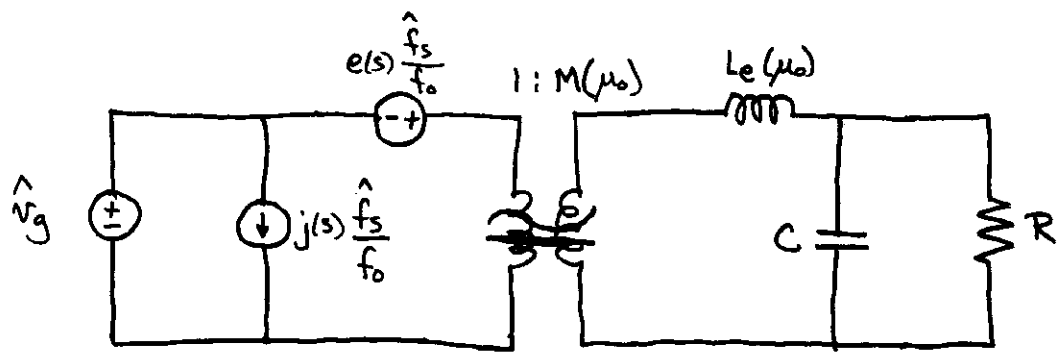
For the full-wave (type-a) ZCS,

$$\mu \cong F$$

so let $\quad \mu = \mu_0 + \hat{\mu} = \dfrac{F_s}{f_0} + \dfrac{\hat{f}_s}{f_0}$

$$\hat{\mu} = \frac{\hat{f}_s}{f_0}$$

the canonical model becomes



– same as in PWM case, but substitute

$$\frac{F_s}{f_0} \longrightarrow D_0$$

$$\frac{\hat{f}_s}{f_0} \longrightarrow \hat{d}$$