Double layer antireflective coatings

MacLeod, Thin Film Optical Filters, p. 78

- Disadvantage of single layer AR coating is limited # of adjustable parameters
  - cannot easily "engineer" refractive index

- 2 Layer coating (designed to operate at $\lambda_0$)

\[
\begin{align*}
  &n_0 \\
  &\frac{n_1}{\lambda/4n_1} = t_1 \quad \text{Layer 1} = \lambda/4 \\
  &(n_2n_3n_0) \quad \frac{\lambda/2n_2}{\lambda} = t_2 \quad \text{Layer 2} = \lambda/2 \\
  &n_3
\end{align*}
\]

Making substrate look higher index at neighboring wavelengths

\[\delta_1 = \text{phase shift} = \frac{2\pi}{\lambda} n_1 \frac{t_1}{\lambda} = \frac{\pi}{2} \frac{\lambda_0}{\lambda}\]

\[\delta_2 = \frac{2\pi}{\lambda} n_2 t_2 = \frac{\pi}{2} \frac{\lambda_0}{\lambda}\]

At $\lambda = \lambda_0$, example refractive indices

\[
\begin{align*}
  n_0 &= 1 \\
  n_1 &= 1.4 \\
  n_2 &= 2 \\
  n_3 &= 1.63
\end{align*}
\]

In this case, transmission matrices yield $R$ (reflectivity) = 0.990 (not much better than single-layer AR)
Analysis of mutual air glass interface with polarizing beam splitter cube

$\eta_b = \text{high index}$
$\eta_l = \text{low index}$

Brewster's angle

$$\tan \theta = \frac{\eta_2}{\eta_1}$$

Solve for $\theta_h$ in high index medium (for $\theta_h$ to be Brewster's)

$$\tan \theta_h = \frac{\eta_l}{\eta_h}$$

$$\frac{\eta_l^2}{\eta_h^2} = \frac{\sin^2 \theta_h}{1 - \sin^2 \theta_h}$$

$$\Rightarrow \sin^2 \theta_h = \frac{\eta_l^2}{\eta_l^2 + \eta_h^2}$$

Solve for $\theta_e$ in low index medium (Brewster's $\Delta$)

$$\tan \theta_e = \frac{\eta_h}{\eta_l} \Rightarrow \frac{\eta_h^2}{\eta_l^2} = \frac{\sin^2 \theta_e}{1 - \sin^2 \theta_e}$$

$$\sin^2 \theta_e = \frac{\eta_h^2}{\eta_l^2 + \eta_h^2}$$

Snell's law

$$n_g \sin \theta_g = n_h \sin \theta_h = n_e \sin \theta_e$$
See mirror/filter examples on slide

**Polarizing beam splitters**

- based on difference between s & p at enhanced Brewster's angle

- Input from air gives non Brewster reflection & background
Since $\theta_g = 45^\circ$, we can solve for $n_g$ that refracts at proper angle although not at Brewster's for glass - it or glass - low interface.

$$n_g^2 \sin^2 45 = \frac{n_e^2}{n_h^2} = n_h^2 \sin^2 \Theta_h$$

$$= n_h^2 \frac{n_e^2}{n_e^2 + n_h^2}$$

$$= n_e^2 \sin^2 \Theta_e$$

$$= \frac{n_e^2 - n_h^2}{n_e^2 + n_h^2}$$

$$\Rightarrow n_g = \sqrt{2} \frac{n_e n_h}{\sqrt{n_e^2 + n_h^2}}$$

must use glass of at least $n = 1.62$

for $n_e = 1.35 \& n_h = 2.35$

To analyze for polarization degree, must calculate reflectivity of $s \& p$ waves (m = # layer pairs).

$$R_{\text{power}} (\text{reflectivity}) = \left( \frac{U_G - (\frac{U_H}{U_G})(U_H/U_L)^{m-1}}{U_G + (\frac{U_H}{U_G})(U_H/U_L)^{m-1}} \right)^2$$
S waves
\[ u_s^G = n_G \cos \theta_G \quad u_s^H = n_H \cos \theta_H \quad u_s^L = n_L \cos \theta_L \]

P waves
\[ u_p^G = n_G / \cos \theta_G \quad u_p^H = n_H / \cos \theta_H \quad u_p^L = n_L / \cos \theta_L \]

Since at Brewster's angle
\[ u_p^H = u_p^L \]

\[ R_p = \left( \frac{u_G - (u_H^2 / u_G)}{u_G + (u_H^2 / u_G)} \right)^2 = \left( \frac{n_G^2 \cos^2 \theta_H}{n_H^2 \cos^2 \theta_G} - 1 \right)^2 \]

S waves
Use
\[ n_H \cos \theta_H = n_L \cos \theta_L \]

\[ \frac{n_H \cos \theta_H}{n_L \cos \theta_L} = \frac{n_H^2}{n_L^2} \]

\[ R_s = \left[ \frac{n_G^2 \cos^2 \theta_G - n_H^2 \cos^2 \theta_H \left( \frac{n_H}{n_L} \right)^{2(n-1)}}{n_G^2 \cos^2 \theta_G + n_H^2 \cos^2 \theta_H \left( \frac{n_H}{n_L} \right)^{2(n-1)}} \right]^2 \]

calculate degree of polarization for transmission & reflection

\[ P_T = \frac{T_p - T_s}{T_p + T_s} = \frac{1 - R_p - (1 - R_s)}{1 - R_p + 1 - R_s} = \frac{R_s - R_p}{2 - R_p - R_s} \]

\[ P_R = \frac{R_s - R_p}{R_s + R_p} \]

Small # layers, pol purity better in reflection

Large # layers, transmission is better
Optical propagation of light through matter

- Model light as EM wave
- Atoms or molecules modeled as dipole oscillators
- Phenomena include absorption, dispersion, polarization, electro-optical and magneto-optic effects

\[ \Rightarrow \text{Focus on absorption and dispersion} \]

3 Kinds of oscillators in medium

1. Bound electrons oscillating within atoms (most important contribution)
   \[ \Rightarrow \text{Atomic oscillators} \]

2. Vibrational oscillators - oscillate in infrared region, resonate at lower frequencies (responsible for long λ cutoff in transmission many materials)

3. Free electron oscillators responsible for properties of metals

References:
- Fowles - Introduction to Modern Optics
- Fox - Optical Properties of Solids
Atomic oscillators

Oscillating electric dipole emits E&M waves.

Model: electron held in stable orbit w/ respect to nucleus

Spring = restoring force for small displacements from equilibrium

Natural resonant frequency

\[ \frac{1}{\omega} = \frac{1}{m_0} + \frac{1}{M_N} \text{ nucleus} \]

Reduced mass

\[ m_N \gg m_0 \Rightarrow \frac{1}{\omega} \approx \frac{1}{m_0} \]

\[ \omega_0 = \sqrt{\frac{k_s}{m}} \]

Dipole moment

\[ P = q \times \xi \text{ separation between } +1 \text{- charge} \]

For atomic oscillator

\[ P = -e \times \xi(t) \text{ time varying displacement of electron from nucleus} \]

Radiate E&M waves at resonance frequency \( \omega_0 \)
Incident light

1. On resonance (corresponds to resonance frequencies of atom) → induces large amplitude oscillations and transfers energy from wave to atom → absorption

2. Off resonance → medium transparent and will not absorb oscillations of atoms follow those of driving wave but with phase lag.
   Phase lag typical for forced oscillations & caused by damping
   Oscillating atoms reradiate instantaneously but 0 lag accumulates through medium and retards propagating wave front
   → propagation velocity smaller than free space

   → reduction in velocity in medium is characterized by refractive index

Quantitative model

- Start w/ EM light wave & atom w/ single resonance frequency
- Model displacement of atomic dipoles as damped harmonic oscillators
Damping harmonic oscillations can lose energy through collisions

An electric field induces oscillations of dipole through driving force exerted on electrons

\( m_N \gg m_0 \Rightarrow \) Ignore motion of nucleus

(1) \( m_0 \frac{d^2x}{dt^2} + m_0 \delta \frac{dx}{dt} + m_0 \omega_0^2 x = -e \overline{E} \)

\( \delta = \) damping rate \( e = \) charge on electron

(2) \( \overline{E}(t) = E_0 \text{ Re}\{e^{-i(\omega t + \varphi)}\} \)

Look for solutions of form

(3) \( x(t) = X_0 \text{ Re}\{e^{-i(\omega t + \varphi)}\} \)

Put (2) \( \rightarrow \) (1)

\(-m_0 \omega^2 X_0 e^{-i\omega t} - i m_0 \delta \omega X_0 e^{-i\omega t} + m_0 \omega_0^2 X e^{-i\omega t} = -e E_0 e^{-i\omega t} \)

\( \Rightarrow \)

\[ X_0 = \frac{-e E_0 / m_0}{\omega_0^2 - \omega^2 - i \delta \omega} \]
Displacement electrons $\Rightarrow$ time varying dipole moment

\[ \vec{P} = \vec{N} \vec{p} = -N e \]

\[ \frac{Ne^2}{m_0} \left[ \frac{1}{w_0^2 - \omega^2 - i \delta \omega} \right] \vec{E} \]

\( \vec{P} \) small unless close to \( w_0 \)
General property forced oscillator (response small unless close to resonance)

\[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi) \vec{E} \]

\[ = \varepsilon_r \varepsilon_0 \vec{E} = \varepsilon_0 n^2 \vec{E} \]

\[ \vec{P} = \varepsilon_0 \chi \vec{E} \]

\[ \chi = \frac{Ne^2/m_0}{(w_0^2 - \omega^2 - i \delta \omega)} \]

\[ n^2 = 1 + \chi \]

\[ n^2 = 1 + \frac{Ne^2}{\varepsilon_0 m (w_0^2 - \omega^2 - i \delta \omega)} \]

\[ \tilde{n} = n + i \chi \]

\[ \tilde{n}^2 = n^2 - \chi^2 + 2i \chi \]
\[ n^2 - k^2 = 1 + \frac{Ne^2}{m \varepsilon_0} \left( \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \delta^2 \omega^2} \right) \]

\[ \omega_0^2 = \frac{Ne^2}{m \varepsilon_0} \left( \frac{\delta \omega}{(\omega_0^2 - \omega^2)^2 + \delta^2 \omega^2} \right) \]

**Dilute media**

Use \( \sqrt{1 + \delta} \approx 1 + \frac{\delta}{2} \)

Approximate

\[ n \approx 1 + \frac{Ne^2}{2 \varepsilon_0 m (\omega_0^2 - \omega^2 - i \delta \omega)} \]

\[ n \approx 1 + \frac{Ne^2 (\omega_0^2 - \omega^2)}{2 \varepsilon_0 m \left[ (\omega_0^2 - \omega^2)^2 + \omega^2 \delta^2 \right]} + \frac{i Ne^2 \delta \omega}{2 \varepsilon_0 m \left[ (\omega_0^2 - \omega^2)^2 + \omega^2 \delta^2 \right]} \]
Multiple Resonances

In general, an optical medium will have many characteristic resonances.

Total polarization given by:

$$\vec{P} = \frac{Ne^2}{m_0} \sum \frac{f_j}{(\omega_j^2 - \omega^2 - i\delta_j \omega)}$$

$f_j =$ oscillator strength

$\Rightarrow$ absorption different between different atomic transitions

$\Rightarrow$ variation of quantum mechanical transition probability

$$n^2 = 1 + \frac{Ne^2}{m_0} \sum \left( \frac{f_j}{\omega_j^2 - \omega^2 - i\delta_j \omega} \right)$$

Index of refraction & extinction coefficient for substance w/ d in IR, VIS, UV