Electromagnetic Wave Theory

JIN AU KONG
3. Propagation and Guidance

3.4 Reflection and Transmission

\[ e^{-ik_z z} \]

![Transmission line model.](image)

The wave impedance in region 0 is defined by the ratio of (3.4.15) to (3.4.16), which gives

\[ Z(z) = \frac{E_y}{H_x} = \frac{\omega \mu_0}{k_z} \frac{1 + Re^{i2k_z z}}{1 - Re^{i2k_z z}} \]

For a perfect conductor occupying region \( t \), \( R = -1 \). We observe that as \( 2k_z z \) varies from 0 to \( 2\pi \), the wave impedance repeats its value at \( z = 0 \). This is also related to the periodic variation of the field amplitudes in the \( z \) direction. As seen from (3.4.15),

\[ E_y = E_0 e^{ik_z z-i\phi} (1 + Re^{i2k_z z}) \]

the expression in the parentheses repeats its values over every interval of \( 2\pi \) for \( 2k_z z \). This phenomenon is best described in terms of the transmission line theory to be described in 3.5. The transmission line model for (3.4.15) and (3.4.16) is shown in Figure 3.4.11, where \( Z_t \) is the wave impedance for region \( t \).

C. Reflection and Transmission by a Layered Medium

Consider a plane wave incident on a stratified isotropic medium with boundaries at \( z = -d_0, -d_1, \ldots, -d_n \) [Fig. 3.4.12]. The \((n + 1)\) th region is semi-infinite and is labeled region \( t \), \( t = n + 1 \). The permittivity and permeability
in each region are denoted by $\varepsilon_l$ and $\mu_l$. The plane wave is incident from region 0 and has the plane of incidence parallel to the $x-z$ plane. All field vectors are dependent on $x$ and $z$ only and independent of $y$. Since $\partial/\partial y = 0$, the Maxwell equations in any region $l$ can be separated into TE and TM components governed by $E_{ly}$ and $H_{ly}$. We obtain

\begin{align*}
    H_{lx} &= \frac{1}{i\omega\mu_l} \frac{\partial}{\partial z} E_{ly} \\
    H_{lz} &= \frac{1}{i\omega\mu_l} \frac{\partial}{\partial x} E_{ly} \\
    \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_l \varepsilon_l \right) E_{ly} &= 0 \\
    E_{lx} &= \frac{1}{i\omega\varepsilon_l} \frac{\partial}{\partial z} H_{ly} \\
    E_{lz} &= \frac{1}{i\omega\varepsilon_l} \frac{\partial}{\partial x} H_{ly} \\
    \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_l \varepsilon_l \right) H_{ly} &= 0
\end{align*}

(3.4.46) (3.4.47) (3.4.48) (3.4.49) (3.4.50) (3.4.51)

We note that in region 0 where $l=1$,

\begin{align*}
    E_{ly} &= A_0 e^{ik_{0}z} \\
    H_{lx} &= -\frac{k_{0}}{\omega\mu_0} (A_0 e^{i\phi_0}) \\
    H_{lz} &= \frac{k_{0}}{\omega\mu_0} (A_0 e^{i\phi_0})
\end{align*}

Obviously (3.4.52) satisfies the phase-matching condition. Transmissions in each layer $l$. The components that have a propagating $\hat{z}$ direction, and $B_l$ represents the field in the negative $\hat{z}$ direction.

In region $t$ where $l=n+1=t$, because region $t$ is semi-infinite the velocity component in the positive direction by $T$.

The wave amplitudes $A_t$ for neighboring regions by the boundary conditions require that $E_y$ and $H_y$

\begin{align*}
    A_t e^{-ik_l z} + B_t e^{ik_l z} = A_{t+1}
\end{align*}
in each region are denoted by $\varepsilon_l$ and $\mu_l$. The plane wave is incident from region 0 and has the plane of incidence parallel to the $x - z$ plane. All field vectors are dependent on $x$ and $z$ only and independent of $y$. Since $\partial / \partial y = 0$, the Maxwell equations in any region $l$ can be separated into TE and TM components governed by $E_{ly}$ and $H_{ly}$. We obtain

$$H_{lx} = -\frac{1}{i\omega \mu_l} \frac{\partial}{\partial z} E_{ly}$$

$$H_{lz} = \frac{1}{i\omega \mu_l} \frac{\partial}{\partial x} E_{ly}$$

$$0 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_l \varepsilon_l \right) E_{ly}$$

$$E_{lx} = -\frac{1}{i\omega \varepsilon_l} \frac{\partial}{\partial z} H_{ly}$$

$$E_{lz} = -\frac{1}{i\omega \varepsilon_l} \frac{\partial}{\partial x} H_{ly}$$

3.4 Reflection and Transmission

The TE waves are completely determined by (3.4.49)–(3.4.51). The two under the replacement $E_l \rightarrow H_l$.

For a TE plane wave, $E_y = 0$ medium, the total field in region $l$

$$E_{ly} = (A_l e^{ik_x x} e^{ik_z z})$$

$$H_{lz} = -\frac{k_z}{\omega \mu_l} (A_l$$

$$H_{lz} = \frac{k_z}{\omega \mu_l} (A_l e^{ik_z z})$$

Obviously (3.4.52) satisfies the Heun equation of (3.4.52) in (3.4.48) yield

$$k_z^2 +$$

We do not write a subscript $l$ for the phase-matching conditions transmissions in each layer $l$. TM components that have a propagating $\hat{z}$ direction, and $B_l$ represents the negative $\hat{z}$ direction.

We note that in region 0 wh

In region $t$ where $l = n + 1 = t$

because region $t$ is semi-infinite velocity component in the positive $z$ direction by $T$.

The wave amplitudes $A_l$ between neighboring regions by the boundary conditions require that $E_l$ and
The TE waves are completely determined by (3.4.46)–(3.4.48) and the TM waves by (3.4.49)–(3.4.51). The two sets of equations are duals of each other under the replacements $E_l \to H_l$, $H_l \to -E_l$, and $\mu_l \to \epsilon_l$.

For a TE plane wave, $E_y = E_0 e^{-ik_x z + ik_z z}$, incident on the stratified medium, the total field in region $l$ can be written as

$$E_{ly} = \left( A_l e^{ik_x z} + B_l e^{-ik_x z} \right) e^{ik_z z} \tag{3.4.52}$$

$$H_{lx} = -\frac{k_x}{\omega \mu_l} \left( A_l e^{ik_x z} - B_l e^{-ik_x z} \right) e^{ik_z z} \tag{3.4.53}$$

$$H_{lz} = \frac{k_z}{\omega \mu_l} \left( A_l e^{ik_x z} + B_l e^{-ik_x z} \right) e^{ik_z z} \tag{3.4.54}$$

Obviously (3.4.52) satisfies the Helmholtz wave equation in (3.4.48). Substitution of (3.4.52) in (3.4.48) yields the dispersion relation

$$k_x^2 + k_z^2 = \omega^2 \mu_l \epsilon_l \tag{3.4.55}$$

We do not write a subscript $l$ for the $x$ component of $\vec{k}$ as a consequence of the phase-matching conditions. Truly, there are multiple reflections and transmissions in each layer $l$. The amplitude $A_l$ thus represents all wave components that have a propagating velocity component along the positive $z$ direction, and $B_l$ represents those with a velocity component along the negative $z$ direction.

We note that in region 0 where $l = 0$,

$$A_0 = RE_0 \tag{3.4.56}$$

$$B_0 = E_0 \tag{3.4.57}$$

In region $t$ where $l = n + 1 = t$, we have

$$A_t = 0 \tag{3.4.58}$$

$$B_t = TE_0 \tag{3.4.59}$$

because region $t$ is semi-infinite and there is no wave propagating with a velocity component in the positive $z$ direction. We denote the transmitted amplitude by $T$.

The wave amplitudes $A_l$ and $B_l$ are related to wave amplitudes in neighboring regions by the boundary conditions. At $z = -d_i$, boundary conditions require that $E_y$ and $H_x$ be continuous. We obtain

$$A_l e^{-ik_x d_i} + B_l e^{ik_x d_i} = A_{l+1} e^{-ik_{(l+1)} z d_i} + B_{l+1} e^{ik_{(l+1)} z d_i} \tag{3.4.60}$$
\[ \frac{k_{lz}}{\mu_l} \left[ A_l e^{-ik_l z} d_l - B_l e^{ik_l z} d_l \right] = \frac{k_{l+1} z}{\mu_{l+1}} \left[ A_{l+1} e^{-ik_{l+1} z} d_{l+1} - B_{l+1} e^{ik_{l+1} z} d_{l+1} \right] \]  \hspace{1cm} (3.4.61)

There are \( n+1 \) boundaries which give rise to \((2n+2)\) equations. In region 0, we have an unknown reflection coefficient \( R \). In region \( t \), we have an unknown transmission coefficient \( T \). These are two unknowns \( A_l \) and \( B_l \) in each of the regions \( l = 1, 2, \ldots, n \). Thus we have a total of \((2n+2)\) unknowns. To solve for the \((2n+2)\) unknowns from the \((2n+2)\) linear equations, we can arrange the equations in matrix form with the unknowns forming a \((2n+2)\) column matrix and the coefficients forming a \((2n+2) \times (2n+2)\) square matrix. The solution is then obtained by inverting the square matrix. This procedure is straightforward but tedious. We shall now describe simpler ways to deal with the problem.

### a. Reflection Coefficients

Since we are interested in finding the reflection coefficient for the stratified medium, let us derive a closed-form formula for \( R \). We first solve (3.4.60) and (3.4.61) for \( A_l \) and \( B_l \).

\[
A_l e^{-ik_l z} d_l = \frac{1}{2} \left( 1 + p_{l(l+1)} \right) \left\{ A_{l+1} e^{-ik_{l+1} z} d_{l+1} + R_{l(l+1)} B_{l+1} e^{ik_{l+1} z} d_{l+1} \right\} \hspace{1cm} (3.4.62)
\]

\[
B_l e^{ik_l z} d_l = \frac{1}{2} \left( 1 + p_{l(l+1)} \right) \left\{ R_{l(l+1)} A_{l+1} e^{-ik_{l+1} z} d_{l+1} + B_{l+1} e^{ik_{l+1} z} d_{l+1} \right\} \hspace{1cm} (3.4.63)
\]

where

\[
p_{l(l+1)} = \frac{\mu_l k_{l+1} z}{\mu_{l+1} k_{lz}} \hspace{1cm} (3.4.64)
\]

for TE waves and

\[
R_{l(l+1)} = \frac{1 - p_{l(l+1)}}{1 + p_{l(l+1)}} \hspace{1cm} (3.4.65)
\]

is the reflection coefficient for waves in region \( l \), caused by the boundary separating regions \( l \) and \( l+1 \). We note from (3.4.64) that

\[
p_{l(l+1)} = \frac{1}{p_{l(l+1)}} \hspace{1cm} (3.4.66)
\]

### 3.4 Reflection and Transmission

which also gives

\[
R_{l(l+1)} = \frac{1 - (1/R_{01}^2)}{(1/R_{01}) e^{ik_{l+1} z}} \hspace{1cm} (1/R_{01}) e^{ik_{l+1} z}
\]

With the second equality we fraction, Equation(3.4.68) expression, until the transmitted region

The reflection coefficient \( d \) Making use of the continued fraction solution its continued fractions. Such a solution computation.

For TM plane waves reflect of duality applies and gives rise to

The only difference is that (3.4
which also gives

\[ R_{t(t+1)} = -R_{t/(t+1)} \]  

(3.4.67)

Thus the reflection coefficient in region \( t+1 \), \( R_{t(t+1)} \), caused by the boundary separating regions \( t+1 \) and \( t \), is equal to the negative of \( R_{t/(t+1)} \).

Forming the ratio of (3.4.62) and (3.4.63) we obtain

\[
\frac{A_t}{B_t} = \frac{e^{i2k_{lt}d_t}}{R_{t/(t+1)}} + \frac{\left[ 1 - \left( \frac{1}{R_{t/(t+1)}} \right)^2 \right] e^{i2(k_{lt}+k_z)d_t}}{\left[ 1/R_{t/(t+1)} \right] e^{i2k_{lt}d_t} + (A_{t+1}/B_{t+1})} \]

(3.4.68)

\[
= \frac{e^{i2k_{lt}d_t}}{R_{t/(t+1)}} + \frac{\left[ 1 - \left( \frac{1}{R_{t/(t+1)}} \right)^2 \right] e^{i2(k_{lt}+k_z)d_t}}{\left[ 1/R_{t/(t+1)} \right] e^{i2k_{lt}d_t}} + \frac{A_{t+1}}{B_{t+1}}
\]

With the second equality we introduce a notation for writing a continued fraction. Equation (3.4.68) expresses \( (A_t/B_t) \) in terms of \( A_{t+1}/B_{t+1} \) and so on, until the transmitted region \( t \), where \( A_t/B_t = 0 \), is reached.

The reflection coefficient due to the stratified medium is \( R = A_0/B_0 \). Making use of the continued fractions, we obtain

\[
R = \frac{e^{i2k_{ld_0}}}{R_{01}} + \frac{\left[ 1 - \left( \frac{1}{R_{01}} \right)^2 \right] e^{i2(k_{l0}+k_z)d_0}}{(1/R_{01}) e^{i2k_{ld_0}}} + \frac{e^{i2k_{ld_1}}}{R_{12}} + \frac{\left[ 1 - \left( \frac{1}{R_{12}} \right)^2 \right] e^{i2(k_{l1}+k_z)d_1}}{(1/R_{12}) e^{i2k_{ld_1}}} + \cdots + \frac{e^{i2k_{n-1}d_{n-1}}}{R_{(n-1)n}}
\]

(3.4.69)

\[
+ \frac{\left[ 1 - \left( \frac{1}{R_{(n-1)n}} \right)^2 \right] e^{i2(k_{n-1}+k_z)d_{n-1}}}{(1/R_{(n-1)n}) e^{i2k_{n-1}d_{n-1}}} + R_{nt} e^{i2k_{nzd_n}}
\]

This is a closed-form solution for the reflection coefficient expressed in continued fractions. Such a solution is very easily programmed for numerical computation.

For TM plane waves reflected from the stratified medium, the principle of duality applies and gives rise to the answer for \( R \) identical to (3.4.69). The only difference is that (3.4.64) now becomes

\[
P_{l(t+1)} = \frac{e^{i(k_{l0}+k_z)}}{e_{l0}k_{l0}}
\]

(3.4.70)
for TM waves. Thus, for the definition of the reflection coefficients in (3.4.65), we shall use (3.4.70) instead of (3.4.64).

For a stratified medium with any given number of layers (the number of layers is defined to be equal to \( t \) or equivalently equal to the number of boundaries), the reflection coefficient is derived from (3.4.69) by taking terms, starting from the last one, until the subscript \( l \) of \( R_{l(l+1)} \) becomes zero. Consider, for instance, a two-layer medium, with \( t = 2 \) and \( n = 1 \). We find from (3.4.69), writing \( k_{0z} = k_z \),

\[
R = \frac{e^{i2k_zd_0}}{R_0} + \frac{[1 - (1/R_0^2)]e^{i2(k_{1z} + k_z)d_0}}{(1/R_0)e^{i2k_{1z}d_0} + R_1e^{i2k_zd_1}}
\]

\[
= \frac{R_0 + R_1e^{i2k_{1z}(d_1 - d_0)}}{1 + R_0R_1e^{i2k_{1z}(d_1 - d_0)}}e^{i2k_zd_0} \quad (3.4.71)
\]

Note that when \( R_0 = \pm 1 \), the reflection coefficient in (3.4.69) will have magnitude unity, disregarding the composition of the stratified medium below \( z = -d_0 \). This should be the case, as \(|R_0| = 1\) represents, for instance, a perfectly conducting coating.

From (3.4.52)–(3.4.54) we see that \( A_t/B_t \) is the ratio of the amplitude of the wave propagating in the positive \( \hat{z} \) direction to that of the wave propagating in the negative \( \hat{z} \) direction.

---

**Example 3.4.1**

Define a space-dependent complex reflection coefficient \( \Gamma_t(z) \) such that

\[
\Gamma_t(z) = \frac{A_t}{B_t} e^{i2k_{1z}z}
\]

On the complex \( \Gamma_t(z) \) plane [Fig. E3.4.1.1], as the phase \( \phi = 2k_{1z}z \) increases with \( z \), \( \Gamma_t(z) \) varies in a counterclockwise manner. If \( k_{1z} \) is complex, \( \Gamma_t(z) \) decreases with increasing \( z \).

Define a wave impedance \( Z_t(z) \) in the negative \( \hat{z} \) direction:

\[
Z_t(z) = \frac{E_{t\hat{z}}}{H_{t\hat{z}}} = \frac{\omega\mu_t}{k_{1z}} \frac{1 + \Gamma_t(z)}{1 - \Gamma_t(z)}
\]

which is complex. For a plane wave propagating in free space in the absence of any medium, the wave impedance in the direction of wave propagation is \( \eta = \omega\mu_0/k = (\mu_0/\varepsilon_0)^{1/2} \approx 377 \Omega \).

With the definition of the complex wave impedance, the ratio of (3.4.60) to (3.4.61) gives \( Z_t(z = -d_l) = Z_{l+1}(z = -d_l) \). Thus at each interface the wave impedance are continuous across the boundary.

---

**Example 3.4.2**

To illustrate the use of the wave medium composed of \( 2N + 2 \) isotropic boundaries) with alternating high and low, \( \varepsilon_t \), \( \mu_t \), \( \eta_t \) are high-permittivity and high-permeability layers. Region \( 0 \) has permittivity \( \varepsilon_0 \) and permeability \( \mu_0 \). If each layer is a quarter-wavelength thick, then \( \mu_t \) is \( \mu_t = (\mu_t/\varepsilon_t)^{1/2} \).

Consider a wave normally incident \( \omega \sqrt{\mu_0\varepsilon_t} \) for all \( t \). The wave impedance \( Z_t = (\mu_t/\varepsilon_t)^{1/2} \). Because of the continu
3. Propagation and Guidance

the reflection coefficients in (3.4.65),

even number of layers (the number equivalently equal to the number is derived from (3.4.69) by taking the subscript \( t \) of \( R_{l(t+1)} \) becomes medium, with \( t = 2 \) and \( n = 1 \).

\[
\frac{R_{01}^2}{\beta_{k_1d_0} + \beta_{k_2d_1}} \frac{d_0}{d_0} e^{2k_2z_0} \frac{d_0}{d_0}
\]

(3.4.71)

fection coefficient in (3.4.69) will imposition of the stratified medium, as \( |R_{01}| = 1 \) represents, for \( l_1/B_1 \) is the ratio of the amplitude \( z \) direction to that of the wave

\[
\Gamma_l(z) \text{ such that } 2k_{l2}z
\]

.1), as the phase \( \phi = 2k_{l2}z \) increases manner. If \( k_{l2} \) is complex, \( \Gamma_l(z) \) de-

gative \( z \) direction:

\[
\frac{1 + \Gamma_l(z)}{1 - \Gamma_l(z)}
\]

ng in free space in the absence of any of wave propagation is \( \eta = \omega \mu_0/k = \)

impedance, the ratio of (3.4.60) to \( l \). Thus at each interface the wave \( l \).

3.4 Reflection and Transmission

![Complex \( \Gamma \) plane.](image)

**Figure E3.4.1.1 Complex \( \Gamma \) plane.**

On the complex \( \Gamma \) plane, \( Z_{l2}(z) \) can be interpreted as the ratio of the two lengths as shown in Figure E3.4.1.1. The magnitude of \( Z_{l2}(z) \) is maximum when \( \Gamma_l \) is real and positive. We define the dimensionless relative wave impedance as

\[
z_l = \frac{Z_{l2}}{Z_{l2}} = \frac{1 + \Gamma_l}{1 - \Gamma_l}
\]

For all possible complex values of \( \Gamma_l(z) \), we can map the corresponding \( z_l(z) \) values onto the complex \( \Gamma_l(z) \) plane. The result is in the form of the Smith chart, which is frequently used in transmission line studies.

--- END OF EXAMPLE 3.4.1 ---

**EXAMPLE 3.4.2**

To illustrate the use of the wave impedance concept, consider a stratified medium composed of \( 2N + 2 \) isotropic dielectric layers (corresponding to \( 2N + 2 \) boundaries) with alternating high and low permittivities, \( \epsilon_h \) and \( \epsilon_l \); regions 1, 3, 5, ..., \( 2N + 1 \) are high-permittivity layers, and regions 2, 4, 6, ..., \( 2N \) are low-permittivity layers. Region 0 has permittivity \( \epsilon \) and permeability \( \mu \). The thickness of each layer is a quarter-wavelength inside the dielectric. The transmitted region is \( 2N + 2 = t \) and has permittivity \( \epsilon_t \). Permeabilities for all layers are equal to \( \mu \) [Fig. E3.4.1.2].

Consider a wave normally incident upon the stratified medium, \( k_x = 0 \), \( k_{l2} = \omega \sqrt{\mu \epsilon_l} \) for all \( l \). The wave impedance of region \( t \), since there is no reflection, is

\[
Z_t = (\mu/\epsilon_t)^{1/2}
\]

Because of the continuity of wave impedance across the boundary,
3. Propagation and Guidance

3.4 Reflection and Transmission

reflective. Such structures are useful as are subject to corrosion and tarnishing.

b. Propagation Matrices and $T_n$

For a plane wave incident on a boundary conditions of continuity at each interface $z = -d_i$, with the wave amplitudes in regions $l$ and

$$A_{l+1} e^{-ik(l+1)x} d_i + B_{l+1} e^{ik(l+1)x} d_i$$

$$A_{l+1} e^{-ik(l+1)x} d_i - B_{l+1} e^{ik(l+1)x} d_i$$

where for TE waves

$$P(l+1)$$

and for TM waves

$$P(l+1)$$

Note that

$P(l+1)$

Equations (3.4.74) and (3.4.75) for $A_l$ and $B_l$ denote amplitudes of denote amplitudes of tangential section we determined the reflector's boundary conditions. We will $T = B_l/B_0$ can be obtained by the

We solve for $A_{l+1}$ and $B_{l+1}$ (3.4.73) and obtain

$$A_{l+1} e^{-ik(l+1)x} d_i = \frac{1}{2} (1 + p(l))$$

$$B_{l+1} e^{ik(l+1)x} d_i = \frac{1}{2} (1 + p(l))$$

Expressing in the form of matrix

$$\begin{pmatrix}
A_{l+1} e^{-ik(l+1)x} d_{l+1} \\
B_{l+1} e^{ik(l+1)x} d_{l+1}
\end{pmatrix} = \begin{pmatrix}
A_l e^{-ik(l)x} d_l & B_l e^{ik(l)x} d_l
\end{pmatrix}$$

Figure E3.4.1.2 Layered medium with alternating high and low permittivities.

the impedance across the interface separating regions $2N + 1$ and $t$ is $Z_{2N+1} = (\mu/\varepsilon_t)^{1/2}$. The relative impedance is $\varepsilon_{2N+1} = (\mu/\varepsilon_t)^{1/2} / (\mu/\varepsilon_h)^{1/2} = (\varepsilon_h/\varepsilon_t)^{1/2}$.

Making use of the Smith chart concept, and noting the periodicity of the structure, we determine the wave impedance at $z = 0$:

$$Z_0 = \left(\frac{\varepsilon_t}{\varepsilon_h}\right)^{1/2} \left(\frac{\varepsilon_t}{\varepsilon_h}\right)^{N} \left(\frac{\mu}{\varepsilon_h}\right)^{1/2}$$

The reflection coefficient $R$ at $z = 0$ is found to be

$$R_0 = \frac{Z_0 (\mu/\varepsilon)^{1/2} - 1}{Z_0 (\mu/\varepsilon)^{1/2} + 1} = \frac{\varepsilon_t (\varepsilon_t/\varepsilon_h)^{1/2} (\varepsilon_t/\varepsilon_h)^N (\mu/\varepsilon_h)^{1/2} - 1}{\varepsilon_t (\varepsilon_t/\varepsilon_h)^{1/2} (\varepsilon_t/\varepsilon_h)^N (\mu/\varepsilon_h)^{1/2} + 1}$$

We observe that, for a high $\varepsilon_t/\varepsilon_h$ ratio and for a larger number of layers, the reflection coefficient $R_0$ approaches the value $-1$, and the structure is highly
3.4 Reflection and Transmission

Reflective. Such structures are useful at optical frequencies since metallic reflectors are subject to corrosion and tarnishing problems.

b. Propagation Matrices and Transmission Coefficients

For a plane wave incident on a stratified medium, we have obtained the boundary conditions of continuity of tangential electric and magnetic fields at each interface \( z = -d_l \), with the two equations (3.4.60)–(3.4.61) relating wave amplitudes in regions \( l \) and \( l + 1 \):

\[
A_{l+1} e^{-ik_{l+1} z d_l} + B_{l+1} e^{ik_{l+1} z d_l} = A_l e^{-ik_{l} z d_l} + B_l e^{ik_{l} z d_l}
\tag{3.4.72}
\]

\[
A_{l+1} e^{-ik_{l+1} z d_l} - B_{l+1} e^{ik_{l+1} z d_l} = p_{l(l+1)} A_l e^{-ik_{l} z d_l} - B_l e^{ik_{l} z d_l}
\tag{3.4.73}
\]

where for TE waves

\[
p_{l(l+1)} = \frac{\mu_{l+1} k_{l+1} z}{\mu_l k_{l+1} z}
\tag{3.4.74}
\]

and for TM waves

\[
p_{l(l+1)} = \frac{\varepsilon_{l+1} k_{l+1} z}{\varepsilon_l k_{l+1} z}
\tag{3.4.75}
\]

Note that

\[
p_{l(l+1)} = \frac{1}{p_{l(l+1)}}
\tag{3.4.76}
\]

Equations (3.4.74) and (3.4.75) follow from duality but bear in mind that \( A_l \) and \( B_l \) denote amplitudes of tangential electric fields for TE waves and denote amplitudes of tangential magnetic fields for TM waves. In the last section we determined the reflection coefficients \( R = A_0 / B_0 \) from the \( (2n + 2) \) boundary conditions. We will now show that the transmission coefficient \( T = B_l / B_0 \) can be obtained by the use of propagation matrices.

We solve for \( A_{l+1} \) and \( B_{l+1} \) in terms of \( A_l \) and \( B_l \) from (3.4.72)–(3.4.73) and obtain

\[
A_{l+1} e^{-ik_{l+1} z d_l} = \frac{1}{2} (1 + p_{l(l+1)}) \left( A_l e^{-ik_{l} z d_l} + R_{l(l+1)} B_l e^{ik_{l} z d_l} \right)
\]

\[
B_{l+1} e^{ik_{l+1} z d_l} = \frac{1}{2} (1 + p_{l(l+1)}) \left( R_{l(l+1)} A_l e^{-ik_{l} z d_l} + B_l e^{ik_{l} z d_l} \right)
\]

Expressing in the form of matrix multiplication, we have

\[
\begin{bmatrix}
A_{l+1} e^{-ik_{l+1} z d_l} \\
B_{l+1} e^{ik_{l+1} z d_l}
\end{bmatrix}
= \overline{p}_{l(l+1)} \cdot
\begin{bmatrix}
A_l e^{-ik_{l} z d_l} \\
B_l e^{ik_{l} z d_l}
\end{bmatrix}
\tag{3.4.77}
\]
3. Propagation and Guidance

where

\[
\overline{V}_{(l+1)L} = \frac{1}{2} [1 + p_{(l+1)L}] \\
\begin{pmatrix}
\exp(-ik_{(l+1)L}(d_{l+1} - d_l)) & R_{(l+1)L}\exp(-ik_{(l+1)L}(d_{l+1} - d_l)) \\
R_{(l+1)L}\exp(ik_{(l+1)L}(d_{l+1} - d_l)) & \exp(ik_{(l+1)L}(d_{l+1} - d_l))
\end{pmatrix}
\] (3.4.78)

is called the forward-propagating matrix. In (3.4.78),

\[
R_{(l+1)L} = \frac{1 - p_{(l+1)L}}{1 + p_{(l+1)L}} = -R_{l(l+1)}
\]

is the reflection coefficient at the boundary separating regions \( l + 1 \) and \( l \), and the first subscript denotes the region with the incident wave. It is to be noted for the forward-propagating matrix between layers \( n \) and \( t = n - 1 \),

\[
\begin{pmatrix}
0 \\
T
\end{pmatrix} = \overline{V}_{tn} \cdot \begin{pmatrix}
A_n e^{-ik_{n}d_n} \\
B_n e^{ik_{n}d_n}
\end{pmatrix}
\]

with

\[
\overline{V}_{tn} = \frac{1}{2} (1 + pt_n) \begin{pmatrix}
\exp(-ik_{tn}d_n) & R_{tn}\exp(-ik_{tn}d_n) \\
R_{tn}\exp(ik_{tn}d_n) & \exp(ik_{tn}d_n)
\end{pmatrix}
\]

By the same token, we may express \( A_t \) and \( B_t \) in terms of \( A_{t+1} \) and \( B_{t+1} \) by using (3.4.62)–(3.4.63) and define a backward-propagating matrix.

The propagation matrices can be used to determine wave amplitudes in any region in terms of those in any other region. For \( m > l \), we make use of the forward propagation matrix to obtain

\[
\begin{pmatrix}
A_m e^{-ik_{m}d_m} \\
B_m e^{ik_{m}d_m}
\end{pmatrix} = \overline{V}_{m(m-1)} \cdot \overline{V}_{(m-1)(m-2)} \cdots \overline{V}_{(l+1)L} \cdot \begin{pmatrix}
A_l e^{-ik_{l}d_l} \\
B_l e^{ik_{l}d_l}
\end{pmatrix}
\]

Similarly, backward-propagating matrices can be used to express wave amplitudes in any region \( j \) in terms of those in region \( l \) for \( l > j \).

In particular, the transmission coefficient \( T = B_l/B_0 \) for a stratified medium with \( t = n + 1 \) layers can be calculated by the multiplication of \( n + 1 \) propagation matrices. Using the forward-propagating matrices, we have

\[
\begin{pmatrix}
0 \\
B_0
\end{pmatrix} = \overline{V}_{t0} \cdot \begin{pmatrix}
R e^{-ik_{t}d_0} \\
\exp(ik_{t}d_0)
\end{pmatrix}
\]

where

\[
\overline{V}_{t0} = \overline{V}_{tn} \cdot \overline{V}_{n(n-1)} \cdots \overline{V}_{10}
\]

includes all information about the stratified medium. Once \( \overline{V}_{t0} \) is known, both the reflection and transmission coefficients can be calculated from its matrix elements.

3.4 Reflection and Transmission

Example 3.4.3

As a first example, we calculate the (half-space) medium. From (3.4.78) we obtain

\[
\begin{pmatrix}
0 \\
T
\end{pmatrix} = \frac{1}{2} (1 + pt_0)
\]

which gives

\[
T = \frac{1}{2} (1 + pt_0)
\]

where we made use of the fact that \( p_{t0} = \) of these reflection and transmission coefficient discussed in the last section.

For a two-layer medium with \( d_0 = \),

\[
\begin{pmatrix}
0 \\
T
\end{pmatrix} = \frac{1}{4} (1 + pt_1)(1 + pt_0) \begin{pmatrix}
R_t \\
T_t
\end{pmatrix}
\]

which yields

\[
T = \frac{4}{9} (1 + pt_0)(1 + pt_1) \frac{1}{1 - 4\exp(k_{t1}d_1)}
\]

The reflectivity and the transmissivity \( k_1 \) and (3.4.32), are found to be

\[
r = \frac{1}{1 - 4\exp(k_{t1}d_1)}
\]

\[
t = \frac{1}{1 - 4\exp(k_{t1}d_1)}
\]

It is straightforward to prove that \( r + t \) (E3.4.3.2).
3.4 Reflection and Transmission

Example 3.4.3

As a first example, we calculate the transmission coefficient for a one-layer (half-space) medium. From (3.4.78) and by letting $d_0 = 0$, we find

$$
\begin{pmatrix}
0 \\
T
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 + p_{t0} \\
1 - p_{t0}
\end{pmatrix} \begin{pmatrix}
R_{t0} \\
1
\end{pmatrix} \begin{pmatrix}
R_{0t} \\
1
\end{pmatrix}
$$

which gives

$$
T_{t0} = \frac{1}{2} \frac{(1 - p_{t0})(1 - R_{0t}^2)}{1 + p_{t0}} \tag{E3.4.3.1}
$$

where we made use of the fact that $p_{t0} = 1/p_{ot}$ and $R_{0t} = -R_{ot}$. The implications of these reflection and transmission coefficients for both TE and TM waves have been discussed in the last section.

For a two-layer medium with $d_0 = 0$, we obtain the transmission coefficient from

$$
\begin{pmatrix}
0 \\
T
\end{pmatrix} = \frac{1}{4} (1 + p_{t1})(1 + p_{t0}) \begin{pmatrix}
e^{ik_{t1} d_1} & R_{t1} e^{ik_{t2} d_1} \\
R_{t1} e^{-ik_{t2} d_1} & e^{-ik_{t1} d_1}
\end{pmatrix} \begin{pmatrix}
R_{1t} e^{ik_{t1} d_1} \\
R_{1t} e^{-ik_{t1} d_1}
\end{pmatrix} \begin{pmatrix}
R \\
1
\end{pmatrix}
$$

which yields

$$
T_{t1} = \frac{1}{4} (1 + p_{t1})(1 + p_{t0}) \frac{(1 - R_{01}^2)(1 - R_{11}^2)}{1 + R_{01} R_{11} e^{2ik_{t1} d_1}} e^{i(k_{t1} - k_{t2}) d_1} \tag{E3.4.3.2}
$$

The reflectivity and the transmissivity for the two-layer medium, in view of (3.4.31) and (3.4.32), are found to be

$$
r = |R|^2$$

$$
t = p_{t0} |T|^2$$

It is straightforward to prove that $r + t = 1$ with $R$ and $T$ given by (3.4.71) and (E3.4.3.2).

— END OF EXAMPLE 3.4.3 —
3. Propagation and Guidance

EXAMPLE 3.4.4 Use the forward-propagation matrix formalism, we can determine the transmission and the reflection coefficients for a periodic medium made of $2N + 2$ isotropic dielectric layers with alternating high and low permittivities $\varepsilon_h$ and $\varepsilon_l$ [Fig. E3.4.1.2]. The thickness of each layer is a quarter-wavelength inside the dielectric. The transmitted region is $t = 2N + 2$ and has permittivity $\varepsilon_t$. Consider normal incidence, $k_z = 0$. The reflection coefficient $R$ has been calculated with the wave impedance approach. Using the forward-propagation matrix formalism, we have

\[
\begin{pmatrix}
0 \\
T
\end{pmatrix} = \overline{V}_{th} \cdot \left( \overline{V}_{hl} \cdot \overline{V}_{th} \right)^N \cdot \overline{V}_{ho} \cdot \begin{pmatrix}
R
\end{pmatrix}
\]

Since in region $m + 1$, $k_{(m+1)z} = k_{m+1}$ and $(d_{m+1} - d_m)$ is a quarter-wavelength thick, we have $k_{(m+1)z} = \frac{\pi}{2}$. Note also the fact that $\mu_{m+1} k_{m+1} / \mu_m k_{m+1} = (\varepsilon_{m}/\varepsilon_{m+1})^{1/2}$. The forward-propagation matrices become

\[
\overline{V}_{ho} = -\frac{i}{2} \begin{pmatrix}
1 + \sqrt{\varepsilon/l\varepsilon_h} & 1 - \sqrt{\varepsilon/l\varepsilon_h} \\
1 - \sqrt{\varepsilon/l\varepsilon_h} & 1 + \sqrt{\varepsilon/l\varepsilon_h}
\end{pmatrix}
\]

\[
\overline{V}_{hl} \cdot \overline{V}_{th} = -\frac{1}{2} \begin{pmatrix}
\sqrt{\varepsilon_{l}/\varepsilon_h} + \sqrt{\varepsilon_{h}/\varepsilon_l} & \sqrt{\varepsilon_{l}/\varepsilon_h} - \sqrt{\varepsilon_{h}/\varepsilon_l} \\
\sqrt{\varepsilon_{l}/\varepsilon_h} - \sqrt{\varepsilon_{h}/\varepsilon_l} & \sqrt{\varepsilon_{l}/\varepsilon_h} + \sqrt{\varepsilon_{h}/\varepsilon_l}
\end{pmatrix}
\]

\[
\overline{V}_{th} = \frac{1}{2} \begin{pmatrix}
(1 + \sqrt{\varepsilon_{h}/\varepsilon_l})e^{ikzd} & (1 - \sqrt{\varepsilon_{h}/\varepsilon_l})e^{-ikzd} \\
(1 - \sqrt{\varepsilon_{h}/\varepsilon_l})e^{-ikzd} & (1 + \sqrt{\varepsilon_{h}/\varepsilon_l})e^{ikzd}
\end{pmatrix}
\]

where $d$ is the total thickness of the periodic medium. The term $(\overline{V}_{hl} \cdot \overline{V}_{th})^N$ can be calculated by making use of the matrix identity

\[
\begin{pmatrix}
a + b & a - b \\
a - b & a + b
\end{pmatrix}^N = 2^{N-1} \begin{pmatrix}
a^N + b^N & a^N - b^N \\
a^N - b^N & a^N + b^N
\end{pmatrix}
\]

It follows that

\[
R = \frac{(\varepsilon_{l}/\varepsilon_h)^N - \sqrt{\varepsilon_{l}/\varepsilon_h} \varepsilon_{l}/\varepsilon_t}{(\varepsilon_{l}/\varepsilon_h)^N + \sqrt{\varepsilon_{l}/\varepsilon_h} \varepsilon_{l}/\varepsilon_t}
\]

and

\[
T = \frac{2i(-1)^N (\varepsilon_t/\varepsilon_h)^{1/2} e^{-ikzd}}{\sqrt{\varepsilon/l\varepsilon_h} \varepsilon_{l}/\varepsilon_h)^{N/2} + \sqrt{\varepsilon_{l}/\varepsilon_h} \varepsilon_{l}/\varepsilon_t)^{N/2}}
\]

In view of (3.4.31) and (3.4.32) we find that the reflectivity $r = |R|^2$, and the transmissivity $t = (\varepsilon_t/\varepsilon)^{1/2} |T|^2$. Again it can be shown that $r + t = 1$. Note

3.4 Reflection and Transmission

that, although both TE and TM waves be use $p_{0l} = k_{lz}/k_z = (\varepsilon_t/\varepsilon)^{1/2}$ because her transmission coefficients for electric field

Prob

P3.4.1

The ionosphere extends from approxi earth radii (mean earth radius is about 65 density at about 300 km. For simplicity, 40 km thick E layer with electron density layer with $N = 6 \times 10^{11} \text{ m}^{-3}$.

(a) What are the plasma frequencies of ti

(b) Consider a plane wave of 10 MHz inci from below the E layer, what is the $E$ layer?

(c) Let $\theta = 30^\circ$, below what frequency w E layer and below what frequency wil

P3.4.2

Let a plane wave be incident on a plane anisotropic crystal. Consider the special case to the plane of incidence. Find the range of reflection for the ordinary wave and the tran

P3.4.3

A plane wave is totally reflected when this incident angle, another piece of glass so that there is a very small air gap between transmission coefficients as a function of the is now possible.

P3.4.4

Sun light glares caused by reflections fr polarized.

(a) Plot the reflectivities for the TE and T.

\[
r_{TE} = |R_{TE}|^2 = \frac{1}{1 + }
\]

\[
r_{TM} = |R_{TM}|^2 = \frac{1}{1 + }
\]