- Assume medium is homogeneous, non-absorbing & magnetically isotropic.

\[ U_e = \text{energy density} = \frac{1}{2} E \cdot D = \frac{1}{2} E_i E_j \]

\[ \frac{dU_e}{dt} = \frac{1}{2} E_i j \left( \frac{dE_i}{dt} E_j + E_i \frac{dE_j}{dt} \right) \]

From Poynting vector theorem
\[ S = E \cdot \frac{dE}{dt} + \nabla \cdot \mathbf{D} \]

\[ + \mathbf{J} \cdot (\mathbf{E} \times \mathbf{H}) = E_i E_j \frac{dE_i}{dt} + \mathbf{H} \cdot \frac{dB}{dt} \]

Quasi looking energy in electric field
\[ \frac{1}{2} E_i j \left( \frac{dE_i}{dt} E_j + E_i \frac{dE_j}{dt} \right) = E_i j E_i \frac{dE_i}{dt} \]

Haus at most six independent elements

- Dielectric tensor symmetric

- Given direction of propagation, there are two eigen waves w/ well defined eigen-phase velocities & polarization directions

- Assume monochromatic plane wave

\[ E = \exp \left( i (\omega t - \mathbf{k} \cdot \mathbf{r}) \right) \]

\[ H = \exp \left( i (\omega t - \mathbf{k} \cdot \mathbf{r}) \right) \]

\[ \mathbf{k} = \frac{\omega}{c} \mathbf{n} \mathbf{s} \]

\[ \mathbf{s} = \text{unit vector in direction of propagation} \]
Substitute into Maxwell equations:

\[
\overline{E} \times \overline{E} = \mu \overline{H}
\]

\[
\overline{H} \times \overline{H} = -\omega \varepsilon \overline{E}
\]

Eliminate \( \overline{H} \)

\(1\) \( \overline{E} \times (\overline{E} \times \overline{E}) + \omega^2 \mu \varepsilon \overline{E} = 0 \)

with principal coordinate system:

\(2\) \( \varepsilon = \begin{pmatrix}
\varepsilon_x & 0 & 0 \\
0 & \varepsilon_y & 0 \\
0 & 0 & \varepsilon_z
\end{pmatrix} \)

Combining \(1\) & \(2\)

\[
\begin{pmatrix}
\omega^2 \mu \varepsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\
k_y k_x & \omega^2 \mu \varepsilon_y - k_x^2 - k_z^2 & k_y k_z \\
k_z k_x & k_z k_y & \omega^2 \mu \varepsilon_z - k_x^2 - k_y^2
\end{pmatrix} \begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix} = 0
\]

(a) \( \det \begin{vmatrix}
\omega^2 \mu \varepsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\
k_y k_x & \omega^2 \mu \varepsilon_y - k_x^2 - k_z^2 & k_y k_z \\
k_z k_x & k_z k_y & \omega^2 \mu \varepsilon_z - k_x^2 - k_y^2
\end{vmatrix} = 0 \)
- represent by normal surface

3D surface in k space (k surface or wavevector surface)

To construct surface, consider any coordinate plane ex. x-y, k_z = 0 & determinant becomes

\[
\begin{vmatrix}
\left(\frac{n_z w}{c}\right)^2 - k_x^2 - k_y^2 \\
\left(\frac{n_y w}{c}\right)^2 - k_x^2 - k_y^2 \\
\left(\frac{n_x w}{c}\right)^2 - k_x^2 - k_y^2
\end{vmatrix} = 0
\]

Either or both factors must be zero

1st factor = 0
eqn of circle
\[k_x^2 + k_y^2 = \left(\frac{n_z w}{c}\right)^2\]

2nd factor = 0
eqn of ellipse
\[\frac{k_x^2}{\left(\frac{n_x w}{c}\right)^2} + \frac{k_y^2}{\left(\frac{n_y w}{c}\right)^2} = 1\]

Intersect of k surface with each coordinate plane consists of one circle & one ellipse

Show slide
- any direction of wave vector \( \mathbf{k} \Rightarrow \) two possible values for wavenumber \( k \) 
\( \Rightarrow \) two values of phase velocity that correspond to two values of polarization

- optic axis: inner + outer sheets intersect
  - one optic axis = uniaxial
  - two optic axes = biaxial

- only one corresponding electric field direction

\[
\begin{align*}
\begin{pmatrix}
k_1 \\
k_2
\end{pmatrix} &= \begin{pmatrix}
\frac{1}{k_x^2 - w^2 \varepsilon_{x} x} & \frac{1}{k_y^2 - w^2 \varepsilon_{y} y} \\
\frac{k_y}{k_x^2 - w^2 \varepsilon_{x} x} & \frac{k_x}{k_y^2 - w^2 \varepsilon_{y} y}
\end{pmatrix}
\end{align*}
\]

- optic axes are directions in crystal where two values of phase velocity are equal

... (A) \( \Rightarrow \) (B) can be written in terms of circumscribed cosines of wave vector

\( k = \frac{\omega}{c} n \hat{s} \)

Fresnel equation of wave normals

\[
\begin{align*}
\frac{S_x^2}{n^2 - \varepsilon_x \varepsilon_\text{o}} + \frac{S_y^2}{n^2 - \varepsilon_y \varepsilon_\text{o}} + \frac{S_z^2}{n^2 - \varepsilon_z \varepsilon_\text{o}} &= 1
\end{align*}
\]

with

\[
\begin{pmatrix}
\frac{S_x}{n^2 - \varepsilon_x \varepsilon_\text{o}} \\
\frac{S_y}{n^2 - \varepsilon_y \varepsilon_\text{o}} \\
\frac{S_z}{n^2 - \varepsilon_z \varepsilon_\text{o}}
\end{pmatrix}
\]
- Fresnel equation of wave normals as quadratic in \( n^2 \) gives two solutions for each \((s_x, s_y, s_z)\).

\( \Rightarrow \) complete solution \( \Rightarrow \) use values of \( n^2 \) one at a time in (c) \( \Rightarrow \) polarization

\( \Rightarrow \) linearly polarized in a non-absorbing medium.

Maxwell's equations

\( D = \frac{n}{c} \cdot \bar{s} \times \bar{H} \quad \nabla \cdot \bar{D} = 0 \)

\( H = \frac{n}{u \cdot c} \cdot \bar{s} \times \bar{E} \)

\( \Rightarrow \bar{D} \& \bar{H} \) both \perp \text{ to direction of propagation} \( \bar{s} \)

\( \Rightarrow \) direction of energy flow given by Poynting vector \( \bar{S} \times \bar{H} \) not collinear with direction of propagation \( \bar{s} \) \( \Rightarrow \) energy not flow in direction of propagation if polarization of wave not eigen-

\( \Rightarrow \) substitute \((\bar{E}) \Rightarrow (D) \) and using \( \bar{S} \times (\bar{E} \times \bar{C}) = \frac{\bar{B}}{\bar{B}} (\bar{S} \cdot \bar{C}) - \bar{C} (\bar{S} \cdot \bar{B}) \)

\( \Rightarrow \bar{D} = -\frac{n^2}{c^2 u} \cdot \bar{s} \times (\bar{s} \times \bar{E}) = \frac{n^2}{c^2 u} \bar{E} \text{ transverse} \)

\( \bar{s} \cdot \bar{D} = 0 \quad \& \quad \frac{n^2}{c^2 u} = n^2 \varepsilon_0 \)

\( \Rightarrow D^2 = \frac{n^2}{c^2 u} \bar{E} \cdot \bar{D} = n^2 \varepsilon_0 \bar{E} \cdot \bar{D} \)
- $E, F, G, S$ are all on the same plane (all orthogonal to $H$)
- Orthogonally related to eigen modes of propagation

$\mathbf{S} \cdot (\mathbf{E}_1 \times \mathbf{H}_2) = 0$

Power flow in anisotropic medium is the sum of the power carried by each mode individually.

$\mathbf{S} = \mathbf{S}_0 (\mathbf{E}_1 \times \mathbf{H}_1) + \mathbf{S}_0 (\mathbf{E}_2 \times \mathbf{H}_2)$

Index Ellipsoid / Optical indicatrix / Ellipsoid of wave normals:

Surfaces of constant energy density $\omega$ in $D$ space can be written as:

$$\frac{Dx^2}{\varepsilon_x} + \frac{Dy^2}{\varepsilon_y} + \frac{Dz^2}{\varepsilon_z} = 2\omega$$

or (in principal coordinate system) $E_{22} E_1^2 = E_{22} E_2^2 + E_{33} E_3^2 = 2\omega$.

$E_x, E_y, E_z$ are principal dielectric constants.

$\sqrt{D\varepsilon_0 \omega} \left[ \frac{(Dx^2)}{\varepsilon_x} + \frac{(Dy^2)}{\varepsilon_y} + \frac{(Dz^2)}{\varepsilon_z} \right] = \omega$

- Replace $D$ by $-\frac{1}{\sqrt{\varepsilon_0 \omega}}$

- Define principal indices of refraction $n_x, n_y, n_z$

$$n_i^2 = \frac{\varepsilon_i}{\varepsilon_0}$$

Can write (G) as:

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

Defines an ellipsoid

Intersect $x$ at $+/- n_x$, $y$ at $+/- n_y$, $z$ at $+/- n_z$.
Index ellipsoid

- Equation of general ellipsoid with major axes parallel to x/y/z directions with lengths 2nx, 2ny, 2nz

Direction of propagation

1) Linearly polarized plane wave with E field aligned with one of principal axes
   (i) propagates with phase velocity c/n

2) Arbitrary direction
   - Normal modes are linearly polarized
   - Find modes from index ellipsoid
     \[ \bar{D} = -\frac{n^2}{c^2} \bar{E} \times (\bar{S} \times \bar{E}) \] (from Maxwell)

Using index ellipsoid to solve problems
- Used to find E vectors & corresponding indices of refraction
- Find intersection ellipse between a plane through origin that is normal to direction of propagation \( \bar{S} \) & index ellipsoid
- Two axes of intersection ellipse
  - Length 2n1 & 2n2
  - Axes are parallel to directions of \( \bar{D}_{1,2} \) that are solutions

Steps

1) Given propagation direction of ray, draw on x-y-z plane

2) Draw plane normal to ray & containing origin

3) Intersection gives ellipse of plane w/ ellipsoid
4) Principal axes of ellipse gives direction of a vector of 2 linear eigenvectors, length at semimajor & semiminor axes gives index of refraction along eigenpolarizations.

For proof of index ellipsoid, see Varir & Gun, "Optics of Liquid Crystal Displays" or Varir & Yeh, "Optical Waves in Crystals".

* Normal surface is uniquely determined by principal indices of refraction $n_x, n_y, \text{ & } n_z$

* Uniaxial & Biaxial Crystals

Equation for normal surface (reduces down from det $3 \times 3$):

$$\left(\frac{k_y^2 + k_y^2}{n_e^2} + \frac{k_z^2}{n_0^2} - \frac{w^2}{c^2}\right)\left(\frac{k^2}{n_0^2} - \frac{w^2}{c^2}\right) = 0.$$
sphere & ellipsoid are normal surface

\[ \Rightarrow \text{touch at 2 points on } z \text{ axis } \Rightarrow \]

\[ z \text{ axis is only optic axis } \Rightarrow \text{uniaxial} \]

\[ n_x < n_y < n_z \quad \text{positive uniaxial} \quad n_x > n_y > n_z \quad \text{negative uniaxial} \quad \text{isotropic} \]

\[ n_x = n_y = n_z \]

normal surface becomes a sphere

ex: cubic crystal

Biaxial crystal

\[ n_x < n_y < n_z \]

- with this convention, optic axes in \( x - z \) plane

Intersection of normal surface with \( xz \) plane

![Diagram showing biaxial and uniaxial crystals]

light propagation in uniaxial crystals

ex: quartz, calcite, lithium niobate

index ellipsoid

\[ \frac{x^2}{n_0^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1 \]

axes of symmetry chosen to be \( z \) axis (convention)
- $\overline{OA}$ length = $n_e \ell \theta$ of extraordinary ray, whose electric displacement vector $\overline{DE}$ is parallel to $\overline{OA}$

- Ordinary ray is polarized (P vector) along $\overline{OB}$, index of refraction = $n_0$

\[
\overrightarrow{d}_o = \frac{\overrightarrow{k} \times \overrightarrow{c}}{|\overrightarrow{k} \times \overrightarrow{c}|} \quad \overrightarrow{k} = \text{wavevector} \quad \overrightarrow{c} = \text{unit vector in direction of c axis (optic axis) or z axis)}
\]

\[
\overrightarrow{d}_{te} = \frac{\overrightarrow{d}_o \times \overrightarrow{k}}{|\overrightarrow{d}_o \times \overrightarrow{k}|}
\]

(polarization of displacement vectors)

\[
\overrightarrow{D} = -\frac{n}{c} \overrightarrow{s} \times \overrightarrow{H}
\]

\[
\overrightarrow{H} = \frac{n}{\omega c} \overrightarrow{s} \times \overrightarrow{E}
\]

\[
\overrightarrow{s} \cdot \overrightarrow{D} = 0
\]

Remember:

- $\theta$ (angle between optic axis and direction of propagation) changes
- Direction of polarization of ordinary ray remains fixed if $n = n_0$
- $\overline{De}(\theta)$ and index of refraction ranges from $n_0(\theta) = n_0$ for $\theta = 0^\circ$ to $n_e(\theta) = n_e$ for $\theta = 90^\circ$

- $n_e(\theta) = 0$, which can be written as

\[
\frac{1}{n_e^2(\theta)} = \frac{\cos^2(\theta)}{n_0^2} + \frac{\sin^2(\theta)}{n_e^2}
\]

Alternate solution: Determine index of refraction from normal surfaces.

\[
\left( \frac{k_x^2 + k_y^2}{n_e^2} + \frac{k_z^2}{n_0^2} - \frac{w^2}{c^2} \right) \left( \frac{k_x^2}{n_0^2} - \frac{w^2}{c^2} \right) = 0
\]

Substitute $k_z = n \left( \frac{w}{c} \right) \cos \theta$

Obtain

\[
\left\{ \begin{array}{l}
\frac{1}{n_e^2(\theta)} = \frac{\cos^2(\theta)}{n_0^2} + \frac{\sin^2(\theta)}{n_e^2} \\
n_0(\theta) = n_0 \text{ (2nd factor)}
\end{array} \right.
\]

Direction of polarization for extraordinary electric field

\[
\left( \begin{array}{c}
k_x \\
k_y \\
k_z
\end{array} \right) = \left( \begin{array}{c}
\frac{\sin \theta}{n_e^2(\theta) - n_0^2} \\
\frac{\cos \theta}{n_e^2(\theta) - n_0^2}
\end{array} \right)
\]

$Ex = 0$ since direction of propagation in $y/z$ plane & $k_y = 0$
- In general, electric field is not \( \parallel \) to propagation vector

\[ \text{E}_x \ y \ z \text{ plane intersects w/ normal surface for positive uniaxial crystal} \]

\[ \text{Summary} \]
- Propagation of light in uniaxial crystal consists of ordinary and extraordinary wave

  - **Ordinary wave**  $\mathbf{E} \parallel \mathbf{D}$ are perpendicular to c-axis & propagation vector
    - $\mathbf{E} \parallel \mathbf{D}$
    - Phase velocity always $c/\eta_0$ regardless of direction

  - **Extraordinary wave**  $\mathbf{D}_e \perp \mathbf{k}$
    - $\mathbf{E}_e$ is not perpendicular to propagation vector
    - In plane formed by $\mathbf{k}$ & $\mathbf{D}_e$

- $\mathbf{E}_e \parallel \mathbf{E}_0$ are orthogonal

\[ \mathbf{S} = \mathbf{E} \times \mathbf{H} \]
Phase, Energy & Group Velocity

- Normal surface (surface of constant \( w \) in \( k \) space) contains information about phase & group velocity

  - Phase velocity
    \[ u_p = \frac{w}{k} \hat{S} \]

  - Group velocity
    \[ u_g = \nabla_k w(k) \text{ vector normal to normal surface} \]

  - Velocity of energy flow
    \[ u_e = \frac{\hat{S}}{w} \text{ Poynting vector energy density} \]

Proof: \( u_g = u_e \) for anisotropic media.

Wavepacket is superposition of many monochromatic plane waves, but each component satisfies following:

1. \( k \times \vec{E} = w u \hat{H} \)
2. \( k \times \hat{H} = -w e \vec{E} \)

Change quantities by: \( \delta k, \delta w, \delta \vec{E}, \delta \hat{H} \)

* Apply chain rule

1. \( \delta k \times \vec{E} + k \times \delta \vec{E} = \delta w u \hat{H} + w u \delta \hat{H} \)
2. \( \delta k \times \hat{H} + k \times \delta \hat{H} = -\delta w e \vec{E} - w e \delta \vec{E} \)
Scalar
Multiply 1 by \( \vec{f} \) and 2 by \( \vec{E} \)
and use
Identity \( \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \)

Result

3. \( 8 \vec{k} \cdot (\vec{E} \times \vec{H}) + \vec{k} \cdot (\vec{S} \vec{E} \times \vec{H}) = \delta w (\vec{H} \cdot \vec{u} \vec{H}) + \omega (\vec{H} \cdot \vec{n} \vec{H}) \)

4. \( -8 \vec{k} \cdot (\vec{E} \times \vec{H}) + \vec{k} \cdot (\vec{S} \vec{H} \times \vec{E}) = -\delta w (\vec{E} \cdot \vec{e} \vec{E}) - \omega (\vec{E} \cdot \vec{e} \vec{d} \vec{E}) \)

Subtract: \( 3 - 4 \)

5. \( 2 \delta \vec{k} \cdot (\vec{E} \times \vec{H}) - \delta w (\vec{E} \cdot \vec{e} \vec{E} + \vec{H} \cdot \vec{n} \vec{H}) \)

\[= \delta \vec{H} \cdot (\vec{w} \vec{u} \vec{H} - \vec{k} \times \vec{E}) + \delta \vec{E} \cdot (\omega \vec{e} \vec{E} + \vec{R} \times \vec{H}) \]

Use Symmetry
\( \vec{H} \cdot \vec{u} \vec{H} = \delta \vec{H} \cdot \vec{n} \vec{H} \)

Right hand side of 5 vanishes (see 3 & 4)

\( \delta \vec{k} \cdot (\vec{E} \times \vec{H}) = \delta w \left( \frac{1}{2} (\vec{E} \cdot \vec{e} \vec{E} + \vec{H} \cdot \vec{n} \vec{H}) \right) \)

\( \delta w = \frac{\delta k \cdot \mathbf{S}}{\nu} = \frac{\delta \vec{k} \cdot \vec{u} \vec{e}}{\nu} \)

Definition of group velocity \( \delta w = (\nabla_k \nu) \cdot \delta \vec{u} \Rightarrow \delta \vec{u} = \delta \vec{u} \cdot \vec{u} \delta \)

\[\Rightarrow \vec{V}_g = \vec{V}_e \]
- Poynting vector is parallel to group velocity (energy flow perpendicular to k surface)

- Normal surfaces for uniaxial crystal

\[
\left( \frac{k x^2 + k y^2}{n_e^2} + \frac{k z^2}{n_o^2} - \frac{w^2}{c^2} \right) \left( \frac{k_e^2}{n_o^2} - \frac{w^2}{c^2} \right) = 0
\]

- yz plane

\[\frac{k x^2 + k y^2}{n_e^2} + \frac{k z^2}{n_o^2} - \frac{w^2}{c^2} = 0\]

ordinary (spherical surface)

extraordinary (non-spherical surface)

Ray velocity (u) along ray direction (visualize velocity of propagation of the light energy, slowness rays in all directions)

\[u = \frac{v}{\cos \Theta}\]

v = phase velocity, \(\Theta\) is angle between \(k\) & \(s\)
Ray velocity surface gives magnitude of ray velocity for any given direction of the ray. Start with wave equation:

\[ \mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{E}) + \omega^2 \mathbf{\nabla} \mathbf{E} = 0 \]

Substitute \( \mathbf{m} \) for \( \mathbf{D} \):

\[ \mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{E}) = -\frac{\omega^2}{c^2 \varepsilon_0} \mathbf{D} \quad (\mathbf{D} \perp \mathbf{k}) \]

Using identity:

\[ \mathbf{\nabla} (\mathbf{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\omega^2}{c^2 \varepsilon_0} \mathbf{D} \]

Take dot product with \( \mathbf{D} \) \( (\mathbf{E}, \mathbf{D} = 0) \):

\[ k^2 \mathbf{E} \cdot \mathbf{D} = \frac{\omega^2}{c^2 \varepsilon_0} \mathbf{D} \cdot \mathbf{D} \]

Since \( \nu = \frac{\omega}{k} \):

\[ \mathbf{E} \cdot \mathbf{D} = ED \cos \theta = \frac{\nu^2}{c^2 \varepsilon_0} D^2 \]

If coordinate axes are principal axes of crystal:

\[ \varepsilon_0 E_x = \frac{D_x}{\varepsilon_{11}} = \frac{D_x}{n^2} \quad \text{for similarly for } y \text{ and } z \text{ components} \]

This equation gives 3 scalar equations.
\[ D_x \left( \frac{c^2}{n_x^2} - u_y^2 - u_z^2 \right) + D_y u_x u_y + D_z u_x u_z = 0 \]

\[ D_x u_y u_x + D_y \left( \frac{c^2}{n_y^2} - u_x^2 - u_z^2 \right) + D_z u_y u_z = 0 \]

\[ D_x u_z u_x + D_y u_y u_z + D_z \left( \frac{c^2}{n_z^2} - u_x^2 - u_y^2 \right) = 0 \]

For non-trivial solutions, determinant of coefficients must vanish.

\[
\begin{vmatrix}
\frac{c^2}{n_x^2} - u_y^2 - u_z^2 & u_x u_y & u_x u_z \\
u_y u_x & \frac{c^2}{n_y^2} - u_x^2 - u_z^2 & u_y u_z \\
u_z u_x & u_z u_y & \frac{c^2}{n_z^2} - u_x^2 - u_y^2
\end{vmatrix} = 0
\]

Ray velocity surface

- Equations of intercepts in x-y plane are obtained by setting \( u_z = 0 \)
- Result gives circle & ellipse
\[
\begin{align*}
    u_x^2 + u_y^2 &= \frac{c^2}{n_z^2} \\
    n_y u_x^2 + n_x u_y^2 &= c^2
\end{align*}
\]

Can do the same for other coordinate planes.

Ray velocity surface consists of 2 sheets (inner & outer one) corresponding to two possible values of \( \omega \) for a given ray direction.

- As with phase velocity surface, \( \omega \) is constant.

Ray velocity surface sheets touch at point \( Q \) that defines a direction where two ray velocities are equal ⇒ ray axis of crystal.

- Biaxial crystal: Ray axis that are distinct from optic axes of crystal.

- Uniaxial crystal: Ray axis is coincident with optic axis of crystal.
Phase velocity surface: reciprocal to \( \overline{k} \) surface, where \( w(k) = \text{constant} \)

\[
V = \frac{w}{k}
\]

\[
\overline{k} = \frac{V}{w^2}
\]

Can write as 3 scalar equations:

\[
k_x = \frac{V_x w}{V^2} \quad k_y = \frac{V_y w}{V^2} \quad k_z = \frac{V_z w}{V^2}
\]

Substitute into 3x3 matrix for \( k \) surface:

\[
\begin{bmatrix}
\frac{n_x^2 v^4}{c^2} - v_y^2 - v_z^2 & v_x v_y & v_x v_z \\
v_y v_x & \frac{n_y^2 v^4}{c^2} - v_x^2 - v_z^2 & v_y v_z \\
v_z v_x & v_z v_y & \frac{n_z^2 v^4}{c^2} - v_x^2 - v_y^2
\end{bmatrix} = 0
\]

Intercepts with planes consist of circles & 4th degree ovals.

See slide.
Double Refraction

- Plane wave incident on anisotropic media
  - Refracted wave is mixture of two eigenmodes
  - Boundary conditions: all wave vectors lie in plane of incidence & tangential components along boundary be same

\[
k_0 \sin \Theta_0 = k_1 \sin \Theta_1 = k_2 \sin \Theta_2
\]

- Incident wave
- 1st propagation vector of refracted wave
- 2nd propagation vector of refracted wave

- **NOT Snell's Law because:**
  - \( k_1 \) & \( k_2 \) are not constant but vary with directions \( \overline{k_1} \) & \( \overline{k_2} \)

Uniaxial crystals

- One shell or normal surface is a sphere \( \Rightarrow \) ordinary wave

\[
N_i \sin \Theta_0 = N_0 \sin \Theta_1 \quad \text{Snell's Law}
\]

- \( N_i \) is index of refraction of incident medium
- \( N_0 \) is ordinary refractive index of crystal

- Second shell is ellipsoid
  - \( k \) depends on direction of propagation
  - Snell's Law is not valid, but can solve graphically
Illustration of double refraction

intersection of normal surface with plane of incidence

See slide 2

Special cases
- Normally incident light on crystal face cut \perp\ to optic axis \Rightarrow 2 ray velocities are equal
  \Rightarrow no double refraction
Light propagation in biaxiral crystals

- Normal surface \Rightarrow [show slide] \quad (\text{w/k}) = \text{constant}
- Consider intersections w/ 3 coordinate planes

\text{Ex}l \quad \text{set } k_y = 0

\begin{align*}
\frac{k_x^2 + k_z^2}{n_y^2} &= \left(\frac{w}{c}\right)^2 \quad \text{circle with radius } n_y w/c \\
\frac{k_x^2}{n_z^2} + \frac{k_z^2}{n_x^2} &= \left(\frac{w}{c}\right)^2 \quad \text{ellipse with semi-axes } n_x w/c \text{ and } n_z w/c
\end{align*}

- Same for other two coordinate planes
- Circle & ellipse only intersect for \( k_y = 0 \) as a result of choice of coordinate axes \( n_x < n_y < n_z \)
- 4 points of intersection determine two optic axes of crystal

See slide
Consider \( k_y = 0 \)

- \( \alpha \) is direction of propagation
- modes associated with circle are polarized \( \perp \) to plane \( k_y = 0 \)
- modes associated \( \perp \) ellipse are polarized in direction of ellipse
- optic axes light propagating along has a unique phase velocity regardless of state of polarization
- \( \nu_g \) is undefined because two shells of normal surface degenerate to point

**Conical retraction**

- Draw unit vectors \( \perp \) normal surface close to singularity \( \Rightarrow \) unit vectors indicating flow of energy
- Vectors form a cone!

**Normal surface near optic axis \( (k_0) \)**

\[
\bar{k}_0 = \hat{x} k_{x0} + \hat{y} k_{y0} + \hat{z} k_{z0}
\]

\[
k_{x0} = n_y \frac{\nu}{c} \sin \theta
\]

\[
k_{y0} = n_x \frac{\nu}{c} \cos \theta
\]

\[
k_{z0} = 0
\]
\( \theta \) is angle between optic axis and \( z \) axis.

- \( \tan \theta = \frac{n_z}{n_x} \left( \frac{n_y^2 - n_z^2}{n_x^2 - n_y^2} \right)^{1/2} \)

- Taylor expand around \( k_0 \)
  \[
  \begin{aligned}
  k_x &= k_{x0} + \xi \\
  k_y &= k_{y0} + \eta \\
  k_z &= k_{z0} + \varsigma
  \end{aligned}
  \]

- Substitute into determinant for normal surface

Neglect higher order terms in \( \xi, \eta, \varsigma \)

\[
\begin{aligned}
4 \left( k_{x0} \xi + k_{z0} \varsigma \right) \\
&\left( n_x^2 k_{x0} \xi + n_z^2 k_{z0} \varsigma \right) \\
&+ \eta^2 \left( n_y^2 - n_x^2 \right) \left( n_y^2 - n_z^2 \right) = 0
\end{aligned}
\]

Represents a cone with vertex at \( k_0 \) (see slide)

(\( \xi, \eta, \varsigma = 0 \)); each direction represents linearly polarized state

\[\text{EX} \] Plate of biaxial crystal (mica)

cut so \( 2 \) \( 11 \) surfaces are \( \pm \) to one optic axis

- If illuminated with unpolarized collimated monochromatic light