and, consequently, \( \chi \) must be expressed as a tensor. We shall find that the \( \chi \) tensor of a crystal summarizes most of its optical properties.

### 6.3 The General Wave Equation

In our study of solid-state optics we shall be concerned only with nonmagnetic, electrically neutral media. Hence \( M \) and \( p \) are both zero. Maxwell's equations, in the form expressed by Equations (6.1) to (6.4), then reduce to the following:

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} & (6.10) \\
\nabla \times \mathbf{H} &= \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} & (6.11) \\
\n\nabla \cdot \mathbf{E} &= -\frac{1}{\varepsilon_0} \nabla \cdot \mathbf{P} & (6.12) \\
\n\nabla \cdot \mathbf{H} &= 0 & (6.13)
\end{align*}
\]

The general wave equation for the \( \mathbf{E} \) field is obtained by taking the curl of Equation (6.10) and the time derivative of Equation (6.11) and eliminating \( \mathbf{H} \). The result is

\[
\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial \mathbf{P}}{\partial t} - \mu_0 \frac{\partial \mathbf{J}}{\partial t} & (6.14)
\]

The two terms on the right-hand side of the above equation are called source terms. They stem from the presence of polarization charges and conduction charges, respectively, within the medium. The way in which the propagation of light is affected by the sources is revealed by the solution of the wave equation when the source terms are included. In the case of nonconducting media the polarization term \(-\mu_0 \partial \mathbf{E}/\partial t^2\) is of importance. It turns out that this term leads to an explanation of many optical effects, including dispersion, absorption, double refraction, and optical activity to mention only a few. In the case of metals it is the conduction term \(-\mu_0 \partial \mathbf{J}/\partial t\) that is important, and the resulting solutions of the wave equation explain the large opacity and high reflectance of metals. Both source terms must be taken into account in the case of semiconductors. The result is a rather complicated wave equation and the solutions are somewhat difficult to interpret. Nevertheless, a qualitative description of many of the optical properties of semiconductors is furnished by classical theory. A rigorous treatment of semiconductor optics must await the application of quantum theory.

### 6.4 Propagation of Light in Isotropic Dielectrics. Dispersion

In a nonconducting, isotropic medium, the electrons are permanently bound to the atoms comprising the medium and there is no preferential direction. This is what is meant by a simple isotropic dielectric such as glass. Suppose that each electron, of charge \(-e\), in a dielectric is displaced a distance \( r \) from its equilibrium position. The resulting macroscopic polarization \( \mathbf{P} \) of the medium is given by

\[
\mathbf{P} = -N e \mathbf{r} & (6.15)
\]

where \( N \) is the number of electrons per unit volume. If the displacement of the electron is the result of the application of a static electric field \( \mathbf{E} \), and if the electron is elastically bound to its equilibrium position with a force constant \( K \), then the force equation is

\[
-e \mathbf{E} = K \mathbf{r} & (6.16)
\]

The static polarization is therefore given by

\[
\mathbf{P} = \frac{N e}{K} \mathbf{E} & (6.17)
\]

However, if the impressed field \( \mathbf{E} \) varies with time, the above equation is incorrect. In order to find the true polarization in this case, we must take the actual motion of the electrons into account. To do this we consider the bound electrons as classical damped harmonic oscillators. The differential equation of motion is

\[
m \frac{d^2 \mathbf{r}}{dt^2} + m \omega_0 \frac{d \mathbf{r}}{dt} + K \mathbf{r} = -e \mathbf{E} & (6.18)
\]

The term \( m \omega_0 \frac{d \mathbf{r}}{dt} \) represents a frictional damping force that is proportional to the velocity of the electron, the proportionality constant being written as \( m \omega_0 \).

Now suppose that the applied electric field varies harmonically with time according to the usual factor \( e^{i \omega t} \). Assuming that the motion of the electron has the same harmonic time dependence, we find that Equation (6.18) becomes

\[
(-m \omega^2 - i \omega m \omega_0 + K) \mathbf{r} = -e \mathbf{E} & (6.19)
\]

Consequently, the polarization, from Equation (6.15), is given by

\[
\mathbf{P} = \frac{N e^2}{m \omega^2 - i \omega m \omega_0 + K} \mathbf{E} & (6.20)
\]

1 The magnetic force \( e \mathbf{x} \times \mathbf{B} \) is neglected here. For electromagnetic waves, this force is normally much smaller than the electric force \( \mathbf{E} \).
It reduces to the static value, Equation (6.17), when \( \omega = 0 \). Thus for a given amplitude of the impressed electric field, the amount of polarization varies with frequency. The phase of \( P \) relative to that of the electric field, also depends on the frequency. This is shown by the presence of the imaginary term in the denominator.

A more significant way of writing Equation (6.20) is

\[
P = \frac{N\varepsilon_0\mu_0 e}{\omega^2 - \omega^2 - i\gamma\omega} E
\]

(6.21)
in which we have introduced the abbreviation \( a_0 \) given by

\[
a_0 = \frac{\gamma}{\sqrt{\kappa}}
\]

(6.22)

This is the effective resonance frequency of the bound electrons.

The polarization formula (6.21) is similar to the amplitude formula for a driven harmonic oscillator, as indeed it should be, since it is the displacement of the elastically bound electrons that actually constitutes the polarization. We should therefore expect to find an optical resonance phenomenon of some kind occurring for light frequencies in the neighborhood of the resonance frequency \( a_0 \). As we shall presently see, this resonance phenomenon is manifest as a large change in the index of refraction of the medium and also by a strong absorption of light at or near the resonance frequency.

To show how the polarization affects the propagation of light, we return to the general wave equation (6.14). For a dielectric there is no conduction term. The polarization is given by Equation (6.21). Hence we have

\[
\nabla \times (\nabla \times E) + \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = -\frac{\mu_0 N\varepsilon_0}{m} \left( \omega_0^2 - \omega^2 - i\gamma\omega \right) \frac{\partial^2 E}{\partial t^2}
\]

(6.23)

Also, from the linear relationship between \( P \) and \( E \), it follows from (6.12) that \( \nabla \cdot E = 0 \). Consequently, \( \nabla \times (\nabla \times E) = -\nabla \phi \), and the above wave equation reduces to the somewhat simpler one

\[
\nabla^2 E = \frac{1}{c^2} \left( 1 + \frac{N\varepsilon_0}{m\varepsilon_0} \right) \frac{\partial^2 E}{\partial t^2}
\]

(6.24)

after rearranging terms and using the relation \( 1/c^2 = \mu_0\varepsilon_0 \).

Let us seek a solution of the form

\[
E = E_0 e^{i(\omega t - kx)}
\]

(6.25)

This trial solution represents what are called homogeneous plane harmonic waves. Direct substitution shows that this is a possible solution provided that

\[
\omega^2 - k^2 = 1 + \frac{N\varepsilon_0}{m\varepsilon_0} \left( \omega_0^2 - \omega^2 - i\gamma\omega \right)
\]

(6.26)

The presence of the imaginary term in the denominator implies that the wavenumber \( k \) must be a complex number. Let us inquire as to the physical significance of this. We express \( k \) in terms of its real and imaginary parts as

\[
k = k + ik
\]

(6.27)

This amounts to the same thing as introducing a complex index of refraction

\[
N = n + ik
\]

(6.28)

where

\[
N = \frac{\omega_0}{c} k
\]

(6.29)

Our solution in Equation (6.25) can then be written as

\[
E = E_0 e^{i(kx - \omega t)}
\]

(6.30)

The factor \( e^{-\gamma x} \) indicates that the amplitude of the wave decreases exponentially with distance. This means that as the wave progresses, the energy of the wave is absorbed by the medium. Since the energy in the wave at a given point is proportional to \( |E|^2 \), then the energy varies with distance as \( e^{-2\gamma x} \). Hence \( 2\gamma \) is the coefficient of absorption of the medium. The imaginary part \( \kappa \) of the complex index of refraction is known as the extinction index. The two numbers \( \alpha \) and \( \kappa \) are related by the equation

\[
\alpha = \frac{\gamma}{\kappa}
\]

(6.31)

The phase factor \( e^{i\kappa x} \) indicates that we have a harmonic wave in which the phase velocity is

\[
u = \frac{\omega}{k} = \frac{c}{n}
\]

(6.32)

From Equations (6.26) and (6.29) we have

\[
n^2 = (n + ik)^2 = 1 + \frac{N\varepsilon_0}{m\varepsilon_0} \left( \frac{\omega_0^2}{\omega^2} - \omega^2 - i\gamma\omega \right)
\]

(6.33)

Equating real and imaginary parts yields the following equations:

\[
n^2 - \kappa^2 = 1 + \frac{N\varepsilon_0}{m\varepsilon_0} \left( \frac{\omega_0^2}{\omega^2} - \omega^2 - i\gamma\omega \right)
\]

(6.34)

and

\[
2nk = \frac{N\varepsilon_0}{m\varepsilon_0} \left( \frac{\omega_0^2}{\omega^2} - \omega^2 + i\gamma\omega \right)
\]

(6.35)

from which the optical parameters \( n \) and \( \kappa \) may be found.
6.4 - PROPAGATION OF LIGHT IN ISOTROPIC DIELECTRICS

\[ n^2 = 1 + \frac{Ne^2}{m_e} \sum \frac{f_j}{(\omega_j^2 - \omega^2 - i\gamma_j\omega)} \]  \hfill (6.36)

The summation extends over all the various kinds of electrons indicated by the subscript $j$. The fractions $f_j$ are known as oscillator strengths. The damping constants associated with the various frequencies are denoted by $\gamma_j$. Figure 6.2 shows graphically the general dependence of the real and imaginary parts of $n^2$ as determined by Equation (6.36). This graph is intended to show qualitatively the case for a substance, such as glass, which is transparent in the visible region and has absorption bands in the infrared and ultraviolet regions of the spectrum. In the limit of zero frequency, the square of the index approaches the value $1 + (Ne^2/m_e) \Sigma f_j/\omega_j^2$. This is just the static dielectric constant of the medium.

In the high-frequency region, the theory predicts that the index should dip below unity and then approach unity from below as $\omega$ becomes infinite. This effect is actually seen experimentally. The case of quartz is shown in Figure 6.3. Here the measured index of refraction of quartz is plotted as a function of wavelength for the appropriate region of the spectrum (x-ray region).

If the damping constants $\gamma_j$ are sufficiently small so that the terms $\gamma_j\omega$ can be neglected in comparison to the quantities $\omega_j^2 - \omega^2$ in Equation (6.36), then the index of refraction is essentially real and its square is given by

\[ n^2 = 1 + \frac{Ne^2}{m_e} \sum \frac{f_j}{\omega_j^2 - \omega^2} \]  \hfill (6.37)
6.5 Propagation of Light in Conducting Media

The decay of a transient current is governed by the associated homogeneous equation

\[ \frac{dJ}{dt} + r^{-1}J = 0 \]  

whose solution is \( J = J_0 e^{-rt} \). Thus a transient current will decay to \( e^{-1} \) of its initial value in a time \( r \). This is called the relaxation time. Now for a static electric field, Equation (6.40) becomes

\[ \frac{\tau}{m}J = \frac{Ne^2}{m}E \]

The static conductivity \( \sigma \) is therefore given by

\[ \sigma = \frac{Ne^2}{m} \tau \]

Let us now assume a harmonic time dependence \( e^{-i\omega t} \) for both electric field \( E \) and the resulting current \( J \) in our differential Equation (6.40). It follows that

\[ (-i\omega + \frac{1}{\tau})J = \frac{Ne^2}{m}E = e^{-i\omega t}E \]

Solving for \( J \), we find

\[ J = \frac{\sigma}{1 - i\omega\tau} E \]

When \( \omega = 0 \), the above equation reduces to \( J = \sigma E \), which is the correct equation for the static case.

Using the dynamic expression for \( J \), we find that the general wave Equation (6.14) reduces to

\[ \nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} + \frac{\rho_0}{\varepsilon_0 \varepsilon} \frac{\partial E}{\partial t} \]

For a trial solution we take a simple homogeneous plane-wave solution of the type

\[ E = E_0 e^{i(kx - \omega t)} \]

where, as in Equation (6.26), \( k \) is assumed to be complex. It is easily found that \( k \) must then satisfy the relation

\[ k^2 = \frac{\omega^2}{c^2} + \frac{i\omega\mu\sigma}{1 - i\omega\tau} \]

For very low frequencies the above formula reduces to the approximate formula

\[ k^2 = \frac{i\omega\mu\sigma}{1 - i\omega\tau} \]
so that \( \mathcal{H} = \sqrt{\mu_0 \mu_r \sigma} = (1 + i) \sqrt{\mu_0 \mu_r \sigma} \frac{2}{\omega_0} \). In this case the real and imaginary parts of \( \mathcal{H} = k + i\alpha \) are equal and are given by
\[
k = \alpha = \sqrt{\frac{\omega_0 \sigma}{2}} \tag{6.50}
\]

Similarly the real and imaginary parts of \( \mathcal{N} = n + i\kappa \) are equal and are given by
\[
n = \kappa = \sqrt{\frac{\sigma}{\omega_0 \epsilon_0}} \tag{6.51}
\]

The so-called “skin depth” \( \delta \) of a metal is that distance at which the amplitude of an electromagnetic wave drops to \( e^{-1} \) of its value at the surface. Thus
\[
\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega_0 \sigma \mu_0}} = \sqrt{\frac{\lambda_0}{\omega_0 \sigma \mu_0}} \tag{6.52}
\]

where \( \lambda_0 \) is the vacuum wavelength. This shows why good conductors are also highly opaque. A high value of the conductivity \( \sigma \) gives a large coefficient of absorption \( \alpha \) and a correspondingly small skin depth. For example, the skin depth in copper \( (\sigma = 5.8 \times 10^7 \text{ mho/m}) \) for 1-mm microwaves is about 10⁻³ mm.

Let us return to the more accurate expression for \( \mathcal{H} \) given in Equation (6.48). The equivalent form of this equation written in terms of the complex index of refraction, as defined by Equation (6.29), is
\[
\mathcal{N}^2 = \frac{\omega_0^2}{\omega^2} + \frac{\omega_0}{\omega - i\tau} \tag{6.53}
\]

Here we have introduced plasma frequency for the metal. It is defined by the relations
\[
\omega_p = \sqrt{\frac{Ne^2}{m_e}} = \sqrt{\frac{\mu_0 e^2}{\tau}} \tag{6.54}
\]

By equating real and imaginary parts in Equation (6.53), we find
\[
n^2 - \kappa^2 = 1 - \frac{\omega_p^2}{\omega^2} \tag{6.55}
\]
\[
2\kappa = \frac{\omega_p^2}{\omega^2} + \frac{1}{\omega \tau} \tag{6.56}
\]

from which the optical “constants” \( n \) and \( \kappa \) may be obtained. An explicit algebraic solution of the above pairs of equations is very cumbersome, hence the solutions are usually solved numerically for \( n \) and \( \kappa \). According to the above theory, these are determined entirely by the plasma frequency \( \omega_p \), the relaxation time \( \tau \), and the frequency \( \omega \) of the light wave.

Typical relaxation times for metals, as deduced from conductivity measurements, are of the order of \( 10^{-13} \) s, which corresponds to frequencies in the infrared region of the spectrum. On the other hand plasma frequencies of metals are typically around \( 10^9 \) s⁻¹, corresponding to the visible and near ultraviolet regions. Figure 6.4 shows the behavior of \( n \) and \( \kappa \) plotted as functions of \( \omega \) from Equations (6.53) and (6.56). As seen from the figure, the index of refraction \( n \) is less than unity for a wide range of frequencies in the region of the plasma frequency. The extinction coefficient \( \kappa \) is very large at low frequencies (long wavelengths). It decreases monotonically with increasing frequency, becoming very small for frequencies greater than the plasma frequency. The metal thus becomes transparent at high frequencies. Qualitative agreement with these predictions of classical theory is obtained in the case of the alkali metals and some of the better conductors such as silver, gold, and copper.

For poor conductors and semiconductors, both free electrons and bound electrons can contribute to the optical properties. Classical theory would, accordingly, yield an equation of the type
\[
\mathcal{N}^2 = 1 - \frac{\omega_p^2}{\omega^2} - \frac{Ne^2}{m_e} \sum \left( \frac{1}{\omega_j^2 - \omega^2} - \frac{1}{\omega_j^2 - \gamma_j \omega} \right) \tag{6.57}
\]

for the complex index of refraction. It turns out that quantum theory gives a similar relation and, in addition, can predict the values of the various parameters \( \gamma_j \) and \( \gamma \) and so forth. The theoretical calculations are difficult, however, as are also the experimental measurements.