POLARIZATION OF LIGHT

Clever uses for polarized light are not restricted to just the field of photonics. Devices for manipulating polarized light can be found in a wide range of settings, from advanced research laboratories to the common household [1,2]. Perhaps one of the most familiar household polarizers is a pair of sunglasses, a necessity for many car drivers on a sunny day. Sunglasses filter the light specularly reflected from a flat paved surface. The reflected light from the flat pavement is predominantly horizontally polarized, so that a polarizer with a vertical transmission axis rejects the specular reflection.

Another example of polarizing glasses are those worn for viewing a stereoscopic motion picture. In this case, the transmission axis of the polarizer covering the right eye is orthogonal to that of the polarizer for the left eye. Likewise, the motion picture scenes for the right and left eyes are projected using orthogonally polarized light. The right eye polarizer passes the light for the right eye scene and rejects the left eye scene. Similarly, the left eye polarizer passes the left eye scene and rejects the right eye scene. The viewer enjoys a stereoscopic picture.

While the polarizing glasses of the previous examples are normally constructed with linearly polarizing material, antiglare screens frequently employ circularly polarizing sheets. Displays, such as radar screens, use these circularly polarizing sheets to suppress glare. Light that enters the circular polarizer, and subsequently undergoes reflection at some other surface, is blocked from reemerging from the circular polarizer because of the reversal of handedness, while the light generated by the screen passes through.

Scientists in many disciplines use polarized light as a tool for their investigations. Physicists are still trying to unfold the mysteries of the invariance of the state of polarization of a photon before and after collisions with high-speed particles. Polarization also presents puzzles such as: “What happens when only one photon polarized at 45° with respect to the birefringent crystal axis enters the crystal?” Is the photon, which is considered the smallest unit of light, further split into horizontally and vertically polarized half-photons?

While puzzles such as these make polarization itself an interesting study, applications of polarization devices and phenomena to other disciplines are extensive. Spectroscopists...
use the Lyot–Ohman filter [1] made out of a combination of polarizers and retarders for their work. With this filter, resolving powers as high as 0.01 nm can be achieved.

Astrophysicists study the pattern of magnetic fields in nebulae by mapping the pattern of perturbation of the state of polarization of light from a nebula. These perturbations result from the Faraday rotation caused by the magnetic field of the nebula.

Many organic materials rotate the direction of light polarization as light passes through them. Chemists use this fact for analyzing the structure of new organic molecules. One of the most familiar examples is the determination of the sugar content of a sugar solution by measuring the rotation of the polarization.

Mechanical engineers use the strain birefringence pattern of a plastic model as an aid to strain analysis. Colorful strain patterns in the plastic model can be viewed under a polariscope.

Biologists are certainly beneficiaries of polarization microscopes [3] that enable them to observe microbes that are transparent and invisible under normal light. The polarization microscope sees the pattern of the retardance that the microbes create. Biologists also know that the direction of polarization of the illuminating light controls the direction of growth of some fungi is used for navigation by certain animals such as bees and horseshoe crabs.

Principles of operation of many liquid crystal displays are based on the manipulation of the polarized light as detailed in Chapter 5.

In the field of fiber-optic communication, many electrooptic devices are polarization dependent. Coherent optical communication systems detect the received light by mixing it with local oscillator light. Fluctuations in the state of polarization of the received light or the local oscillator light will cause the output power of the intermediate frequency IF signal to fluctuate. Countermeasures have to be exploited. The concepts in this chapter establish the foundation for understanding polarization. In Chapter 12, we will deal with issues such as countermeasures for polarization jitter in coherent communication systems.

6.1 INTRODUCTION

The types of waves that have so far appeared in this book have been linearly polarized waves. The \( \mathbf{E} \) field component did not change direction as the wave propagated. As shown in Fig. 6.1a, this type of wave is called a linearly polarized wave, and the direction of the \( \mathbf{E} \) field is called the direction of polarization.
In this chapter, waves whose directions of polarization rotate as the waves propagate will be described. The \( \mathbf{E} \) vector rotates around the propagation direction \( \mathbf{k} \), as the wave propagates, as shown in Figs. 6.1b and 6.1c. When the cross section of the helix is an ellipse, the wave is said to be elliptically polarized. When the cross section is circular as in Fig. 6.1b, it is naturally called a circularly polarized wave. If the \( \mathbf{E} \) vector rotates in a clockwise sense when observed at a distant location in the propagation path while looking toward the light source, as in Fig. 6.1b, the handedness of the polarization is right-handed rotation. Similarly, if the \( \mathbf{E} \) vector rotates in a counterclockwise direction, as in Fig. 6.1c, the rotation is left-handed. If there is no repetition in the pattern of the \( \mathbf{E} \) field as the wave propagates, as shown in Fig. 6.1d, the wave is said to be unpolarized or depolarized.

For handedness to be meaningful, both the direction of observation and the direction of propagation have to be specified. By convention, the handedness is specified by looking into the source of light.

Information describing the pattern and orientation of the polarized light is called the state of polarization. Any given state of polarization can be decomposed into

\[ \text{Figure 6.1 Various states of polarization (SOP). (a) Linearly (horizontally) polarized. (b) Right-handed circularly polarized. (c) Left-handed circularly polarized. (d) Depolarized.} \]
two linearly polarized component waves in perpendicular directions. The state of polarization is determined by the relative amplitude and difference in phase between the two component waves. This relative phase difference is termed retardance.

The three most basic optical components that are used for manipulating or measuring the state of polarization are the (1) retarder, (2) linear polarizer, and (3) rotator.

In this chapter, the prime emphasis is placed on how to use these optical components. The circle diagrams are predominantly used for explaining the operation. In the next chapter, however, the Poincaré sphere will be used for explaining the operation.

6.2 CIRCLE DIAGRAMS FOR GRAPHICAL SOLUTIONS

Graphical and analytical methods for finding the state of polarization complement each other. The graphical method is fail-safe and is often used to confirm the results obtained by analytical methods. The graphical method helps visualize the state of polarization for a given set of parameters and also makes it easier to visualize intermediate stages. On the other hand, analytical methods provide higher accuracy and are easier to generalize. This chapter begins with a look at graphical solutions to common polarization problems.

6.2.1 Linearly Polarized Light Through a Retarder

A retarder can be made from any birefringent material, that is, any material whose refractive index depends on direction. As an example, let us take the uniaxial crystal characterized by refractive indices $n_e$ and $n_o$ as described in Chapter 4. The orthogonal linearly polarized component waves are the e-wave and the o-wave. It is further assumed that the front and back surfaces of the retarder are parallel to the optic axis of the crystal, and the propagation direction of the incident light is normal to the front surface of the retarder. In this situation, the directions of the component e-wave and o-wave do not separate as they propagate through the retarder; rather, they emerge

Left-handed?  
Right-handed?

Ha! Right-handed wave

Right- or left-handedness can only be determined after both the direction of propagation and the direction of observation have been specified.
together. Depending on which is smaller, \( n_e \) or \( n_o \), one of the component waves moves through the retarder faster than the other. The relative phase difference is the retardance \( \Delta \). The polarization direction of the faster component wave is called the fast axis of the retarder, and the polarization direction of the slower component wave is called the slow axis. The emergent state of polarization is the superposition of the two component waves and will depend on the relative amplitudes of the two component waves, as well as the retardance.

A circle diagram will be used to find the state of polarization as the incident linearly polarized light transmits through the retarder. Figure 6.2a shows the configuration. A linearly polarized wave at azimuth \( \theta = 55^\circ \) is incident onto a retarder with retardance

![Graphical solution. (a) Geometry. (b) Circle diagram.](Image)
The difference in the usage of the words retardance and retardation is analogous to that between transmittance and transmission. Retardance is a measurable quantity representing the difference in phase angles.

$\Delta = 60^\circ$. The direction of the fast axis of the retarder is designated by an elongated $F$ and in this case is oriented in the $x$ direction. The direction of the slow axis is perpendicular to that of the fast axis and is taken as the $y$ direction. The $z$ direction is the direction of propagation.

The incident light $E$ is decomposed into the directions of the fast and slow axes, that is, in the $x$ and $y$ directions. In complex notation, the component waves are

$$E_x = A e^{i(-\omega t + \beta z)}$$
$$E_y = B e^{i(-\omega t + \beta z + \Delta)}$$

with

$$E = E_x \hat{i} + E_y \hat{j}$$
$$A = |E| \cos 55^\circ$$
$$B = |E| \sin 55^\circ$$

and the corresponding real expressions are

$$E_x = A \cos(-\omega t + \beta z)$$
$$E_y = B \cos(-\omega t + \beta z + \Delta)$$

The phasor circle $C_1$ in Fig. 6.2 represents Eq. (6.1) and $C_2$ represents Eq. (6.2). As time progresses, both phasors rotate at the same angular velocity as $e^{-j\omega t}$ (for now a fixed $z$), or clockwise as indicated by $0, 1, 2, 3, \ldots, 11$. The phase of $E_y$, however, lags by $\Delta = 60^\circ$ because of the retarder. The projection from the circumference of circle $C_1$ onto the $x$ axis represents $E_x$, and that from the $C_2$ circle onto the $y$ axis represents $E_y$. It should be noted that the phase angle $-\omega t$ in $C_1$ is with respect to the horizontal axis and $-\omega t + \Delta$ in $C_2$ is with respect to the vertical axis.

By connecting the cross points of the projections from $0, 1, 2, 3, \ldots, 11$ on each phasor circle, the desired vectorial sum of $E_x$ and $E_y$ is obtained. The emergent light is elliptically polarized with left-handed or counterclockwise rotation.

Next, the case when the fast axis is not necessarily along the $x$ axis will be treated. For this example, a retarder with $\Delta = 90^\circ$ will be used. Referring to the geometry in Fig. 6.3a, the fast axis $F$ is at azimuth $\Theta$ with respect to the $x$ axis, and linearly polarized light with field $E$ is incident at azimuth $\theta$. Aside from the new value of $\Delta$ and the azimuth angles, the conditions are the same as the previous example.

The only difference in the procedure from that in the previous case is that the fast axis is no longer in the $x$ direction and the incident field has to be decomposed into components parallel to the fast and slow axes, rather than into $x$ and $y$ components. Figure 6.3b shows the circle diagram for this case.
It is important to recall that the only allowed directions of polarization inside the crystal are along the fast and slow axes; no other directions in between the two axes are allowed. This is the reason why the incident field is decomposed into components along the fast and slow axes.

6.2.2 Sign Conventions

As mentioned in Chapter 1, this book has employed the convention of

\[ e^{-j\omega t} \]

rather than

\[ e^{j\omega t} \]

which appears in some textbooks. This section attempts to clarify some of the confusion surrounding signs and the choice of Eq. (6.5) or (6.6). Let us take the example of the retarder in Fig. 6.2 as the basis for discussion. The expression for the state of polarization depends critically on the difference between \( \phi_x \) and \( \phi_y \) of the \( E_x \) and \( E_y \) component waves. With the convention of Eq. (6.5), the phases of \( E_x \) and \( E_y \) for the
positive $z$ direction of propagation are

\[ \phi_x = \beta z - \omega t \]  
\[ \phi_y = \beta z - \omega t + \Delta \]  

where $\Delta = 60^\circ$. We will now explain why $\phi_y$ was expressed as $\beta z - \omega t + \Delta$ rather than $\beta z - \omega t - \Delta$. The example was defined such that the $x$ direction is the direction of the fast axis, which means $E_x$ advances faster than $E_y$, and hence $E_x$ leads $E_y$. Let us examine Eqs. (6.7) and (6.8) more closely to see if it is indeed the case that $E_x$ leads $E_y$. For simplicity, the observation is made on the $z = 0$ plane. Both $\phi_x$ and $\phi_y$ are becoming large negative values as time elapses. At the time when $\phi_x = 0$, $\phi_y$ is still a positive number, namely, $\phi_y = 60^\circ$ and $\phi_y$ leads $\phi_x$ by $60^\circ/\omega$ seconds in the movement toward large negative values. Hence, one can say that $E_y$ is lagging $E_x$ by $60^\circ$ or $E_x$ is leading $E_y$ by $60^\circ$. This confirms that $\phi_y = \beta z - \omega t + \Delta$ was the correct choice to represent $E_x$ leading $E_y$, for the convention of Eq. (6.5). When $E_x$ leads $E_y$, a left-handed polarization results for $\Delta = 60^\circ$, as shown in Fig. 6.2.

Now, let us look at the other convention of using $e^{j\omega t}$ instead of $e^{-j\omega t}$. The same example of the retarder in Fig. 6.2 will be used. When $E_x$ and $E_y$ are propagating in the positive $z$ direction, the signs of the $\beta z$ and $\omega t$ terms are opposite (Chapter 1). Let $\phi'_x$ and $\phi'_y$ denote the phases of the $x$ and $y$ components using the $e^{j\omega t}$ convention.

\[ \phi'_x = -\beta z + \omega t \]  
\[ \phi'_y = -\beta z + \omega t - \Delta \]  

where $\Delta = 60^\circ$. As the $x$ direction was specified as the direction of the fast axis, Eqs. (6.9) and (6.10) have to represent the case where $E_x$ leads $E_y$ by $60^\circ$. Let us verify that this is true. Taking $z = 0$ as the plane of observation, both $\phi'_x$ and $\phi'_y$ become large negative numbers as time elapses. At the time $\phi'_x = 0$, the phase of $\phi'_y$ is a negative number and is behind $\phi'_x$ by $60^\circ/\omega$ seconds in the movement toward large positive numbers. This is consistent with $E_x$ leading $E_y$ by $60^\circ$.

To conclude this example, if $E_x$ leads $E_y$ by $60^\circ$, the resulting polarization is left-handed, regardless of the choice of convention of Eq. (6.5) or (6.6). However, for this previous statement to be true, the sign of $\Delta$ does depend on the choice of convention.

In Problem 6.1, the same reasoning is applied to the geometry of Fig. 6.3.

Next, the state of polarization of the emergent wave will be investigated as a function of retardance. For simplicity, the amplitudes of $E_x$ and $E_y$ are kept the same, that is, $B/A = 1$, and the fast axis is kept along the $x$ axis. The case of $B/A \neq 1$ is left for Problem 6.2. A series of circle diagrams were drawn to obtain the states of polarization with $\Delta$ as a parameter. The results are summarized in Fig. 6.4.

With $\Delta = 0^\circ$ (360°), the state of polarization is linear with azimuth $\theta = 45^\circ$. As $\Delta$ is increased from $0^\circ$ to $90^\circ$, the shape becomes elliptical, growing fatter and fatter while keeping the major axis always at $45^\circ$, and the rotation of polarization always left-handed. When $\Delta$ reaches $90^\circ$, the wave becomes a circularly polarized wave, still with left-handed rotation. As soon as $\Delta$ passes $90^\circ$, the radius in the $45^\circ$ direction starts to shrink while that in the $135^\circ$ direction expands, and the state of polarization becomes elliptically polarized with its major axis at $135^\circ$, but still with left-handed rotation. This trend continues until $\Delta = 180^\circ$. 

![CIRCLE DIAGRAMS FOR GRAPHICAL SOLUTIONS](image-url)
Figure 6.4 Elliptical polarizations with $A = B$. The fast axis is in the $x$ direction, and $\Delta$ is the parameter. $\rightarrow\Rightarrow$ represents left-handedness, $\Leftarrow\Rightarrow$ represents right-handedness, and $\Leftarrow\Rightarrow$ represents linear polarization.

Table 6.1 Summary of states of polarization with fixed $B/A = 1$ for various retardance values

<table>
<thead>
<tr>
<th>Retardance $\Delta$</th>
<th>0°</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shape</td>
<td>$\rightarrow\Rightarrow$</td>
<td>$\Leftarrow\Rightarrow$</td>
<td>$\Leftarrow\Rightarrow$</td>
<td>$\rightarrow\Rightarrow$</td>
<td></td>
</tr>
<tr>
<td>Inclination $\theta$</td>
<td>45°</td>
<td>135°</td>
<td>135°</td>
<td>45°</td>
<td></td>
</tr>
<tr>
<td>Sign of $\sin \Delta$</td>
<td>+</td>
<td></td>
<td></td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Handedness</td>
<td>Left</td>
<td></td>
<td></td>
<td>Right</td>
<td></td>
</tr>
</tbody>
</table>

With $\Delta = 180^\circ$, the wave becomes linearly polarized again, but this time the direction of polarization is at $135^\circ$. As soon as $\Delta$ exceeds $180^\circ$, the wave starts to become elliptically polarized but with right-handed rotation. As $\Delta$ increases between $180^\circ$ and $360^\circ$, the state of polarization changes from linear ($135^\circ$) to right-handed elliptical (major axis at $135^\circ$) to right-handed circular to right-handed elliptical (major axis at $45^\circ$) to linear ($45^\circ$). In this region of $\Delta$, the handedness is always right-handed. The results are summarized in Table 6.1.
Next, a method of classification other than Table 6.1 will be considered. As one may have already realized, it is the combination of two numbers — $B/A$ and $\Delta$ — that determines the state of polarization of the emergent light from the retarder. These two numbers, however, are obtainable from the quotient of Eqs. (6.1) and (6.2), namely,

\[ E_y/E_x = \left( \frac{B}{A} \right) e^{i\Delta} \tag{6.11} \]

Each point on the complex number plane of $E_y/E_x$ corresponds to a state of polarization. As a matter of fact, this representation will be extensively used in the next chapter. Figure 6.5 shows such a complex plane. For this example, $B/A = 1$ and the states of polarization are drawn on a unit circle for various values of $\Delta$. Figure 6.5 summarizes the results in Fig. 6.4.

**Example 6.1** A quarter-waveplate, commonly written as a $\lambda/4$ plate, is a retarder with $\Delta = 90^\circ$. As shown in Fig. 6.6, horizontally linearly polarized light is intercepted by a $\lambda/4$ plate whose orientation $\Theta$ is rotated. Draw the sequence of elliptically polarized waves of the emergent light as the fast axis of the $\lambda/4$ plate is rotated at $\Theta = 0^\circ, 22.5^\circ, 45^\circ, 67.5^\circ, 90^\circ, 112.5^\circ, 135^\circ, 157.5^\circ, 180^\circ$, and $202.5^\circ$.

**Solution** The series of circles is drawn in Fig. 6.7. As the correct numbering of the circles $C_1$ and $C_2$ is crucial to the final result, a few tips are given here on how to set up the numbering. The convention of $e^{-j\omega t}$ is being used, so that numbering of both circles $C_1$ and $C_2$ is in a clockwise sense. Refer to the drawings with $\Theta = 22.5^\circ$.
and $\Theta = 157.5^\circ$ as examples. Point $P$ on the horizontal axis represents the tip of the incident light polarization at $t = 0$. $P$ is decomposed into points $P_1$ and $P_2$ on circles $C_1$ and $C_2$, respectively, as shown in Fig. 6.7. If the retardance had been zero ($\Delta = 0^\circ$), then $P_1$ and $P_2$ would correspond to point 1 for each of the circles $C_1$ and $C_2$. Because of the retardance of the $\lambda/4$ plate, $C_2$ is delayed $90^\circ$ with respect to $C_1$, which corresponds to a rotation of $C_2$ by $90^\circ$ in the counterclockwise direction. $P_2$ now lines up with point 2 of $C_2$. Observe in the case of the $\lambda/4$ plate, for all diagrams in Fig. 6.7, the line drawn from point 1 of $C_1$ and the line drawn from point 2 of $C_2$ intersect along the horizontal axis at $P$. This is a good method for obtaining the correct numbering.

With $\Theta = 0^\circ$, the radius of $C_2$ becomes zero, and with $\Theta = 90^\circ$, that of $C_1$ becomes zero. The emergent light is identical to the incident light for these cases.

The results are summarized in Fig. 6.8. The major or minor axis is always along the direction of the fast axis. This is a characteristic of a quarter-waveplate when the incident light is linearly polarized. First, the major axis follows the fast axis, and then the minor axis, and then the major axis. They alternate at every $45^\circ$. In the region $0 < \Theta < \pi/2$, the emergent light is right-handed, while in the region $\pi/2 < \Theta < \pi$ the emergent light is left-handed. It is worthwhile remembering that the handedness of the emergent circularly polarized wave alternates every $90^\circ$ of rotation of the retarder. At

With the case of $\theta = 157.5^\circ$, two circles $C_2$ are drawn, one on each side. Either circle $C_2$ can be used, as long as one makes sure that point 1 on circle $C_1$ as well as on circle $C_2$ correspond to point $P$ if the retardance is momentarily reduced to zero.
Figure 6.7  Circle diagrams as the $\lambda/4$ plate is rotated. The incident light is horizontally polarized.

Note: Handedness changes at the azimuth of the retarder, $\Theta = 90^\circ$ and $\Theta = 180^\circ$. 
\[ \Theta = 45^\circ, \text{ right-handed circular polarization is obtained, and at } \Theta = 135^\circ \text{ left-handed circular polarization is obtained. This is a quick convenient way of obtaining a right circularly or left circularly polarized wave.} \]

At every 90°, the emergent light becomes identical with the incident light and is horizontally linearly polarized. Note that the orientation with \( \Theta = 180^\circ \) is identical to that of \( \Theta = 0^\circ \), and the orientation with \( \Theta = 202.5^\circ \) is identical to that of \( \Theta = 22.5^\circ \).

### 6.2.3 Handedness

The question of how the direction of the handedness is determined will be resolved. The instantaneous value \( \theta(t) \) of the direction of polarization with respect to the \( x \) axis is

\[
\theta(t, z) = \tan^{-1} \left( \frac{E_\lambda(t)}{E_x(t)} \right)
\]  

(6.12)
From Eqs. (6.3) and (6.4), $\theta(t, z)$ is expressed as

$$\theta(t, z) = \tan^{-1} \left( \frac{B}{A} \left( \cos \Delta + \tan(\omega t - \beta z) \sin \Delta \right) \right)$$  \hspace{1cm} (6.13)

For $\sin \Delta = 0$, the azimuth $\theta(t, z)$ becomes independent of time and location, and a linearly polarized wave results. With other values of $\sin \Delta$, $\theta(t, z)$ depends on time and location.

The direction of the movement of $\theta(t, z_0)$ at a fixed point $z = z_0$ is found from the derivative of Eq. (6.13):

$$\frac{d\theta(t, z_0)}{dt} = \sin \Delta$$  \hspace{1cm} (6.14)

The factor in the large parentheses of Eq. (6.14) is always positive, and $d\theta(t, z_0)/dt$ is the same sign as $\sin \Delta$. When $\sin \Delta$ is positive, the azimuth $\theta(t, z_0)$ increases with time and if negative, $\theta(t, z_0)$ decreases. These results are summarized in Eq. (6.15):

- Counter clockwise (left-handed) when $\sin \Delta > 0$
- Linearly polarized when $\sin \Delta = 0$
- Clockwise (right-handed) when $\sin \Delta < 0$

(6.15)

For the case of circularly polarized emergent light, that is,

$$B/A = 1 \quad \text{and} \quad \sin \Delta = \pm 1$$  \hspace{1cm} (6.16)

the derivative simplifies to

$$\frac{d\theta(t, z_0)}{dt} = \pm \omega$$  \hspace{1cm} (6.17)

The angular velocity of the rotation of the polarization is the same as that of the component wave. The sense of rotation of $d\theta(t, z_0)/dt$ also matches the sign of $\sin \Delta$.

It should be noted that Eq. (6.15) is true only when the incident wave is linearly polarized.

### 6.2.4 Decomposition of Elliptically Polarized Light

The graphical method for constructing an elliptical polarization has been described. Up to this point, the incident light has been decomposed into components along the fast and slow axes of the retarder. In this section, the graphical method will be generalized to allow decomposition of a given elliptical polarization into an arbitrary set of mutually perpendicular component waves.

The values $B^\prime/A^\prime$ and $\Delta^\prime$ of the newly decomposed waves, however, vary according to the desired orientation of the decomposed component waves. An example will be used for explaining the decomposition.
Example 6.2  Graphically decompose the elliptically polarized wave with $B/A = 1$ and $\Delta = 45^\circ$ shown in Fig. 6.4 into $E_x'$ polarized at $22.5^\circ$ and $E_y'$ polarized at $112.5^\circ$ and then determine the values of $B'/A'$ and $\Delta'$ of the newly decomposed component waves.

Solution  Referring to Fig. 6.9, the decomposition is performed as follows:

1. Draw coordinates $x' - y'$ in the desired directions.
2. Determine the radius of circles $C_1$ and $C_2$ from the points of the extrema on the ellipse in the $x'$ and $y'$ directions.
3. Extend a line downward parallel to the $y'$ axis from point 1 on the ellipse to point 1 on circle $C_1$. Similarly, extend a line to the right parallel to the $x'$ axis from the same point on the ellipse to intersect point 1 and point 1' on the circle $C_2$ (set the point 1' aside for now).

Figure 6.9  Decomposition of an elliptically polarized wave into component waves along arbitrary orthogonal coordinates.
4. From point 1 on the ellipse $O$, follow the ellipse in the direction of rotation of polarization to point 2. Point 2 is the intersection of the ellipse $O$ with the $y'$ axis. Draw the extension line from point 2 on the ellipse parallel to the $x'$ axis to make point 2 on circle $C_2$.

5. Find point 3, which is the tangent to the ellipse parallel to the $y'$ axis. Draw an extension line parallel to the $x'$ axis from point 3 on the ellipse $O$ to point 3 on circle $C_2$.

6. Find point 4, which is the intersection of the ellipse $O$ with the $y'$ axes, and draw an extension parallel to the $x'$ axis to point 4 on circle $C_2$.

7. Now, $B'/A'$ and $\Delta'$ can be obtained from this graph. The ratio of the radii of $C_1$ and $C_2$ gives

$$B'/A' = 0.59$$

and the angle $\Delta'$ on circle $C_2$ gives

$$\Delta' = 55^\circ$$

The $E_y$ phasor rotates clockwise from the $y'$ axis, so that $E'_y$ is lagging by $\Delta' = 55^\circ$ from that of $E'_x$, consistent with the left-handed rotation of the ellipse given by Eq. (6.15). As a matter of fact, examination of Fig. 6.9 shows that $\Delta'$ can be calculated directly from the intersections of $B'$ and $C'$ of the ellipse on the $y'$ axis:

$$\cos(90^\circ - \Delta') = \frac{OC'}{OB'}$$

and

$$\sin \Delta' = \frac{OC'}{OB'}$$

8. Note that if the phasor on $E'_y$ starts from point 1' and both $E'_x$ and $E'_y$ rotate in the clockwise sense ($-\omega t$), then the intersections will not form the original ellipse, and point 1' has to be discarded. □

### 6.2.5 Transmission of an Elliptically Polarized Wave Through a $\lambda/4$ Plate

Previous sections dealt with the transmission of linearly polarized light through a retarder. This section treats the more general case of transmission of an elliptically polarized incident wave through a retarder.

As shown in Fig. 6.10, let the azimuth and ellipticity of the incident wave be $\theta$ and $\epsilon$, respectively. The retarder is again a $\lambda/4$ plate. The circle diagram method starts with the decomposition of the incident elliptic field into the field parallel to the fast axis of the $\lambda/4$ plate and that parallel to the slow axis. The former component field is represented by phasor circle $C_1$ and the latter by phasor circle $C_2$.

Next, the retardance is considered. The endpoint $P$ of the phasor vector of the incident wave is projected onto point 1 of circle $C_1$ and projected onto point 0 of circle $C_2$. Point 0 on circle $C_2$ is delayed by $90'$ with respect to point 1 in order to account for transmission through the $\lambda/4$ plate. The circumference of the phasor circles
is divided into four and numbered 1, 2, 3, and 4 sequentially. The intercepts of the projections from each of the circles parallel to the fast and slow axes are numbered 1, 2, 3, and 4. The emergent ellipse, shown as a dotted line in Fig. 6.10, is completed by connecting 1, 2, 3, 4 and the handedness is in the direction of 1, 2, 3, and 4.

6.3 VARIOUS TYPES OF RETARDERS

A waveplate is a retarder with a fixed retardance. Waveplates providing retardances of 360°, 180°, and 90° are called full-waveplates, half-waveplates, and quarter-waveplates, respectively. They are also written simply as λ, λ/2, and λ/4 plates. Retarders with adjustable retardance are called compensators.
6.3.1 Waveplates

Waveplates can be fabricated either from a single piece of birefringent crystal or from a combination of two pieces of crystal. The difficulty with fabricating a single crystal waveplate is that the plate has to be made extremely thin. The thickness $d$ for a $\lambda/4$ plate is calculated as

$$\frac{2\pi}{\lambda} |n_e - n_o| = \frac{\pi}{2}$$

(6.18)

Taking $\lambda = 0.63$ $\mu$m, the values of $d$ for typical birefringent crystals are:

- For calcite, $d = 0.92$ $\mu$m ($n_e = 1.4864$, $n_o = 1.6584$).
- For quartz, $d = 17.3$ $\mu$m ($n_e = 1.5443$, $n_o = 1.5534$).
- For mica, $d = 31.5$ $\mu$m ($n_e = 1.594$, $n_o = 1.599$).

The thickness is in the range of tens of microns.

Even though calcite has a cleavage plane and need not be polished, its brittleness makes it hard to handle thin pieces. Quartz is not as brittle but requires polishing because it does not have a cleavage plane. Mica has more favorable properties. It is not only flexible, but also possesses cleavage planes; however, there is some difficulty in cleaving at exactly the right thickness. The plate with the desired thickness is selected among many cleaved pieces.

The stringent requirement of excessively small thicknesses can be avoided by taking advantage of the rollover of the retardance at every $2\pi$ radians. The retardance of a $\lambda/4$ plate, for instance, is designed as

$$\frac{2\pi}{\lambda} |n_e - n_o| d = \left(2\pi N + \frac{\pi}{2}\right)$$

(6.19)

where the value of $N$ is normally a few hundred.

With $N = 100$, the thickness of a quartz $\lambda/4$ plate is 7 mm. The drawbacks of a retarder with a large $N$ are a higher sensitivity to temperature and to the angle of incidence. The increase in the path of a ray with incident angle $\theta$ compared to the ray normal to the plate of thickness $d$ is

$$\Delta d = \left(\frac{1}{\cos \theta} - 1\right) \div d \frac{\theta^2}{2}$$

(6.20)

The value of $\theta$ that creates a retardance error of $\pi/2$ radians is

$$\frac{2\pi}{\lambda} |n_e - n_o| d \frac{\theta^2}{2} = \frac{\pi}{2}$$

(6.21)

Inserting Eq. (6.19) into (6.21) gives

$$\theta \div \frac{1}{\sqrt{2N}} \text{ rad}$$

(6.22)

With $N = 100$, the cone of the allowed angle of incidence is narrower than $4^\circ$. 
The second approach to alleviating the thinness requirement is the combination of two plates with their optical axes perpendicular to each other. Figure 6.11 shows the construction of a waveplate of this kind. The optical paths $\phi_x$ and $\phi_y$ for $E_x$ and $E_y$ are

$$\phi_x = \frac{2\pi}{\lambda} (d_1 n_e + d_2 n_o)$$

and

$$\phi_y = \frac{2\pi}{\lambda} (d_1 n_o + d_2 n_e)$$

and the retardance $\Delta = \phi_y - \phi_x$ is

$$\Delta = \frac{2\pi}{\lambda} (d_1 - d_2)(n_o - n_e) \text{ rad}$$

What matters in Eq. (6.25) is the difference in thickness, rather than the total thickness so that the thickness of each plate can be comfortably large to facilitate polishing.

### 6.3.2 Compensator

Figure 6.12a shows the structure of a Babinet compensator. Two wedge-shaped birefringent crystals are stacked such that their optical axes are perpendicular to each other. The apex angle, however, is made small so that the separation of the o- and e-waves is negligibly small. They can be slid against one another so that the difference in thicknesses $d_1 - d_2$ is adjustable.

Depending on the location of the incident ray, the retardance is varied. At the location where $d_1 = d_2$, regardless of the values of $n_e$ and $n_o$, both waves $E_x$ and $E_y$
Figure 6.12 Comparison between Babinet-type and Soleil-type compensators. (a) Babinet compensator. Depending on the horizontal locations of the incident light, the emergent state of polarization varies. (b) Soleil compensator. For a given $d_1$ and $d_2$, the emergent state of polarization does not change as the incident light position is varied.
go through the same amount of phase shift and the retardance is zero. However, at the locations where \(d_1 \neq d_2\), the two waves do not go through the same phase shift. If the crystal is a positive uniaxial crystal and \(n_e > n_o\) like quartz and if \(d_1 < d_2\), then the phase of \(E_y\) lags behind that of the \(E_x\) wave. The amount of phase lag, or retardation, can continuously be adjusted by either shifting the location of the incident light while keeping the relative positions of the crystal stack fixed, or sliding one crystal against the other by means of a micrometer while keeping the location of the incident light fixed.

The compensator can be used at any wavelength provided that the light at that wavelength is not significantly absorbed by the crystal. The compensator also provides a full range of retardation so that it can be used as a zero-wave, quarter-wave, half-wave, or full-wave retarder. The sense of circular polarization can also be changed by changing plus \(\lambda/4\) to minus \(\lambda/4\) of retardation. In some applications, the compensator is used to measure the retardation of a sample material. Light of a known state of polarization enters the sample, which causes a change in the polarization of the emergent light. The emergent light is passed through the compensator and the thickness \(d_2\) is adjusted to regain the initial state of polarization. From this adjustment of \(d_2\), which is precalibrated in terms of retardance, the retardation of the sample can be determined.

Figure 6.12b shows the structure of a Soleil compensator. The difference between the Babinet and Soleil compensators is that the Soleil compensator has another block of crystal so that the thickness \(d_2\) is independent of the location of the incident wave. The top two blocks consist of two wedge-shaped crystals. They can be slid by a micrometer against one another so that the thickness \(d_2\) is adjustable. The thickness \(d_1\) of the bottom block is fixed. With the Soleil compensator, the emergent state of polarization is independent of the location of incidence.

As mentioned earlier, when \(N\) is large, there are stringent requirements on the collimation of the light entering the retarder, as well as a larger temperature dependence. However, for the Soleil or Babinet compensator, it is possible to reduce \(N\) to zero.

### 6.3.3 Fiber-Loop Retarder

Bending an optical fiber creates birefringence within the fiber. This is the basis of the fiber-loop retarder whereby a controlled amount of bend-induced birefringence is used to change the retardance. An advantage of the fiber-loop retarder controller over conventional optical devices, such as quarter-waveplates and half-waveplates, is that the polarization control is achieved without interrupting transmission of light in the fiber.

When an optical fiber is bent, the fiber is compressed in the radial direction of the bend and is expanded in the direction perpendicular to it, as shown in Fig. 6.13. The refractive index of glass is lowered where it is compressed and raised where it is expanded. In the coordinates shown in Fig. 6.13, the difference between the change \(\Delta n_x\) in the \(x\) direction before and after the bending, and the change \(\Delta n_y\) in the \(y\) direction before and after the bending is calculated as [4,5]

\[
\Delta n_y - \Delta n_x = -0.0439n^3 \left(\frac{r}{R}\right)^2
\]  

(6.26)

where \(n\) is the index of refraction of the core, \(2r\) is the outer diameter of the fiber (diameter of the cladding), and \(R\) is the radius of curvature of the loop as indicated in
Fig. 6.13. Bending of a fiber to create birefringence.

Fig. 6.14. The coefficient 0.0439 was calculated from Poisson’s ratio and the strain-optical coefficients for a silica glass fiber.

Figure 6.15 shows the structure of a fiber-loop polarization controller based on the bend-induced birefringence in the fiber [6]. It combines both a $\lambda/4$ and $\lambda/2$ fiber-loop retarder and a polarizer loop. The retarders are made of ordinary single-mode fibers. The radius of the left-hand side retarder spool is designed such that the phase of the $y$-polarized wave is $\pi/2$ radians behind that of the $x$-polarized wave. This spool is the fiber equivalent of a “quarter-waveplate” and it will be referred to as the $\lambda/4$ loop. The right-hand side retarder spool is designed to create a $\pi$-radian phase shift, analogous to a “half-waveplate”, and will be referred to as the $\lambda/2$ loop. Usually, the two spools have the same radius, and the $\lambda/2$ spool has twice the number of fiber turns as the $\lambda/4$ spool.

The orientation of the fiber loops can be changed as indicated by the arrows in Fig. 6.15. When conversion of elliptic to linear polarization is desired, the $\lambda/4$ loop is oriented so that the elliptically polarized wave going into the loop is converted into a linearly polarized wave upon exiting the loop (see Section 6.4.3.2). The $\lambda/2$ loop is then oriented to rotate the direction of the linear polarization to the desired direction.

**Example 6.3** Find the loop radius of a fiber-loop polarization controller for a $\lambda = 1.30\ \mu m$ wavelength system. The number of turns is $N = 4$ for the $\lambda/4$ loop and $N = 8$ for the $\lambda/2$ loop. The diameter of the fiber is 125 $\mu m$, and the index of refraction of the core is $n = 1.55$.

**Solution** The radius $R$ of the $\lambda/4$ loop is first calculated. The difference $\Delta \beta$ in propagation constants in the $x$ and $y$ directions is

$$\Delta \beta = k_0(n + \Delta n_x) - k_0(n + \Delta n_y)$$

where $k_0$ is the free-space propagation constant. Combining Eq. (6.26) with (6.27),

$$\Delta \beta = 0.0439 \ k_0 n^3 \left( \frac{r}{R} \right)^2 \ \text{rad/m}$$

(6.28)
Figure 6.14  Fiber-loop-type retarder using a single-mode fiber.

Figure 6.15  Fiber-loop polarization controller.
The $\lambda/4$ condition requires

$$2\pi RN \cdot \Delta\beta = \frac{\pi}{2}$$  \hspace{1cm} (6.29)

Equations (6.28) and (6.29) give

$$R = 0.176 \ k_0 n^3 r^2 N$$  \hspace{1cm} (6.30)

With the parameters provided, the loop radius is

$$R = 4.95 \text{ cm}$$

A choice of radius smaller than $R = 1.5 \text{ cm}$ is not recommended, because the transmission loss (as will be mentioned in Section 6.5.3) becomes increasingly significant as $R$ is decreased. □

6.4 HOW TO USE WAVEPLATES

The waveplate is one of the most versatile optical components for manipulating the state of polarization. Various applications of the waveplate will be summarized from the viewpoint of laboratory users.

6.4.1 How to Use a Full-Waveplate

A full-wave plate ($\lambda$ plate) combined with an analyzer (polarizer) functions like a wavelength filter. The retardance of any thick waveplate critically depends on the wavelength of light. This can be seen from Eq. (6.19), which gives the expression for the retardance of a thick quarter-waveplate. The expression for the retardance of the thick full-waveplate can be obtained by substituting $2pEM$ for $pEM/2$ on the right-hand side of Eq. (6.19).

The operation of the wavelength filter is described as follows. Multiwavelength linearly polarized light is incident onto the full-waveplate. The azimuth of the incident light is chosen so that both fast and slow component waves propagate through the full-waveplate. Only those wavelengths that satisfy the full-wave retardance condition will emerge as linearly polarized waves. The transmission axis of the analyzer is oriented in the same direction as the azimuth of the incident light, so that the emergent linearly polarized waves experience the least amount of attenuation on passing through the analyzer. For all other wavelengths, the state of polarization is changed by going through the full-waveplates, and these will be attenuated on passing through the analyzer.

By adding additional full-waveplate and analyzer pairs, a narrower linewidth wavelength filter is obtainable. Such filters are the Lyot–Ohman and Šolic filters [1,13].

6.4.2 How to Use a Half-Waveplate

The half-waveplate can be used to change the orientation and/or the handedness of a polarized wave. The case of a linearly polarized incident wave is first considered. Let
Polarization of Light

Let the vector \( \overrightarrow{OP} \) of the incident \( E \) wave be in the vertical direction, and let the direction of propagation be into the page. The plane of the half-waveplate is in the plane of the page. Let the fast axis of the half-waveplate (\( \lambda/2 \) plate) be oriented at an angle \( \phi \) from the direction of polarization, as shown in Fig. 6.16a. The incident vector \( \overrightarrow{OP} \) is decomposed into \( \overrightarrow{OQ} \) and \( \overrightarrow{QP} \) in the directions of the fast and slow axes, respectively.

After transmission through the \( \lambda/2 \) plate, the direction of the vector \( \overrightarrow{QP} \) is reversed to \( \overrightarrow{PQ} \), while the vector \( \overrightarrow{OQ} \) remains unchanged. Alternatively, \( \overrightarrow{OQ} \) is reversed and \( \overrightarrow{PQ} \) is unchanged. The resultant vector for the transmitted wave becomes the vector \( \overrightarrow{OP'} \).

The emergent vector \( \overrightarrow{OP'} \) is a mirror image of the incident vector \( \overrightarrow{OP} \) with respect to the fast axis. Another way of looking at the emergent wave is to say that the incident vector has been rotated toward the fast axis by \( 2\phi \). For instance, a vertically polarized incident wave can be converted into a horizontally polarized wave by inserting the \( \lambda/2 \) plate at an azimuth angle of 45°.

Next, the case of elliptically polarized incident light is considered. Let us say that the direction of the major axis is vertical and the handedness is right. The direction of the fast axis is \( \phi \) degrees from the major axis of the ellipse. As illustrated in Fig. 6.16b, the incident elliptically polarized wave is decomposed into components parallel to the fast and slow axes, represented by circles \( C_1 \) and \( C_2 \), respectively. The points 1, 2, 3, 4 on the circle \( C_1 \) correspond to points 1, 2, 3, 4 on the circle \( C_2 \).

After transmission through the \( \lambda/2 \) plate, the points on circle \( C_2 \) (slow axis) lag the points on circle \( C_1 \) (fast axis) by 180°. This is shown in Fig. 6.16b by moving the points 1, 2, 3, 4 on \( C_2 \) diametrically opposite, as indicated by 1', 2', 3', and 4', while the points 1, 2, 3, and 4 on \( C_1 \) remain unchanged. The emergent wave is drawn with the new combination. The emergent wave is elliptically polarized with the same shape but rotated from that of the incident wave. The emergent wave looks like a mirror image with respect to the fast axis of the \( \lambda/2 \) plate. The handedness is reversed. Alternatively, the emergent ellipse also looks as if it were made by rotating the incident ellipse toward the fast axis by \( 2\phi \), then reversing the handedness.

The \( \lambda/2 \) plate does not change the shape of the ellipse, only the orientation and handedness. When just a simple change of handedness is desired, one of the axes of the ellipse is made to coincide with the fast axis of the \( \lambda/2 \) plate.

6.4.3 How to Use a Quarter-Waveplate

The quarter-waveplate is probably the most popular of the waveplates. It has a variety of uses including polarization converter, handedness interrogator, and retardance measuring tool.

6.4.3.1 Conversion from Linear to Circular Polarization by Means of a \( \lambda/4 \) Plate

Figure 6.17 shows three configurations for converting a linearly polarized wave into either a circularly or elliptically polarized wave by means of a quarter-waveplate. In the figure, the incident light is vertically polarized and propagating from left to right. In Fig. 6.17a, looking into the source of light, the direction of the polarization of the incident light is 45° to the left of the fast axis of the quarter-waveplate. With this configuration, the emergent light is a left-handed circularly polarized wave (Problem 6.3).
Figure 6.16 Usage of a $\lambda/2$ plate. (a) Incident wave is linearly polarized. (b) Incident wave is elliptically polarized.
Figure 6.17 Converting a linearly polarized wave by means of a $\lambda/4$ plate. (a) Linear to left-handed circular polarization. (b) Linear to right-handed circular polarization. (c) Linear to elliptical polarization.
Figure 6.17b is similar to Fig. 6.17a except that the direction of polarization of the incident wave is 45° to the right of the fast axis of the quarter-waveplate. The emerging light is a right-handed circularly polarized wave.

In summary, looking toward the direction of the source, if the incident light polarization is oriented 45° to the left of the fast axis, the handedness is also left-handed. On the other hand, if the incident light polarization is oriented 45° to the right of the fast axis, the handedness becomes also right-handed. Thus, for the same linearly polarized incident wave, a change in the handedness of the emergent light is achieved by just rotating the quarter-waveplate by 90° in either direction.

An elliptically polarized wave is generated by orienting the fast axis at an azimuth angle other than 45° with respect to the direction of the incident light polarization, as shown in Fig. 6.17c.

Inspection of the results shown in Fig. 6.17 and the answers to Problem 6.3 reveals a shortcut method to drawing the emergent wave from a $pNAK/\lambda/4$ plate when the incident light is linearly polarized. Figure 6.18 illustrates this shortcut method, and the steps are explained below.

**Step 1.** Draw in the azimuth $\theta$ of the input light and $\Theta$ of the fast axis of the $\lambda/4$ plate.

**Step 2.** Draw the line $\overline{ef}$ perpendicular to the fast axis of the $\lambda/4$ plate.

**Step 3.** Complete the rectangle $efgh$. The center of the rectangle coincides with the origin.

**Step 4.** The ellipse that is tangent to this rectangle represents the polarization of the emergent light. If $E$ is to the left of the fast axis, the emergent wave has left-handed elliptical polarization.

### 6.4.3.2 Converting Light with an Unknown State of Polarization into Linearly Polarized Light by Means of a $\lambda/4$ Plate

Figure 6.19 shows an arrangement for converting an elliptically polarized wave into a linearly polarized wave. The incident light beam goes through a $\lambda/4$ plate, a $\lambda/2$ plate, and an analyzer and finally reaches the photodetector. If the fast axis of the quarter-waveplate is aligned to the major (or minor) axis of the elliptically polarized incident light, the output from the quarter-waveplate will be linearly polarized. This fact is detailed in Fig. 6.20, where elliptically polarized light is decomposed into two component fields perpendicular and parallel to the major axis. The phase of the perpendicular component is delayed from that of the parallel component by 90° (see also Problem 6.4). If the fast axis of the quarter-waveplate is aligned with the delayed perpendicular component, the two component waves become in phase and the emergent wave is a linearly polarized wave as indicated by the vector $\overrightarrow{OP}$.

If the fast axis of the $\lambda/4$ plate is further rotated by 90° in either direction, again the emergent wave is a linearly polarized wave as indicated by vector $\overrightarrow{OP'}$.

A method for aligning the $\lambda/4$ plate in the desired location will now be explained. Besides the $\lambda/4$ plate, a $\lambda/2$ plate and an analyzer are added as shown in Fig. 6.19. The transmission axis (major principal axis) of the analyzer is for now set in the vertical direction. The output of the analyzer is monitored with a photodetector. The function of the $\lambda/2$ plate is to rotate the direction of the light polarization emerging from the $\lambda/4$ plate.
Step 1. Draw $E$ and $F$.

Step 2. Draw $ef$ perpendicular to the $F$ axis.

Step 3. Complete the rectangle $efgh$.

Step 4. Draw the ellipse tangent to the rectangle.

Figure 6.18 A shortcut method of finding the state of polarization when linearly polarized light is incident onto a $\lambda/4$ plate.
Figure 6.19 Converting from elliptical to linear polarization.

Figure 6.20 Converting an elliptically polarized wave into a linearly polarized wave by means of a $\lambda/4$ plate.
First, with an arbitrary orientation of the $\lambda/4$ plate, the $\lambda/2$ plate is rotated. As the $\lambda/2$ plate is rotated, the value of the minimum output from the photodetector is noted. This is the beginning of an iterative procedure aimed at producing a sharp null in the photodetector output. As the next step, the $\lambda/4$ plate is rotated by a small amount in one direction. The $\lambda/2$ plate is again rotated. The new minimum output from the photodetector is compared to the previous minimum. If the new minimum is smaller, the $\lambda/4$ plate was turned in the correct direction. The procedure of rotating the $\lambda/4$ plate by a small amount followed by a rotation of the $\lambda/2$ plate is repeated until the absolute minimum has been found. Only when the input to the $\lambda/2$ plate is linearly polarized is its output linearly polarized, and the photodetector output shows sharp nulls where the linearly polarized output from the $\lambda/2$ plate is perpendicularly polarized to the transmission axis of the analyzer.

Once a linearly polarized wave and a sharp null are obtained, the direction of polarization can be changed to the desired direction by rotating the $\lambda/2$ plate.

The fiber equivalent of Fig. 6.19 is shown in Fig. 6.15. The fiber-loop $\lambda/4$ plate can be treated as a conventional $\lambda/4$ plate whose surface is perpendicular to the plane of the fiber loop. The alignment procedure for the conventional waveplates in Fig. 6.19 applies equally to the fiber-loop waveplates in Fig. 6.15. The direction of the fast axis of the $\lambda/2$ and $\lambda/4$ loops lies in the plane of the fiber loop as explained in Section 6.3.3.

### 6.4.3.3 Measuring the Retardance of a Sample

Figure 6.4 summarizes the sequential change in the state of polarization as the retardance $\Delta$ is increased from 0° to 360° for a linearly polarized initial state with $B/A = 1$. By the same token, if linearly polarized light with $B/A = 1$ is incident on a birefringent sample of known orientation but unknown retardance, the retardance can be determined from the state of polarization of the emergent light.

Figure 6.21 shows an arrangement of Senarmont’s method for measuring the retardance of a sample. The incident light is linearly polarized at $\theta = 45^\circ$ with respect to the $x$–$y$ axes and the amplitudes of the $x$ and $y$ components are the same, namely, $B/A = 1$. Either the fast or slow axis of the crystal is aligned to the $x$ axis (a method for locating the fast or slow axis can be found in Problem 6.10). The emergent light is an elliptically polarized wave with a $45^\circ$ (or $135^\circ$) azimuth angle. The ellipticity $\tan \beta$, which is the ratio of the major axis to the minor axis, depends on the value of the

![Figure 6.21](image-url)
retardance and, as obtained in Problem 6.11,

$$\Delta = \frac{1}{2} \beta$$ (6.31)

The elliptically polarized wave further enters a $\lambda/4$ plate whose fast axis orientation is set at $45^\circ$ or $135^\circ$. The emergent light from the $\lambda/4$ plate is linearly polarized because the fast axis is aligned to the major or minor axis of the ellipse as mentioned in Section 6.4.3.2. The azimuth angle of the light emergent from the $\lambda/4$ plate is $\theta = 45^\circ + \beta$. The value of $\beta$ can be found from the direction of the sharp null when rotating the analyzer. Finally, $\Delta$ is found from $\beta$ by Eq. (6.31).

It is important to realize that the direction of the major or minor axis of the elliptically polarized light incident to the $\lambda/4$ plate is always at $45^\circ$ or $135^\circ$ if $B/A = 1$, and the fast axis of the sample is along the $x$ axis, regardless of $\Delta$. Once the fast axis of the $\lambda/4$ plate is set to $\theta = 45^\circ$ or $135^\circ$, the $\lambda/4$ plate need not be adjusted during the measurement. The only adjustment needed is the direction of the analyzer. This is a noble feature of Senarmont’s method.

### 6.4.3.4 Measurement of Retardance of an Incident Field

The previously mentioned Senarmont’s method used a priori knowledge of the azimuth angle of $45^\circ$ of the emergent light from the sample, but in this case, the retardance between the $x$ and $y$ directions of an incident wave with an arbitrary state of polarization will be measured.

The measurement consists of three steps using a polarizer, a $\lambda/4$ plate, and a photodetector. Let an arbitrary incident wave be represented by Eqs. (6.1) and (6.2). The arrangement is similar to the one shown in Fig. 6.21, but the sample is removed.

**Step 1.** First, only the polarizer, which is used as an analyzer, and the photodetector are installed. With the transmission axis of the analyzer along the $x$ axis, the transmitted power is measured. The transmitted power $I_x$ is expressed as

$$I_x = \frac{1}{2} |E_x|^2 = \frac{1}{2} A^2$$

where an ideal analyzer is assumed (i.e., lossless transmission of the through polarization and complete rejection of the cross polarization). Next, the transmission axis of the analyzer is rotated along the $y$ axis, and the transmitted power is measured. The transmitted power $I_y$ is expressed as

$$I_y = \frac{1}{2} |E_y|^2 = \frac{1}{2} B^2$$

The total transmitted power $I_0$ is

$$I_0 = I_x + I_y$$

Next, the transmission axis of the analyzer is rotated at a $45^\circ$ azimuth angle to the $x$ axis and the transmitted light power is measured. Both $E_x$ and $E_y$ contribute to
the component along the analyzer transmission axis:

$$I_1 = \frac{1}{2} \left| \frac{1}{\sqrt{2}} E_x + \frac{1}{\sqrt{2}} E_y \right|^2$$

$$= \frac{1}{4} |A + Be^{j\Delta}|^2$$

$$= \frac{1}{4} (A + Be^{j\Delta})(A + Be^{-j\Delta})$$

$$= \frac{1}{4} (A^2 + B^2 + 2AB \cos \Delta)$$

**Step 2.** The $\lambda/4$ plate is inserted in front of the analyzer. The fast axis of the $\lambda/4$ plate is parallel to the $y$ axis, and the analyzer is kept with its transmission axis at $45^\circ$ to the $x$ axis. The transmitted power is

$$I_2 = \frac{1}{2} \left| \frac{1}{\sqrt{2}} E_x + \frac{1}{\sqrt{2}} E_y e^{-j90^\circ} \right|^2$$

$$= \frac{1}{4} (A^2 + B^2 + 2AB \sin \Delta)$$

From these measured values of $I_0$, $I_1$, and $I_2$, the retardance is

$$\Delta = \tan^{-1} \left( \frac{2I_2 - I_0}{2I_1 - I_0} \right)$$

### 6.5 LINEAR POLARIZERS

A linear polarizer favorably passes the component of light polarized parallel to the transmission axis and suppresses the component polarized parallel to the extinction axis. The extinction axis is perpendicular to the transmission axis.

The three major types of linear polarizers are the (1) dichroic polarizer, (2) birefringence polarizer, and (3) polarizer based on Brewster’s angle and scattering. Each type has merits and demerits and a choice has to be made considering such parameters as transmission loss, power of extinction, wavelength bandwidth, bulkiness, weight, durability, and cost.

#### 6.5.1 Dichroic Polarizer

Figure 6.22 shows an oversimplified view of the molecular structure of a dichroic sheet polarizer. It is analogous to a lacy curtain suspending an array of long slender conducting molecules.

The dichroic sheet is quite thin and is normally laminated on a transparent substrate for strength. Transmission through the dichroic sheet depends on the direction of polarization of the incident wave [7,8].

When the axis of a conducting molecule is parallel to the $E$ field, the situation is similar to a linear dipole antenna receiving a radio signal. A current is induced in the axial direction and can flow freely along the molecule except at both ends. At the ends, the axial current has to be zero and the direction of the current has to be
The “dichroic” polarizer has to do with “two colors.” Historically [1], certain crystals displaying polarizing properties were observed to change color when they were held up to sunlight and were viewed with a polarizer. The color changes occur when the polarizer is rotated.

Although somewhat of a misnomer, the term dichroic has persisted. At present, any sheet whose absorption depends on the direction of polarization of the incident light is called a dichroic sheet. Another example is the dichroic filter. It reflects at a specified wavelength and transmits at another specified wavelength, while maintaining a nearly zero coefficient of absorption for all wavelengths of interest.

reversed, resulting in a current standing wave $I_z$ with a sinusoidal distribution along the molecular axis as shown in Fig. 6.23a.

When, however, the direction of the $E$ field is perpendicular to the axis of the molecule, the current is induced in a diametrical direction. The current has to be zero at the left and right edges of the molecule. The magnitude of the excited current cannot be large because the zero current boundary conditions are located so close to each other. The distribution of the current $I_x$ has a quasitriangular shape with a short height. The magnitude of the induced current for a perpendicular orientation of the $E$ field is small compared to that for a parallel orientation of the $E$ field, as shown in Fig. 6.23b.

Regardless of the conductivity of the molecule, the transmitted light is attenuated as long as the direction of the $E$ field is parallel to the molecular axis. For resistive-type slender molecules, the induced current is converted into heat and there is no reflected wave. For slender molecules that are conductors, the induced current sets up a secondary cylindrical wave whose amplitude is identical to that of the incident wave but whose phase is shifted by 180° in order that the resultant field on the surface of the molecule vanishes, thereby satisfying the boundary condition of a perfect conductor. In the region beyond the molecule, both the transmitted and the 180° out-of-phase secondary wave propagate in the same direction but in opposite phase, and they cancel each other. When the incident wave is from the left to the right, there is no emergent wave in the region to the right of the molecule, as illustrated in Fig. 6.24. In the region
Sir John A. Fleming (1849–1946) used to explain electromagnetic wave phenomena by making analogies with wagging a rope, and it is tempting to apply a rope analogy to the case of the slender molecule polarizer. Imagine that a rope is stretched horizontally through a set of vertical bars. If on one side of the bars, the rope is shaken up and down to produce a wave propagating with its crests in the vertical direction, the wave will pass through the bars unhindered. On the other hand, if the rope is shaken left and right so that its crests are in the horizontal direction, the propagating wave is blocked by the bars. This analogy is opposite to reality in the case of light transmission through the slender molecule polarizer. The transmission axis is perpendicular to the bars (axis of slender molecules), and the extinction axis is parallel to the bars. The shaking rope analogy is shaky in this case.

in front of the molecule, these two waves are propagating in opposite directions, and there exists a standing wave in the region to the left of the molecule in Fig. 6.24.

As mentioned earlier, when the $E$ field is perpendicular to the molecule axis, the degree of excitation of the induced current $I_x$ on the molecule is small and the wave can propagate through the molecule curtain with minimum attenuation.

The quality of a polarizer is characterized by two parameters: the major principal transmittance $k_1$ and the minor principal transmittance $k_2$. $k_1$ is the ratio of the intensity of the transmitted light to that of the incident light when the polarizer is oriented to maximize the transmission of linearly polarized incident light. $k_2$ is the same ratio but when the polarizer is oriented to minimize transmission. The values of $k_1$ and $k_2$ are defined when the incident light direction is perpendicular to the surface of the polarizer. The performance of the polarizer is optimum at this angle of incidence.

The value of $k_2$ can be reduced by increasing the density of the slender molecules, but always with a sacrifice of a reduction in $k_1$. Figure 6.25 shows the characteristic curves of $k_1$ and $k_2$ for a typical dichroic sheet polarizer. Even though the transmission ratio defined as $R_t = k_1/k_2$ can be as large as $10^5$, it is hard to obtain the ideal value of $k_1 = 1$ with a dichroic polarizer sheet. On the other hand, the birefringent-type polarizer can provide both a large transmission ratio and a value of $k_1$ very close to unity.

The advantages of the dichroic sheet are that it is thin, lightweight, and low-cost, but the disadvantages are low $k_1$ values (70% is common) and relatively low
power handling capability due to absorption. The transmission ratio deteriorates in the ultraviolet region, $\lambda < 300$ nm.

**Example 6.4** Find the state of polarization of the emergent wave $E_i$ for the following combinations of incident field $E_0$ and polarizer.
Figure 6.24 Top view of the fields incident onto and scattered from a slender perfectly conducting molecule.

(a) The incident light is linearly polarized, and a poor-quality polarizer is used with major and minor transmittances $\sqrt{k_1} = 1$ and $\sqrt{k_2} = 0.5$, respectively.

(b) This situation is the same as case (a) but with an elliptically polarized incident wave.

(c) The incident wave is elliptically polarized in the same way as case (b) but the polarizer has ideal characteristics, namely, $\sqrt{k_1} = 1$ and $\sqrt{k_2} = 0$. Draw the locus of the major axis of the emergent light as the polarizer is rotated in its plane.

Solution
The solutions are shown in Fig. 6.27.
(a) Figure 6.27a shows the configuration. The transmission axis of the polarizer is shown by an extended T. The incident field $E_0$ is decomposed into components $E_{01}$.
The concept of an absorption indicatrix is shown in Fig. 6.26. It is used to analyze the transmission of a polarized wave through a bulk medium that possesses an absorboanisotropy like a dichroic crystal. The method for using this indicatrix is similar to that used for the refraction indicatrix presented in Section 4.5.2. Referring to Fig. 6.26, consider the case when light propagating along the direction $ON$ is incident onto an absorboanisotropic crystal. The intercept of the plane containing the origin and perpendicular to $ON$ with the ellipsoid generates the “cross-sectional ellipse.” The lengths of the vectors $a_1$ and $a_2$ of the major and minor axes represent absorbancies in these two directions of polarization.

If the direction of polarization of the incident light is arbitrary, the $E$ field of the incident light is decomposed into components parallel to $a_1$ and $a_2$, which suffer absorbancies $a_1$ and $a_2$, where $a_1$ and $a_2$ are the major and minor axes of the ellipse. The amplitude of the emergent light is the vectorial sum of these two components [9].

The shape of the ellipsoid of the absorption indicatrix is significantly more slender than that of the refraction indicatrix in Section 4.5.2.

and $E_{02}$, which are parallel and perpendicular to the transmission axis of the polarizer. Their phasor circles are $C_1$ and $C_2$. The incident field being linear, the phasors are in phase and points 1, 2, 3, and 4 are numbered accordingly. The $E_{02}$ component suffers an attenuation of $\sqrt{k_2} = 0.5$, which is represented in Fig. 6.27a by shrinking circle $C_2$. On $C_2$, the points 1, 2, 3, and 4 shrink to $1'$, $2'$, $3'$, and $4'$. Successive intersections of
Figure 6.26 Absorption indicatrix used in finding the transmission through an absorboanisotropic crystal.

points 1, 2, 3, and 4 of circle $C_1$ and points 1', 2', 3', and 4' of the shrunken circle $C_2$ produce the state of polarization of the emergent wave.

The emergent wave is linearly polarized but the azimuth angle is not the same as that of the incident wave.

(b) Circles $C_1$ and $C_2$ are set up in a similar fashion to that of part (a), the only difference being that points 1, 2, 3, and 4 of the linear incident light are replaced by points 1, 2, 3, and 4 of the incident ellipse, as shown in Fig. 6.27b. The emergent light polarization is formed from points 1, 2, 3, and 4 of $C_1$ and points 1', 2', 3', and 4' of the shrunken $C_2$.

The azimuth angle of the emergent wave is closer to the azimuth of the polarizer than the incident wave.

(c) Figure 6.27c explains the case with an ideal polarizer. Since $k_2 = 0$, the radius of the circle $C_2$ shrinks to zero, and the amplitude of the emergent wave is determined solely by the radius of circle $C_1$. The emergent light is linearly polarized and the direction of the emergent $E$ field is always along the direction of the transmission axis. Referring to Fig. 6.27c, the solid line ellipse represents the incident ellipse. When the
azimuth angle of the polarizer is at $\Theta$, the direction of the emergent linear polarization is also at $\theta = \Theta$, and the amplitude is represented by $\overline{Oh}$, which is perpendicular to the tangent to the incident ellipse. The dashed line in the figure shows the locus of the field vector as the polarizer is rotated. The position of $h$ in Fig. 6.27c corresponds to $1'$ when $k_2 = 0$ in Fig. 6.27b. It should be noted that the answer is not an ellipse.
6.5.2 Birefringence Polarizer or Polarizing Prism

As mentioned in Chapter 4, there are only two possible directions of polarization (e- and o-waves) inside a uniaxial crystal. No other directions of polarization are allowed. A birefringence polarizer, which is sometimes called a prism polarizer, creates a linearly polarized wave by eliminating one of the two waves. One type makes use of the difference in the critical angles of total internal reflection for e- and o-waves. Another type makes use of the difference in the angle of refraction for the two waves. A significant advantage of the birefringence polarizer over a simple dichroic polymer polarizing sheet is its high transmission coefficient of 90% to 95% or better compared to 70% for the polarizing sheet. Moreover, the polarizing beamsplitter gives access to beams of each polarization.

Figure 6.28 shows a cut-away view of the Nicol prism. A calcite crystal is sliced diagonally and is cemented back together with Canada balsam cement whose index of refraction is $n = 1.55$. Since the indices of refraction of calcite are $n_e = 1.486$ and $n_0 = 1.658$ at $\lambda = 0.58$ µm, the e-wave does not encounter total internal reflection and exits through the crystal to the right. The crystal is sliced at such an angle that total internal reflection of the o-wave takes place at the interface between the calcite crystal and the Canada balsam cement. The reflected o-wave is absorbed by the surrounding dark coating.

Even though the Nicol prism is one of the best-known polarizers because of its long history, the Nicol prism has the following disadvantages: the Canada balsam cement absorbs in the ultraviolet region of the spectrum, the power handling capability is limited by the deterioration of the cement, and the emergent beam is laterally displaced from the position of the incident beam. A favorable characteristic of the Nicol prism is its reasonable field of view of 28°.

The Glan–Foucault or Glan–Air polarizing prism is shown in Fig. 6.29a. This type eliminates the use of Canada balsam cement so as to avoid absorption in the
ultraviolet region and the limitation on the power handling capability of the prism. As shown in Fig. 6.29a, the front surface of the prism is cut parallel to the optic axis and perpendicular to the incident beam. The angle of the slanted airgap is chosen such that total internal reflection takes place for the o-wave. A polarizing prism normally discards the o-wave by the use of an absorptive coating. However, if desired, the side surface of the polarizer can be polished to allow the o-wave to exit. The o-wave from this polarizing beamsplitter can be used for monitoring purposes or for providing an additional source with an orthogonal direction of polarization to

Figure 6.29 Glan–Foucault prism polarizer/beamsplitter and its modification. (a) Cutaway view of the Glan–Foucault prism polarizer/beamsplitter. (b) Same as (a) but modified by Taylor for better transmission.
the e-wave. An additional merit of this prism is its short longitudinal dimension. The Glan–Air prism, however, suffers from the demerits of a narrow acceptance angle of 15° to 17° and multiple images caused by multiple reflections in the airgap.

The Glan–Foucault prism was modified by Taylor who rotated the crystal axis by 90°, as illustrated in Fig. 6.29b. With this orientation Brewster’s angle can be used to minimize the reflection. The value of $k_1$ was significantly increased.

The Glan–Thomson polarizing prism has the same geometry as the Glan–Foucault prism but uses Canada balsam cement in place of the airgap in order to increase the viewing angle to 25° to 28° at the cost of the aforementioned drawbacks of Canada balsam cement.

Several other types of polarizing prisms are similar and all are shown for comparison in Fig. 6.30. The direction of refraction is determined by assuming that a negative birefringent crystal ($n_e < n_o$) like calcite is used. The angular separation of the o- and e-waves is made by different arrangements of the optic axes of two pieces of the same crystal material. The Rochon, Senarmont, and Ahrens polarizing prisms do not deviate the direction of one of the transmitted lightwaves from the direction of the incident light. With reference to Fig. 6.30, the deviated transmitted light from the Rochon prism is vertically polarized while that of the Senarmont prism is horizontally polarized. The Wollaston polarizing prism maximizes the angular separation between the two beams because what is labeled the output o-wave is, in fact, the e-wave in the first prism and both waves are refracted at the interface. The geometry of the Cotton-type prism is almost the same as the bottom piece of the Ahrens type, except for the larger apex angle of the prism for optimization of operation.

Prisms based on refraction create aberrations when they are introduced in a convergent beam. This is because the vertical geometry is not the same as the horizontal geometry, and the angle of refraction from the boundaries in the vertical direction is different from that of the horizontal direction just like a cylindrical lens.

### 6.5.3 Birefringence Fiber Polarizer

Next, the fiber-loop birefringence polarizer will be explained. When an optical fiber is bent too tightly, the light in the core starts to leak out. The amount of leakage, however, depends on the direction of polarization of the light because the change in the refractive index caused by the bending is not isotropic. The fiber-loop-type polarizer makes use of this property. Needless to say, the fiber-loop polarizer is especially advantageous for use in fiber-optic communications because polarization control is achieved without having to exit the fiber and transmission of light in the fiber is uninterrupted.

When an optical fiber is bent, the fiber is compressed in the radial direction of the bend and is expanded in the direction perpendicular to it, as shown at the top of Fig. 6.13. The refractive index of glass is lowered where it is compressed and raised where it is expanded. The differential stress creates anisotropy in the refractive indices in the two aforementioned directions in the fiber.

Both single-mode and polarization-preserving fibers can be used for fabricating a polarizer, but better results are obtained with polarization-preserving fibers which already have birefringence even before bending the fibers. With ordinary single-mode
Figure 6.30 Various types of birefringence polarizers (polarizing prisms) using calcite. (a) Rochon. (b) Wollaston. (c) Senarmont. (d) Ahrens. (e) Cotton.
fibers, the fiber has to be bent over a much tighter radius to achieve the desired effect and, consequently, is prone to breakage. Birefringence in the Panda-type polarization-preserving fiber, shown in the inset in Fig. 6.31, is produced by contraction of the glass with a higher thermal expansion coefficient in the "eyes" region when the drawn fiber solidifies. The index of refraction $n_x$ seen by the wave polarized in the direction of the "eyes" is raised due to the expansion of the core and that of $n_y$ seen by the wave polarized in the direction of the "nose" is lowered due to the contraction of the core. The slow axis is in the direction of the "eyes" and the fast axis is in the direction of the "nose."

Figure 6.31  Refractive index profiles of a Panda fiber. (a) Profile along the $x$ axis. (b) Profile along the $y$ axis. (After K. Okamoto, T. Hosaka, and J. Noda [10].)
By comparing the inset in Fig. 6.31 with Fig. 6.13, one soon notices that the same effect caused by the contraction of the “nose” can be generated by just bending the fiber in the \( y \) direction. As a result, the birefringence of the Panda fiber is even more enhanced when the Panda fiber is bent in the \( y \) direction.

Figure 6.31 shows the calculated profile of the indices of refraction when the Panda fiber is bent [10]. The distribution along the “eyes” direction of the Panda fiber is shown in Fig. 6.31a while that along the “nose” direction is shown in Fig. 6.31b.

As long as one stays on the line connecting the centers of the “eyes” (\( x \) axis) the strains inside the core and the cladding are identical and the difference between \( n_x \) and \( n_y \), which is directly related to the strain, is the same in the core and cladding, as shown in Fig. 6.31a.

Along the “nose” direction (\( y \) axis), however, the amount of strain varies significantly with the distance from the center of the fiber, and the difference between \( n_x \) and \( n_y \) also varies with the radius in the \( y \) direction. As shown in Fig. 6.31b, even though \( n_y \) is smaller than \( n_x \) inside and on the periphery of the core, \( n_y \) grows bigger than \( n_x \) in the region beyond 20 \( \mu \)m. The difference \( \Delta_y \) between \( n_y \) in the core and \( n_y \) in the cladding also decreases with \( y \), whereas the difference \( \Delta_x \) between \( n_x \) in the core and \( n_x \) in the cladding stays the same with \( y \). The evanescent wave exists in these regions and a slight decrease in \( \Delta_x \) significantly increases the bending loss of the \( E_y \) component [10]. Thus, the emergent light is predominantly \( E_x \) polarized in the direction of the “eye.” It is this anisotropic strain distribution that makes the fiber polarizer work.

The tensile stress due to the Panda “eyes” can be enhanced further by increasing the bending of the fiber in the \( y \) direction. Excessive bending, however, starts incurring the transmission loss of the \( x \)-polarized wave. The amount of bending has to be determined from a compromise between the transmittance \( k_1 \) and the extinction ratio (defined as the inverse of the transmission ratio) \( R \). A transmission loss of 0.5 dB with an extinction ratio \( R = -30 \) dB is obtainable for a wide wavelength range by 10 turns of a 3-cm-diameter Panda fiber loop [11].

It is difficult to know the orientation of the Panda eyes unless its cross section is examined under a microscope. The use of a special Panda fiber whose cross section is oval shaped to indicate the orientation of the Panda “eyes” makes it easier to bend the fiber into a coil while maintaining the right bending orientation [10].

### 6.5.4 Polarizers Based on Brewster’s Angle and Scattering

Brewster’s angle is another phenomenon that depends on the direction of polarization of light and can be utilized to design a polarizer. Brewster’s angle of total transmission exists only for a lightwave whose direction of polarization is in the plane of incidence.

Figure 6.32 shows a pile-of-plates polarizer that is based on Brewster’s angle. Brewster’s condition is

\[
\tan \theta_B = \frac{n_1}{n_0}
\]

The wave polarized in the plane of incidence transmits through totally without reflection. The wave polarized perpendicular to the plane of incidence also transmits through, with some loss due to reflection. To be effective as a polarizer, several plates are necessary in order to increase the loss due to reflection of the wave polarized perpendicular to the plane of incidence.
Light with high purity of linear polarization is obtainable from an external cavity-type gas laser such as shown in Fig. 14.1. This type of laser uses a Brewster window, and in the case of the He–Ne laser, the light goes back and forth more than 2000 times before exiting the cavity. This is equivalent to a pile of 2000 plate polarizers, and light with very pure linear polarization is obtained.

### 6.5.5 Polarization Based on Scattering

A rather unconventional polarizer makes use of the nature of Rayleigh scattering. Scattering from a particle smaller than the wavelength of light creates polarized light. Referring to Fig. 6.33, the light scattered in the direction normal to the incident ray is linearly polarized. The vertically polarized component of the incident light cannot
be scattered in the vertical direction because the $E$ field would become parallel to the direction of propagation. The light scattered toward the vertical direction is highly horizontally polarized light, and the light scattered toward the horizontal direction is highly vertically polarized light.

A chamber filled with either $N_2$ or $CO_2$ molecules makes a polarizer. Even though the amount of the scattered light is small, the purity of the polarization is good. The direction of polarization is perpendicular to the plane containing the path of the light from the source to the observer by way of the scatterer, as indicated in Fig. 6.33.

### 6.6 CIRCULARLY POLARIZING SHEETS

A polarizer sheet laminated with a $\lambda/4$ plate sheet is sometimes marketed as a circularly polarizing sheet. Its usages are presented here.

#### 6.6.1 Antiglare Sheet

In this section, a method of preventing glare using a circularly polarizing sheet will be described. Figure 6.34 shows a circularly polarizing sheet being used as an antiglare cover for a radar screen. Figure 6.35 explains the function of the sheet, and for purposes of the explanation, the polarizer and the $\lambda/4$ plate sheet are separated. Figure 6.35a shows the state of polarization of the light incident on to the radar surface, and Fig. 6.35b shows the state of polarization of the reflected wave from the radar screen. In Fig. 6.35a, the direction of the polarization is $45^\circ$ to the left of the fast axis, and left-handed circularly polarized light is incident onto the radar screen.

![Figure 6.34](image_url)

**Figure 6.34** A circularly polarizing sheet, which is a lamination of polarizer and $\lambda/4$ plate sheets, is used for prevention of glare on a radar screen.
The very right top inset in the Figure shows what happens on reflection. If the surface is assumed to be a perfect reflector, at the moment when the field vector of the incident light points in the direction $OA$, the field vector of the induced field should point in the opposite direction $OA'$ to satisfy the boundary condition that the resultant tangential $E$ field is zero on the surface of a perfect conductor. At the next moment, when the incident vector moves to $OB$, the induced vector moves to $OB'$. Although
the incident vector and the induced vector always point in opposite directions, they always rotate in the same direction.

The reflected wave is the expansion of the induced wave. Figure 6.35b shows how the reflected wave propagates toward the operator. Recall that the observer looks toward the source of light, and the reflected wave is right-handed circularly polarized. Likewise, the azimuth angle of the fast axis of the $\lambda/4$ plate now looks to the observer like $\Theta = 135^\circ$.

The light transmitted through the $\lambda/4$ plate is found by the circle diagram to be horizontally polarized. The light cannot go through the polarizer, and the light reflected from the radar surface does not reach the radar operator. The blips originating from the radar screen, which are randomly polarized, reach the operator’s eye with some attenuation.

### 6.6.2 Monitoring the Reflected Light with Minimum Loss

A reflectometer gathers information from reflected light. One of the simplest ways to sample the reflected light is to use a nonpolarizing beamsplitter (NPBS) in the manner shown in Fig. 6.36a. With this configuration, however, the reflected as well as the

---

**Figure 6.36** Comparison between two types of reflectometers. (a) Using a nonpolarizing beamsplitter. (b) Using a polarizing beamsplitter and a $\lambda/4$ plate.
incident beam will be split by the splitter, resulting in light being lost to the system. If a beamsplitter with reflectance $R$ is used, the intensity of the light collected by the system is $I_{in} T R \sigma$, where $I_{in}$, $T$, and $\sigma$ are incident light intensity, the transmittance of the beamsplitter, and the reflectivity of the target, respectively. Because of the constraint $T + R = 1$, the optimum intensity of the collected light occurs when $R = 0.5$, and the collected light intensity is at best $0.25 I_{in} \sigma$. Only one-quarter of the incident light intensity is useful.

The reflected light is often weak, as, for instance, in a system for remotely analyzing the gas contents from a smokestack. The system shown in Fig. 6.36b can be used to maximize the sensitivity. This reflectometer uses the combination of a polarizing beamsplitter (PBS) and a quarter-waveplate.

The arrangement is quite similar to that for preventing the glare explained in Fig. 6.35, where the vertically polarized incident light is converted into a horizontally polarized reflected light after passing through the $\lambda/4$ plate twice. In the reflectometer of Fig. 6.36b, the vertically polarized light transmits through the PBS and is converted into a left-handed circularly polarized wave by the $\lambda/4$ plate whose azimuth is $45^\circ$. The light reflected from the target is a right-handed circularly polarized wave, which in turn is converted into a horizontally polarized light by the same $\lambda/4$ plate. The horizontally polarized wave is reflected by the PBS to the detector.

The power loss due to the transmission loss of the optical components is $10^{-3} - 10^{-5}$, depending on the quality of the components.

6.7 ROTATORS

When a linearly polarized light propagates in quartz along its optic axis, the direction of polarization rotates as it propagates. Similar phenomena can be observed inside other crystals like cinnabar ($\text{HgS}$) and sodium chlorate ($\text{NaClO}_3$), as well as solutions like sucrose ($\text{C}_{12}\text{H}_{22}\text{O}_{11}$), turpentine ($\text{C}_{10}\text{H}_{16}$), and cholesteric liquid crystals. Even some biological substances like amino acids display this effect. This phenomenon of rotation of the direction of polarization is called optical activity. A substance that displays optical activity is called an optically active substance. Each optically active substance has a particular sense of rotation. Media in which the rotation of polarization is right-handed looking toward the source are called dextrorotary (dextro in Latin means right). Media in which the rotation of polarization is left-handed are called levorotary (levo in Latin means left). There are both $d$- and $l$-rotary varieties of quartz.

Fresnel explained the mechanism of optical activity by decomposing a linearly polarized wave into circularly polarized waves. As shown in Fig. 6.37, linearly polarized incident light can be considered as a combination of right- and left-handed circularly polarized waves with equal amplitudes. If these two oppositely rotating circularly polarized waves rotate at the same speed, the direction of the polarization of the resultant wave remains unchanged. However, if the rotation speeds are different, the direction of polarization of the resultant wave rotates as the two waves propagate.

In explaining the difference in rotation speeds of the left and right circular component waves, Fresnel attributed this to the rotational asymmetry of the molecular structure of the optically active medium.

A birefringent material is a material that is characterized by two indices of refraction. If, for example, the refractive indices are $n_x$ and $n_y$, corresponding to $x$ and $y$ linearly
One may wonder about the validity of Fresnel’s explanation of optical activity for optically active liquids because of the random orientation of the molecules. As shown in the figure, a coil spring that looks right-handed is still right-handed even when it is flipped over.

polarized component waves, the material is said to be linearly birefringent. Retarders are examples of linearly birefringent devices. If the refractive indices are $n_l$ and $n_r$, corresponding to left and right circularly polarized component waves, the material is said to be circularly birefringent. Optically active substances are examples of circular birefringence.
Let us compare the emergent polarization for these two different types of birefringence. In the case of linear birefringence, the shape and/or orientation of the emergent polarization may differ from that of the incident light, as illustrated in the retarder examples shown in Figs. 6.7 and 6.8. On the other hand, for a circularly birefringent medium, the orientation of the emergent polarization changes, but the shape remains the same. For example, linearly polarized light incident on an optically active medium will remain linearly polarized, but the direction will rotate, as illustrated in Fig. 6.37.

The angle of rotation in an optically active medium is proportional to the distance of propagation in the medium. The angle of rotation per unit distance is called the rotary power. The rotary power of quartz, for instance, is $27.71^\circ$/mm at the $D$ line of the sodium spectrum ($\lambda = 0.5893$ $\mu$m) and at 20°C. In the case of a liquid substance like natural sugar dissolved in water, the angle of rotation is proportional to both the length of transmission and the concentration of the solute. The saccharimeter detailed in Section 6.7.1 determines the concentration of an optically active sugar solution by measuring the angle of rotation.

The rotary power depends on the wavelength of the light as well as the temperature of the substance. If an optically active medium is placed between the orthogonally oriented polarizer and analyzer shown in Fig. 6.38, and white light is used as the incident light, then the optical spectrum is attenuated for wavelengths whose angles of rotation are an integral multiple of $\pi$ radians and their complimentary colors appear. A beautiful color pattern is observed. This phenomenon is called rotary dispersion.

The Faraday effect causes a substance to behave like an optically active medium when an external magnetic field is applied. This induced optical activity exists only when an external magnetic field is applied. The sense of rotation is solely determined.
by the direction of the magnetic field and does not depend on the direction of light propagation. This is an important distinction between ordinary optical activity (a reciprocal phenomenon) and the Faraday effect (a nonreciprocal phenomenon). In the case of natural optical activity, if the direction of light propagation is reversed in an $l$-rotary material, the rotation is still $l$-rotary. In the case of the Faraday effect, if a material is $l$-rotary when the light propagates in the direction parallel to the magnetic field, the material is $d$-rotary when the light propagates antiparallel to the magnetic field.

**Example 6.5** As shown in Fig. 6.38, the polariscope consists of two polarizer sheets arranged with their transmission axes perpendicular to each other. Find the amplitude $E_3$ of the emergent light when the following components are inserted between the polarizers. Assume $k_1 = 1$ and $k_2 = 0$.

(a) No component.

(b) A polarizer sheet with transmission axis along $x'$ and an azimuth angle of $45^\circ$.

(c) A $\lambda/4$ plate with an azimuth angle of $45^\circ$ (fast axis along $x'$ axis in Fig. 6.38).

(d) A $\lambda/2$ plate with azimuth angle of $45^\circ$.

(e) A full-waveplate with azimuth angle of $45^\circ$.

(f) A $90^\circ$ rotator.

**Solution**

This time, the solutions are found without resorting to circle diagrams.

(a) Nothing is inserted. $E_3 = 0$.

(b) A polarizer is inserted at $45^\circ$. As shown in Fig. 6.39a, $E_1$ is decomposed into $E_{x'}$ and $E_{y'}$. $E_{y'}$ is extinguished. Only the horizontal component of $E_{x'}$ is transmitted through the analyzer.

$$E_3 = E_1 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{E_1}{2}$$

(c) A $\lambda/4$ plate is inserted at $45^\circ$. There are two ways to solve this problem.

1. Decompose $E_1$ into components $E_{x'}$ parallel to the $x'$ axis and $E_{y'}$ parallel to the $y'$ axis. The components are

$$E_{x'} = \frac{E_1}{\sqrt{2}} \quad \text{and} \quad E_{y'} = \frac{E_1}{\sqrt{2}}e^{i90^\circ}$$

These two waves are further decomposed into both horizontal and vertical ($x$ and $y$) components, but one needs to be concerned only with the $x$ component because only the horizontal component passes through the analyzer. The horizontal component of the incident wave to the analyzer is

$$E_x = \frac{1}{\sqrt{2}} \frac{E_1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{E_1}{\sqrt{2}}e^{i90^\circ}$$
where the second term is from \( \mathbf{E}_{y'} \) as shown in Fig. 6.39a and the first term is from \( \mathbf{E}_{x'} \), and

\[
E_3 = \frac{E_1}{2} (1 - e^{j90^\circ}) = \frac{E_1}{\sqrt{2}} e^{-j45^\circ}
\]

(2) Note that emergent wave \( \mathbf{E}_2 \) is a circularly polarized wave with radius \( E_1/\sqrt{2} \). The magnitude of the horizontally polarized wave \( \mathbf{E}_3 \) is \( E_1/\sqrt{2} \).
(d) \( \mathbf{E}_1 \) is decomposed into \( \mathbf{E}_{x'} \) and \( \mathbf{E}_{y'} \). The vector \( \mathbf{E}_{y'} \) is reversed in direction because of the \( \lambda/2 \) retarder, as shown in Fig. 6.39b. The resultant of \( \mathbf{E}_{x'} \) and \( -\mathbf{E}_{y'} \) becomes \( \mathbf{E}_3 \).

\[
E_3 = \frac{2E_1}{\sqrt{2}} \cos 45^\circ
= E_1
\]

Another way of obtaining the same result is to make use of the fact that a \( \lambda/2 \) plate rotates the polarization by \( 2\theta \). The vertical polarization becomes the horizontal polarization.

(e) The full-waveplate does not disturb the state of polarization and the answer is the same as (a):

\[
E_3 = 0
\]

(f) If a vertical vector pointing toward the \( +y \) direction is rotated by \( 90^\circ \), the result is a horizontal vector pointing toward the \( -x \) direction.

\[
E_3 = -E_1 \quad \square
\]

### 6.7.1 Saccharimeter

As a sugar solution is an optically active substance, the concentration of sugar can be determined by measuring the angle of rotation of the transmitted light polarization. The Lausent-type saccharimeter such as shown in Fig. 6.40 is widely used to monitor the sugar concentration of grapes in a vineyard. This is a pocketable outdoor type and uses white light. The combination of a wavelength filter \( F \) and a polarizing beamsplitter (PBS) converts the incident white light into quasimonochromatic linearly polarized light. In the left half of the field, the light passes through a thin quartz rotator \( R \), while in the right half of the field, the light misses the rotator. Thus, a slight difference in the direction of polarization is created between the light \( \mathbf{E}_L \) passing through in the left field and \( \mathbf{E}_R \) in the right field. This slight difference in the direction of polarization is for the purpose of increasing the accuracy of reading the azimuth of the analyzer \( A \) through which the incident light is viewed.

![Figure 6.40 Lausent-type saccharimeter.](image-url)
The first step is the calibration without solution. When the direction \( k_2 \) of the extinction axis is adjusted at exactly the midpoint of the angle between the two directions of polarization, the contrast in the intensities between the left and right fields diminishes. The azimuth angle \( \theta_1 \) of the analyzer of diminishing contrast is noted.

Next, the solution under test is poured into the chamber. The directions of polarization in both left and right fields will rotate by an amount that is proportional to the concentration of the sugar.

The analyzer is again rotated so that the direction \( k_2 \) of the extinction axis lies at the midpoint of the rotated directions of polarizations, and the contrast between the left and right field diminishes. This new azimuth angle \( \theta_2 \) of the analyzer is noted.

The difference \( \theta = \theta_2 - \theta_1 \) of the azimuth angles of the analyzer is the angle of rotation caused by the optical activity of the sugar solution.

The explanation of the operation of the saccharimeter will be repeated referring to Fig. 6.41. As shown in Fig. 6.41a, when the analyzer is not exactly adjusted such that \( k_2 \) is at the midpoint of \( E_L \) and \( E_R \), a contrast between the left and right field intensities can be seen (the right side is darker). As soon as \( k_2 \) of the analyzer is adjusted to the midpoint, as shown in Fig. 6.41b, the contrast disappears. The azimuth angle \( \theta_1 \) is noted. In the field, this calibration is performed prior to introducing the sample, as the power of rotation of the quartz rotator \( R \) is temperature dependent.

As the second step, the test sample is introduced into the chamber. Both \( E_L \) and \( E_R \) rotate by the same amount due to the optical activity of the sample, as shown in Fig. 6.41c, and a contrast in field intensities appears again (the left side is darker). The analyzer is rotated to find the azimuth \( \theta_2 \) that diminishes the contrast, as shown in Fig. 6.41d. The rotation is computed as \( \theta = \theta_2 - \theta_1 \).

The concentration \( P \) of sugar in grams per 100 cc of solution is given by the formula

\[
\theta = \left[ \theta \right]^\circ l \frac{P}{100}
\]

where \( \left[ \theta \right]^\circ \) is the specific rotary power of the substance at a temperature \( t^\circ C \) and a light wavelength \( \lambda \ \mu m \). For sugar, \( \left[ \theta \right]^\circ_{20,5893 \mu m} = 66.5^\circ \) (per length in decimeter \( \times \) concentration in grams per 100 cc). The quantity \( l \) is the length of the chamber in units of 10 cm.

The high-accuracy performance of this type of saccharimeter is attributed to the following:

1. The contrast of two adjacent fields rather than the absolute value of the transmitted light through the analyzer was used. The eyes are quite sensitive to detecting differences in intensities between adjacent fields.
2. The region of minimum rather than maximum light transmission through the analyzer was used. The sensitivity of the eyes to detecting a change in the transmitted light is greater near the minimum of transmission.

Another approach is to eliminate both the quartz rotator and the intensity compensator. The incident light to the sample is not divided. The direction of the polarization of the emergent light is directly measured by a split-field polarizer. The split-field polarizer is made up of two analyzers side by side with a 5° to 10° angle between the extinction axes, as shown in Fig. 6.41e. When the split-field analyzer is rotated so that the \( d'd' \) axis aligns with the direction of polarization of the light, the contrast between the left and right sections disappears. The split-field polarizer again
Figure 6.41  Field view of Lausent-type saccharimeter. (a) Without sample and unadjusted. (b) Without sample and adjusted to eliminate contrast. (c) With sample and unadjusted. (d) With sample and adjusted to eliminate contrast. (e) Split-field polarizer.

uses the contrast of the fields near the minimum of transmission and the precision reaches 0.001°.

6.7.2 Antiglare TV Camera

It is often difficult for a TV reporter to videotape a passenger inside a car due to the light reflected from the surface of the car window.

The geometry of an antiglare camera [12] is shown in Fig. 6.42. When the incident angle to the car window is in the vicinity of Brewster’s angle (56° for glass), reflection of the $p$-polarized light is suppressed, but the $s$-polarized light is not, and thus the light reflected from the car window is strongly linearly polarized. Removal of this particular component of the light minimizes glare to the TV camera. One way of accomplishing this is by means of a liquid crystal rotator such as the one shown in the display pannel in Fig. 5.33, but without the input polarizer $P_1$. 
Figure 6.42  Operation of an antiglare TV camera.

(a) off  
(b) on

(c) off  
(d) on

Figure 6.43  Demonstration of antiglare TV camera. (Courtesy of H. Fujikake et al. [12].)
The total amount of light into the camera is monitored and used as an electrical servosignal. The amount of light for the same scene should be at a minimum when the glare light has successfully been removed. The electrical servosignal rotates the polarization direction of the incident glare light until it becomes blocked by polarizer sheet $P_2$. The servosignal is minimized when the required amount of polarization rotation is achieved. In order to construct a variable rotator, TN liquid crystal rotators of $45^\circ$ and $90^\circ$ are combined. By selecting the appropriate combination of the applied electric field to the two TN liquid crystal rotators, the amount of rotation can discretely be varied from $45^\circ$ to $135^\circ$ at intervals of $45^\circ$. The photographs in Fig. 6.43 demonstrate the effectiveness of the antiglare TV camera. The photographs on the left were taken with an ordinary camera, while those on the right are the same scenes taken with the antiglare camera. With the antiglare camera, the passengers in the car, and the fish in the pond, are clearly visible.

### 6.8 THE JONES VECTOR AND THE JONES MATRIX

A method of analysis based on $2 \times 2$ matrices was introduced by R. Clark Jones [13,14] of Polaroid Corporation to describe the operation of optical systems. Each component of the system has an associated Jones matrix, and the analysis of the system as a whole is performed by multiplication of the $2 \times 2$ component matrices. Moreover, the state of polarization at each stage of the multiplication is easily known.

The state of polarization is described by the Jones vector whose vector components are $E_x$ and $E_y$. From Eqs. (6.1) and (6.2), the Jones vector is

$$ \begin{bmatrix} E_x \\ E_y \end{bmatrix} = e^{i(\beta z - \omega t)} \begin{bmatrix} A \\ B e^{i\Delta} \end{bmatrix} $$

(6.32)

The common factor is generally of no importance and is omitted. Eliminating the common factor $e^{i(\beta z - \omega t)}$, Eq. (6.32) is written as

$$ E = \begin{bmatrix} A \\ B e^{i\Delta} \end{bmatrix} $$

(6.33)

Representative states of polarizations expressed by the Jones vector are shown in Fig. 6.44.

If only the relative phase between $E_x$ and $E_y$ is important, the common factor $e^{i\Delta/2}$ can be removed, and Eq. (6.33) becomes

$$ E = \begin{bmatrix} A e^{-j\Delta/2} \\ B e^{j\Delta/2} \end{bmatrix} $$

(6.34)

If one interprets an optical component as a converter of the state of polarization from $[E_x \ E_y]$ into $[E'_x \ E'_y]$, then the function of the optical component is represented by the $2 \times 2$ matrix that transforms $[E_x \ E_y]$ into $[E'_x \ E'_y]$.

$$ \begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = [2 \times 2] \begin{bmatrix} E_x \\ E_y \end{bmatrix} $$

(6.35)

Such a $2 \times 2$ matrix is called the Jones matrix.
6.8.1 The Jones Matrix of a Polarizer

The Jones matrix of a polarizer whose major principal transmission axis is along the $x$ axis is

$$P = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

(6.36)

For an ideal polarizer, $k_1 = 1$ and $k_2 = 0$.

Next, the case when the polarizer is rotated in its plane will be considered. Let the direction of the transmission axis be rotated by $\Theta$ from the $x$ axis. The incident field $\mathbf{E}$ is expressed in $x-y$ coordinates.

In this case, the incident field $\mathbf{E}$ has to be decomposed into $E_{x1}$, which is along the major principal axis of the polarizer, and $E_{y1}$, which is along the minor principal axis. Referring Fig. 6.45,

$$E_{x1} = E \cos(\theta - \Theta)$$

$$= E \cos \theta \cos \Theta - E \sin \theta \sin \Theta$$

$$= E_x \cos \Theta - E_y \sin \Theta$$

Similarly,

$$E_{y1} = -E_x \sin \Theta + E_y \cos \Theta$$

$E_{x1}$ and $E_{y1}$ can be rewritten in a matrix form as

$$\begin{bmatrix} E_{x1} \\ E_{y1} \end{bmatrix} = \begin{bmatrix} \cos \Theta & \sin \Theta \\ -\sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

(6.37)

The matrix in Eq. (6.37) is a rotation by $\Theta$ degrees from the original coordinates; the incident field $\mathbf{E}$ is expressed in coordinates $x_1$ and $y_1$ that match the directions of the principal axes of the polarizer.
The light emergent from the polarizer is now

\[
\begin{bmatrix}
E'_{x1} \\
E'_{y1}
\end{bmatrix} =
\begin{bmatrix}
k_1 & 0 \\
0 & k_2
\end{bmatrix}
\begin{bmatrix}
E_{x1} \\
E_{y1}
\end{bmatrix}
\]  \hspace{1cm} (6.38)

The emergent wave, however, is in \(x_1\) and \(y_1\) coordinates, and needs to be expressed in the original \(x\) and \(y\) coordinates.

Referring Fig. 6.45, the sum of the projections of \(E'_{x1}\) and \(E'_{y1}\) to the \(x\) axis provides \(E'_x\). A similar projection to the \(y\) axis provides \(E'_y\).

\[
E'_x = E'_{x1} \cos \Theta - E'_{y1} \sin \Theta \\
E'_y = E'_{x1} \sin \Theta + E'_{y1} \cos \Theta
\]

which again can be rewritten in a matrix form as

\[
\begin{bmatrix}
E'_x \\
E'_y
\end{bmatrix} =
\begin{bmatrix}
\cos \Theta & -\sin \Theta \\
\sin \Theta & \cos \Theta
\end{bmatrix}
\begin{bmatrix}
E'_{x1} \\
E'_{y1}
\end{bmatrix}
\]  \hspace{1cm} (6.39)

This is a rotation by \(-\Theta\) degrees from the \(x_1\) and \(y_1\) coordinates. The emergent wave is expressed in the original \(x\) and \(y\) coordinates.
Combining Eqs. (6.37) to (6.39), the Jones matrix \( P_\Theta \) for a polarizer rotated by \( \Theta \) is given by

\[
\begin{bmatrix}
E'_x \\
E'_y
\end{bmatrix} = P_\Theta \begin{bmatrix}
E_x \\
E_y
\end{bmatrix}
\] (6.40)

\[
P_\Theta = \begin{bmatrix}
\cos \Theta & -\sin \Theta \\
\sin \Theta & \cos \Theta
\end{bmatrix}
\begin{bmatrix}
k_1 & 0 \\
0 & k_2
\end{bmatrix}
\begin{bmatrix}
\cos \Theta & \sin \Theta \\
-\sin \Theta & \cos \Theta
\end{bmatrix}
\] (6.41)

\[
P_\Theta = \begin{bmatrix}
k_1 \cos^2 \Theta + k_2 \sin^2 \Theta & (k_1 - k_2) \sin \Theta \cos \Theta \\
(k_1 - k_2) \sin \Theta \cos \Theta & k_1 \sin^2 \Theta + k_2 \cos^2 \Theta
\end{bmatrix}
\]

If the polarizer is ideal and \( k_1 = 1 \) and \( k_2 = 0 \), Eq. (6.41) becomes

\[
P_\Theta = \begin{bmatrix}
\cos^2 \Theta & \sin \Theta \cos \Theta \\
\sin \Theta \cos \Theta & \sin^2 \Theta
\end{bmatrix}
\] (6.42)

### 6.8.2 The Jones Matrix of a Retarder

The Jones matrix of a retarder whose fast axis is oriented along the \( x \) axis is

\[
R = \begin{bmatrix}
1 & 0 \\
0 & e^{i\Delta}
\end{bmatrix}
\] (6.43)

\[
R = \begin{bmatrix}
e^{-i\Delta/2} & 0 \\
0 & e^{i\Delta/2}
\end{bmatrix}
\] (6.44)

The Jones matrix of the half-waveplate is

\[
H = \begin{bmatrix}
-j & 0 \\
0 & j
\end{bmatrix}
\] (6.45)

and that of the quarter-waveplate is

\[
Q = \begin{bmatrix}
e^{-j\pi/4} & 0 \\
0 & e^{j\pi/4}
\end{bmatrix}
\] (6.46)

When a retarder is rotated, the treatment is similar to that of the rotated polarizer. The Jones matrix whose fast axis is rotated by \( \Theta \) from the \( x \) axis is

\[
R_\Theta = \begin{bmatrix}
\cos \Theta & -\sin \Theta \\
\sin \Theta & \cos \Theta
\end{bmatrix}
\begin{bmatrix}
e^{-i\Delta/2} & 0 \\
0 & e^{i\Delta/2}
\end{bmatrix}
\begin{bmatrix}
\cos \Theta & \sin \Theta \\
-\sin \Theta & \cos \Theta
\end{bmatrix}
\] (6.47)

Noting that Eq. (6.47) becomes the same as Eq. (6.40) if \( k_1 \) and \( k_2 \) are replaced by \( e^{-i\Delta/2} \) and \( e^{i\Delta/2} \), respectively, the product of the matrix Eq. (6.47) is obtained as

\[
R_\Theta = \begin{bmatrix}
e^{-i\Delta/2} \cos^2 \Theta + e^{i\Delta/2} \sin^2 \Theta & -j2 \sin \frac{\Delta}{2} \sin \Theta \cos \Theta \\
-j2 \sin \frac{\Delta}{2} \sin \Theta \cos \Theta & e^{-i\Delta/2} \sin^2 \Theta + e^{i\Delta/2} \cos^2 \Theta
\end{bmatrix}
\] (6.48)
The Jones matrix of a retarder with retardance $\Delta$ rotated by $\pm 45^\circ$ is, from Eq. (6.48),

$$
R_{\pm 45^\circ} = \begin{bmatrix}
\cos \frac{\Delta}{2} & \mp j \sin \frac{\Delta}{2} \\
\mp j \sin \frac{\Delta}{2} & \cos \frac{\Delta}{2}
\end{bmatrix} \quad (6.49)
$$

When a half-waveplate is rotated by $\Theta = \pm 45^\circ$, the Jones matrix is

$$
H_{\pm 45^\circ} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \quad (6.50)
$$

where a common factor of $e^{\pm j\pi/2}$ which appears after inserting $\Delta/2 = \pi/2$ and $\Theta = \pm 45^\circ$ into Eq. (6.48), is suppressed.

When a quarter-waveplate is rotated by $\Theta = \pm 45^\circ$, the Jones matrix is

$$
Q_{\pm 45^\circ} = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & \mp j \\
\mp j & 1
\end{bmatrix} \quad (6.51)
$$

### 6.8.3 The Jones Matrix of a Rotator

A rotator changes the azimuth angle without disturbing all other parameters of the state of polarization.

Let the incident linearly polarized field $\mathbf{E}$ be converted into $\mathbf{E}'$ by rotation as shown in Fig. 6.46. Noting that $|\mathbf{E}'| = |\mathbf{E}|$,

$$
\begin{align*}
E'_x &= E \cos(\theta_0 + \theta) = E \cos \theta_0 \cos \theta - E \sin \theta_0 \sin \theta \\
E'_y &= E \sin(\theta_0 + \theta) = E \cos \theta_0 \sin \theta + E \sin \theta_0 \cos \theta
\end{align*} \quad (6.52)
$$

Since

$$
E_x = E \cos \theta_0 \\
E_y = E \sin \theta_0
$$

Eq. (6.52) is equivalent to

$$
\begin{bmatrix}
E'_x \\
E'_y
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
E_x \\
E_y
\end{bmatrix} \quad (6.53)
$$

which is the same expression as that for rotating the coordinates by $-\theta$.

While the above explanation dealt with the rotation of a linearly polarized incident light, the same holds true for elliptically polarized incident light. For elliptical polarization, each decomposed wave rotates by the same amount and the elliptical shape does not change, but the azimuth of the axes rotates by $\theta$.

Regardless of the orientation of a light wave incident onto a rotator, the amount of rotation is the same.

**Example 6.6** Find the answers to Example 6.5 using the Jones matrix.
Solution

(a) No plate is inserted. From Eq. (6.42) with $\Theta = 90^\circ$ and then $0^\circ$, the Jones matrix expression is

$$\begin{bmatrix} E_x' \\ E_y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

There is no output.

(b) A polarizer is inserted at $\Theta = 45^\circ$. The Jones matrix expression from Eq. (6.42) is

$$\begin{bmatrix} E_x' \\ E_y' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

In the above expression, the vector just after the inserted polarizer is linearly polarized at $45^\circ$. The advantage of the Jones matrix is that the state of polarization can be known.
at each stage of manipulation. Performing the final matrix multiplication gives

\[
\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} E_x \\ 0 \end{bmatrix}
\]

which is a linearly polarized wave along the \( x \) direction.

(c) A quarter-waveplate is inserted at \( \Theta = 45^\circ \). From Eqs. (6.42) and (6.51), the Jones matrix expression is

\[
\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left( 1 - j \right) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{e^{-j\pi/2}}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_y \\ E_y e^{j\pi/2} \end{bmatrix}
\]

The intermediate state of polarization after passing through the polarizer and quarter-waveplate is left-handed circular polarization from Fig. 6.44. The emergent wave is

\[
\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \frac{e^{-j\pi/2}}{\sqrt{2}} \begin{bmatrix} E_x \\ 0 \end{bmatrix}
\]

(d) A half-waveplate is inserted at \( \Theta = 45^\circ \). From Eqs. (6.42) and (6.50), the Jones matrix expression is

\[
\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} E_y \\ 0 \end{bmatrix}
\]

The light leaving the half-wave plate is linearly polarized along the \( x \) direction. The emergent wave is

\[
\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} E_y \\ 0 \end{bmatrix}
\]

(e) A full-waveplate is inserted with azimuth \( 45^\circ \). From Eqs. (6.42) and (6.49) with \( \Delta = 2\pi \), the Jones matrix expression is

\[
\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ E_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
(f) A 90° rotator is inserted. From Eqs. (6.42) and (6.53), the Jones matrix expression is

\[
\begin{bmatrix}
E_x' \\
E_y'
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -E_y \\ 0 \end{bmatrix} = \begin{bmatrix} -E_y \\ 0 \end{bmatrix}
\]

Example 6.7 Apply Jones matrices to Senarmont’s method for measuring the retardance \( \Delta \) of a crystal plate.

Solution As shown in Fig. 6.21, a linearly polarized wave inclined at 45° is incident onto the crystal under test. The light emergent from the crystal further goes through a quarter-waveplate at \(-45°\), where the wave is converted into a linearly polarized wave whose azimuth angle determines the retardance of the sample under test.

The output field \( E \) is from Eqs. (6.44) and (6.51)

\[
\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix} \begin{bmatrix} \cos \left( \frac{\Delta}{2} + 45° \right) \\ \sin \left( \frac{\Delta}{2} + 45° \right) \end{bmatrix}
\]

The emergent wave from the quarter-waveplate is linearly polarized with azimuth angle \( \Delta/2 + 45° \).

6.8.4 Eigenvectors of an Optical System

With most optical systems, if the state of polarization of the incident wave is varied, the state of polarization of the emergent wave also varies. However, one may find a particular state of polarization that does not differ between the incident and emergent waves, except for a proportionality constant. The field vector that represents such an incident wave is called an eigenvector and the value of the proportionality constant is called an eigenvalue of the given optical system. For instance, a lasing light beam (see Section 14.2.3) bouncing back and forth inside the laser cavity has to be in the same state of polarization after each trip, over and above the matching of the phase, so that the field is built up as the beam goes back and forth. When the laser system is expressed in terms of the Jones matrix, the eigenvector of such a matrix provides the lasing condition and the eigenvalue, the gain or loss of the system. [15]

Let \( \begin{bmatrix} E_x \\ E_y \end{bmatrix} \) be an eigenvector of the optical system, and let \( \lambda \) be its eigenvalue. The relationship between incident and emergent waves in Jones matrix representation is

\[
\lambda \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}
\]

(6.54)
Equation (6.54) is rewritten as
\[
\begin{bmatrix}
  a_{11} - \lambda & a_{12} \\
  a_{21} & a_{22} - \lambda 
\end{bmatrix}
\begin{bmatrix}
  E_x \\
  E_y 
\end{bmatrix} = 0
\] (6.55)

The eigenvalues and corresponding eigenvectors of the system will be found by solving Eq. (6.55).

For nontrivial solutions for \(E_x\) and \(E_y\) to exist, the determinant of Eq. (6.55) has to vanish:
\[
(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0
\] (6.56)

Equation (6.56) is a quadratic equation in eigenvalue \(\lambda\) and the solution is
\[
\lambda_1 = \frac{1}{2} \left[ a_{22} + a_{11} - \sqrt{(a_{22} - a_{11})^2 + 4a_{12}^2} \right]
\lambda_2 = \frac{1}{2} \left[ a_{22} + a_{11} + \sqrt{(a_{22} - a_{11})^2 + 4a_{12}^2} \right]
\] (6.57)

The convention of choosing \(\lambda_1 < \lambda_2\) will become clear as the analysis progresses (see the discussion surrounding Eq. (6.86)).

Next, the eigenvectors will be found. Inserting \(\lambda_1\) into either the top or bottom row of Eq. (6.55) gives
\[
E_y = \frac{a_{11} - \lambda_1}{-a_{12}} E_x
\] (6.58)
or
\[
E_y = \frac{-a_{21}}{a_{22} - \lambda_1} E_x \tag{6.59}
\]

The equality of Eqs. (6.58) and (6.59) is verified from Eq. (6.56). One has to be careful whenever \(a_{12} = 0\) or \(a_{22} - \lambda_1 = 0\), as explained in Example 6.6.

Eigenvector \(\mathbf{v}_1\), whose components \(E_x\) and \(E_y\) are related by either Eq. (6.58) or (6.59), is rewritten as
\[
\mathbf{v}_1 = \begin{bmatrix}
  E_{x1} \\
  E_{y1}
\end{bmatrix} = \begin{bmatrix}
  -a_{12} \\
  a_{11} - \lambda_1
\end{bmatrix}
\] (6.60)

and similarly for \(\lambda_2\)
\[
\mathbf{v}_2 = \begin{bmatrix}
  E_{x2} \\
  E_{y2}
\end{bmatrix} = \begin{bmatrix}
  -a_{12} \\
  a_{11} - \lambda_2
\end{bmatrix}
\] (6.61)

By taking the inner product of the eigenvectors given by Eqs. (6.60) and (6.61), we will find the condition that makes the eigenvectors orthogonal. Simplification of the product using Eq. (6.57) leads to
\[
\begin{bmatrix}
  E_{x1} & E_{y1}
\end{bmatrix}
\begin{bmatrix}
  E_{x2} \\
  E_{y2}
\end{bmatrix} = a_{12}(a_{12} - a_{21}) \tag{6.62}
\]
Thus, these vectors are orthogonal if

$$a_{12} = a_{21}$$  \hspace{1cm} (6.63)$$

meaning Eq. (6.54) is a symmetric matrix.

**Example 6.8** Find the eigenvalues and eigenvectors of a quarter-waveplate with its fast axis along the x axis.

**Solution** The Jones matrix expression of a quarter-waveplate is, from Eq. (6.46),

$$\lambda \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} e^{-j\pi/4} & 0 \\ 0 & e^{j\pi/4} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$  \hspace{1cm} (6.64)$$

Comparing Eq. (6.64) with (6.54) gives

$$a_{11} = e^{-j\pi/4}$$
$$a_{22} = e^{j\pi/4}$$  \hspace{1cm} (6.65)$$
$$a_{12} = a_{21} = 0$$

Inserting Eq. (6.65) into (6.57) gives

$$\lambda_{1,2} = \frac{1}{\sqrt{2}} (1 \mp j) = e^{\mp j\pi/4}$$  \hspace{1cm} (6.66)$$

As mentioned earlier, if $a_{12}$ or $a_{22} - \lambda_1$ is zero, one has to be careful. Here, the original equation Eq. (6.55) is used,

$$(a_{11} - \lambda)E_x + a_{12}E_y = 0$$
$$a_{21}E_x + (a_{22} - \lambda)E_y = 0$$  \hspace{1cm} (6.67)$$

and is combined with Eqs. (6.65) and (6.66) with $\lambda = \lambda_1$ to give

$$0E_x + 0E_y = 0$$
$$0E_x + \left(2j \sin \frac{\pi}{4}\right)E_y = 0$$

The above two equations are simultaneously satisfied if $E_y = 0$ and $E_x$ is an arbitrary number, meaning a horizontally polarized wave. The output is $\lambda_1E_x$.

Similarly, with $\lambda = \lambda_2$, Eq. (6.55) becomes

$$\left(-2j \sin \frac{\pi}{4}\right)E_x + 0E_y = 0$$
$$E_x + 0E_y = 0$$

which leads to $E_x = 0$ and $E_y$ can be any number. The eigenvector is a vertically polarized wave. The output is $\lambda_2E_y$.

The magnitude of the transmitted light is $|\lambda_{1,2}| = 1$. If the phase of the output is important, the phase factor ($e^{j\Delta/2}$) that was discarded from Eq. (6.34) should be retained in Eq. (6.64).
6.9 STATES OF POLARIZATION AND THEIR COMPONENT WAVES

Relationships existing among ellipticity, azimuth of the major axes of the ellipse, $E_x$ and $E_y$ component waves, and retardance will be derived. Such relationships will help to convert the expression for an elliptically polarized wave into that of $E_x$ and $E_y$ component waves.

6.9.1 Major and Minor Axes of an Elliptically Polarized Wave

The lengths of the major and minor axes will be found from the expressions for the $E_x$ and $E_y$ component waves.

Letting $\phi_0 = -\omega t + \beta z$, Eqs. (6.3) and (6.4) can be rewritten for convenience as

$$\frac{E_x}{A} = \cos \phi_0$$

$$\frac{E_y}{B} = \cos \phi_0 \cos \Delta - \sin \phi_0 \sin \Delta$$

In order to find an expression that is invariant of time and location, $\phi_0$ is eliminated by putting Eq. (6.68) into (6.69):

$$\frac{E_y}{B} = \left( \frac{E_x}{A} \right) \cos \Delta - \sqrt{1 - \left( \frac{E_x}{A} \right)^2} \sin \Delta$$

Rearranged, Eq. (6.70) is

$$\left( \frac{E_x}{A} \right)^2 + \left( \frac{E_y}{B} \right)^2 - 2 \frac{E_x E_y}{AB} \cos \Delta = \sin^2 \Delta$$

In order to facilitate the manipulation, let’s rewrite Eq. (6.71) as

$$g(X, Y) = \sin^2 \Delta$$

$$g(X, Y) = a_{11} X^2 + a_{22} Y^2 + 2a_{12} XY$$

where

$$X = E_\chi, \quad Y = E_y$$

$$a_{11} = \frac{1}{A^2}, \quad a_{22} = \frac{1}{B^2}, \quad a_{12} = -\frac{\cos \Delta}{AB}$$

Let us express Eq. (6.73) in matrix form so that various rules [16] associated with the matrix operation can be utilized. As shown in Fig. 6.47, let the position vector $v'$ of a point $(X, Y)$ on the ellipse be represented by

$$v = \begin{bmatrix} X \\ Y \end{bmatrix} \quad v' = [X \ Y]$$
and define a symmetric matrix $M$ as

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad (6.76)$$

Realize that an equality exists as

$$v' M v = a_{11} X^2 + a_{22} Y^2 + 2a_{12} X Y \quad (6.77)$$

The normal $v_N$ from the circumference of the ellipse is obtained by taking the gradient of Eq. (6.73) (see Eq. (4.89)) and is expressed in vector form as

$$v_N = 2 \begin{bmatrix} a_{11} X + a_{12} Y \\ a_{12} X + a_{22} Y \end{bmatrix} \quad (6.78)$$

Hence,

$$v_N = 2Mv \quad (6.79)$$

As shown in Fig. 6.47, if the vector $v$ were to represent the direction of the major or minor axis of the ellipse, $v_N$ should be parallel to $v$ or $v_N = \lambda v$, and hence, the
condition for $\mathbf{v}$ to be along the major or minor axis is, from Eq. (6.79),

$$M\mathbf{v} = \lambda \mathbf{v}$$

(6.80)

where the factor 2 was absorbed in $\lambda$. Thus, the eigenvectors of matrix $M$ provides the directions of the major and minor axes, and the eigenvectors are given by Eqs. (6.60) and (6.61).

Next, the actual lengths $a$ and $b$ of the major and minor axes of the ellipse will be found. The position vector $\mathbf{v}$ in Fig. 6.47 of a point $(X,Y)$ on the ellipse in the $X-Y$ coordinates is expressed in the new $x-y$ coordinates taken along the major and minor axes as

$$\mathbf{v} = x\hat{\mathbf{v}}_1 + y\hat{\mathbf{v}}_2$$

(6.81)

where $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ are the unit vectors of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Inserting Eq. (6.82) into (6.77) gives

$$\mathbf{v}^T M \mathbf{v} = (x\hat{\mathbf{v}}_1^T + y\hat{\mathbf{v}}_2^T) M (x\hat{\mathbf{v}}_1 + y\hat{\mathbf{v}}_2) = \lambda_1 x^2 + \lambda_2 y^2$$

(6.82)

where use was made of

$$M\hat{\mathbf{v}}_1 = \lambda_1 \hat{\mathbf{v}}_1$$

(6.83)

$$M\hat{\mathbf{v}}_2 = \lambda_1 \hat{\mathbf{v}}_2$$

(6.84)

$$\hat{\mathbf{v}}_1^T \cdot \hat{\mathbf{v}}_2 = 0$$

Combining Eqs. (6.72), (6.73), (6.77) and (6.83) gives

$$\frac{x^2}{(\sin \Delta)^2/\lambda_1} + \frac{y^2}{(\sin \Delta)^2/\lambda_2} = 1$$

(6.85)

Thus, in the new $x-y$ coordinates along $\mathbf{v}_1$ and $\mathbf{v}_2$, the major and minor axes of the ellipse $a$ and $b$ are

$$a = \frac{|\sin \Delta|}{\sqrt{\lambda_1}} \quad \text{and} \quad b = \frac{|\sin \Delta|}{\sqrt{\lambda_2}}$$

(6.86)

Since $a$ is conventionally taken as the length of the major axis, the smaller eigenvalue is taken for $\lambda_1$. That is, the negative sign of Eq. (6.57) will be taken for $\lambda_1$, and the positive sign for $\lambda_2$.

In summary, the eigenvalues and eigenvectors of $M$ have given the lengths as well as the directions of the major and minor axes.

Before going any further, we will verify that Eq. (6.71) is indeed the expression of an ellipse, and not that of a hyperbola, as both formulas are quite alike. Note that if $\lambda_1 \lambda_2 > 0$, then Eq. (6.85) is an ellipse, but if $\lambda_1 \lambda_2 < 0$, it is hyperbola. From Eq. (6.57), the product $\lambda_1 \lambda_2$ is

$$\lambda_1 \lambda_2 = a_{11} a_{22} - a_{12}^2$$

(6.87)
Note that Eq. (6.87) is exactly the determinant of $M$. Thus, the conclusions are

$$\det M > 0, \quad \text{ellipse}$$
$$\det M < 0, \quad \text{hyperbola}$$

(6.88)

The value of the determinant is, from Eqs. (6.74) and (6.76),

$$\lambda_1 \lambda_2 = \det \begin{bmatrix} \frac{1}{A^2} & - \frac{\cos \Delta}{AB} \\ - \frac{\cos \Delta}{AB} & \frac{1}{B^2} \end{bmatrix} = \frac{\sin^2 \Delta}{A^2 B^2}$$

(6.89)

Thus,

$$\lambda_1 \lambda_2 > 0$$

(6.90)

and Eq. (6.71) is indeed the expression of an ellipse.

### 6.9.2 Azimuth of the Principal Axes of an Elliptically Polarized Wave

Figure 6.48 shows the general geometry of an ellipse. Capital letters will be used for the quantities associated with the $X$ and $Y$ components of the $E$ field, and lowercase letters...
for those quantities expressed in $x–y$ coordinates. The $x–y$ coordinates correspond to the directions of the major and minor axes of the ellipse. $E_X$ does not exceed $A$ and $E_Y$ does not exceed $B$, and the ellipse is always bordered by a rectangle $2A \times 2B$. The ratio $B/A$ is often expressed in terms of the angle $\alpha$ as

$$\tan \alpha = \frac{B}{A} \quad (6.91)$$

Since the right-hand side of Eq. (6.91) is a positive quantity, $\alpha$ must lie in the range

$$0 \leq \alpha \leq \pi/2 \quad (6.92)$$

The vector $v_1$ points in the direction of the $x$ axis. The azimuth angle $\theta$ of the major axis with respect to the $X$ axis will be found. From Eq. (6.60), $\tan \theta$ is expressed as

$$\tan \theta = -\frac{(a_{11} - \lambda_1)}{a_{12}} \quad (6.93)$$

Inserting Eqs. (6.57) and (6.74) into Eq. (6.93) gives

$$\tan \theta = (-t + \sqrt{t^2 + \cos^2 \Delta}) \frac{1}{\cos \Delta} \quad (6.94)$$

where

$$t = \frac{A^2 - B^2}{2AB} \quad (6.95)$$

Equations (6.94) and (6.95) will be simplified further. Using the double-angle relationship of the tangent function given by

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \quad (6.96)$$

Eq. (6.94) is greatly simplified as

$$\tan 2\theta = \frac{\cos \Delta}{t} \quad (6.97)$$

Applying the double-angle relationship of Eq. (6.96) to the angle $\alpha$, and making use of Eq. (6.91), Eq. (6.95) becomes

$$t = \frac{1}{\tan 2\alpha} \quad (6.98)$$

The final result is obtained from Eqs. (6.97) and (6.98):

$$\tan 2\theta = \tan 2\alpha \cos \Delta \quad (6.99)$$
As seen from Fig. 6.48, all configurations of the principal axes can be expressed by $0 \leq \theta \leq 180^\circ$.

The following conclusions are immediately drawn from Eq. (6.99):

1. If the amplitudes $A$ and $B$ are identical and $\alpha = 45^\circ$, then the azimuth $\theta$ can only be $45^\circ$ or $135^\circ$, regardless of the value of $\Delta$:

$$\theta = 45^\circ \quad \text{for} \quad \cos \Delta > 0$$
$$\theta = 135^\circ \quad \text{for} \quad \cos \Delta < 0$$

This agrees with previous discussions involving Fig. 6.4.

2. For any value of $A$ and $B$, if $\Delta = 90^\circ$, the azimuth $\theta$ is either $0^\circ$ or $90^\circ$.

6.9.3 Ellipticity of an Elliptically Polarized Wave

Ellipticity is another quantity that describes the shape of an ellipse. The ellipticity $\epsilon$ is defined as

$$\epsilon = \frac{b}{a}$$

(6.100)

where $a$ is the length of the major axis, and $b$ is the length of the minor axis of the ellipse.

From Eqs. (6.57), (6.86), and (6.100) the ellipticity is

$$\epsilon = \sqrt{\frac{1 - Y}{1 + Y}}$$

(6.101)

where

$$Y = \sqrt{\left(\frac{a_{11} - a_{22}}{a_{11} + a_{22}}\right)^2 + \left(\frac{2a_{12}}{a_{11} + a_{22}}\right)^2}$$

(6.102)

From Eq. (6.74), the quantities under the square root of Eq. (6.102) are simplified as

$$\frac{a_{11} - a_{22}}{a_{11} + a_{22}} = -\cos 2\alpha$$
$$\frac{2a_{12}}{a_{11} + a_{22}} = -\sin 2\alpha \cos \Delta$$

(6.103)

where the trigonometric relationships

$$\cos 2\alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$$
$$\sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$$

(6.104)

(6.105)

were used.
Insertion of Eq. (6.103) into (6.102) gives

\[ Y = \sqrt{1 - \sin^2 2\alpha \sin^2 \Delta} \]  

(6.106)

Thus, insertion of Eq. (6.106) into (6.101) gives the ellipticity. A few more manipulations will be made on the expression for \( \epsilon \), but first, observe the following behavior of \( \epsilon \) for given values of \( B/A \) and \( \Delta \):

1. With zero retardance \( \Delta \), the value \( \epsilon \) is always zero and the wave is linearly polarized.
2. Only when \( B/A = 1 \) and \( \Delta = 90^\circ \), can the wave be circularly polarized.

Returning to the manipulations on the ellipticity expression, \( \epsilon \) will be rewritten further in terms of trigonometric functions. Referring to Fig. 6.48, \( \epsilon \) can be represented by the angle \( \beta \):

\[ \tan \beta = \epsilon \]  

(6.107)

Since \( \epsilon \) is a quantity between 0 and 1

\[ 0 \leq \beta \leq \pi/4 \]  

(6.108)

The trigonometric relationship

\[ \sin 2\beta = \frac{2\tan \beta}{1 + \tan^2 \beta} \]  

(6.109)

is applied to Eqs. (6.101) and (6.107) to obtain

\[ \sin 2\beta = \sqrt{1 - Y^2} \]  

(6.110)

Insertion of Eq. (6.106) into (6.110) gives

\[ \sin 2\beta = \sqrt{\sin^2 2\alpha \sin^2 \Delta} \]  

(6.111)

\[ \sin 2\beta = \sin 2\alpha |\sin \Delta| \]  

(6.112)

Because of the restrictions imposed on \( \alpha \) and \( \beta \) in Eqs. (6.92) and (6.108), both \( \sin 2\beta \) and \( \sin 2\alpha \) are positive and the absolute value of \( \sin \Delta \) has to be taken.

6.9.4 Conservation of Energy

When the state of polarization is converted, the light power neither increases nor decreases, aside from the loss due to nonideal optical components. Conservation of energy dictates that

\[ a^2 + b^2 = A^2 + B^2 \]  

(6.113)
Equation (6.113) will now be verified. From Eq. (6.86), \( a^2 + b^2 \) is expressed in terms of the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) and \( \Delta \) as

\[
a^2 + b^2 = \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \sin^2 \Delta \tag{6.114}
\]

From Eqs. (6.57) and (6.74), the sum of the eigenvalues is

\[
\lambda_1 + \lambda_2 = \frac{1}{A^2} + \frac{1}{B^2} \tag{6.115}
\]

Insertion of Eqs. (6.89) and (6.115) into Eq. (6.114) finally proves the equality of Eq. (6.113).

Next, area relationships will be derived from Eq. (6.86). The product \( ab \) is

\[
ab = \frac{\sin^2 \Delta}{\sqrt{\lambda_1 \lambda_2}} \tag{6.116}
\]

With Eq. (6.89), a substitution for \( \sqrt{\lambda_1 \lambda_2} \) is found and

\[
ab = AB|\sin \Delta| \tag{6.117}
\]

where the absolute value sign was used because all other quantities are positive. Note area \( \pi ab \) of the ellipse becomes zero when \( \Delta = 0 \), and a maximum when \( \Delta = \pm \pi/2 \), for given values of \( A \) and \( B \).

Furthermore, the difference \( a^2 - b^2 \) will be calculated. From Eq. (6.86), \( a^2 - b^2 \) is

\[
a^2 - b^2 = \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \sin^2 \Delta \tag{6.118}
\]

With Eq. (6.89), Eq. (6.118) becomes

\[
a^2 - b^2 = (\lambda_2 - \lambda_1)A^2B^2 \tag{6.119}
\]

In the following, \( \lambda_2 - \lambda_1 \) will be calculated. From Eq. (6.57), the difference \( \lambda_2 - \lambda_1 \) is

\[
\lambda_2 - \lambda_1 = (a_{22} - a_{11}) \sqrt{1 + \left( \frac{2a_{12}}{a_{22} - a_{11}} \right)^2} \tag{6.120}
\]

Manipulation of Eqs. (6.74), (6.95), (6.98), and (6.99) gives

\[
\frac{2a_{12}}{a_{22} - a_{11}} = \tan 2\theta \tag{6.121}
\]

From Eq. (6.120) and (6.121), the difference becomes

\[
\lambda_2 - \lambda_1 = \frac{A^2 - B^2}{A^2B^2} \frac{1}{\cos 2\theta} \tag{6.122}
\]
Inserting Eq. (6.122) back into (6.119) gives the final result of

\[(a^2 - b^2) \cos 2\theta = A^2 - B^2\] (6.123)

This relationship is used later on in converting between \(X-Y\) and \(x-y\) components.

### 6.9.5 Relating the Parameters of an Elliptically Polarized Wave to Those of Component Waves

So far, parameters such as \(\epsilon (= \tan \beta)\) and \(\theta\) have been derived from \(B/A (= \tan \alpha)\) and \(\Delta\). In this section, \(\alpha\) and \(\Delta\) will conversely be obtained from \(\beta\) and \(\theta\).

Using the trigonometric identity of Eq. (6.104) for \(\beta\) instead of \(\alpha\) and using Eqs. (6.107), (6.113), and (6.123), \(\cos 2\beta\) is expressed as

\[
\cos 2\beta = \frac{A^2 - B^2}{A^2 + B^2} \cdot \frac{1}{\cos 2\theta}
\] (6.124)

Dividing both numerator and denominator by \(A^2\), and using the trigonometric relationship Eq. (6.104) for \(\tan \alpha\), Eq. (6.124) becomes

\[
\cos 2\alpha = \cos 2\theta \cos 2\beta
\] (6.125)

Next, the expression for the retardance \(\Delta\) will be derived. The derivation makes use of Eq. (6.112), which contains \(|\sin \Delta|\), and the absolute value cannot be ignored.

\[
|\sin \Delta| = \sin \Delta \quad \text{for } \sin \Delta > 0; \text{ left-handed}
\]

\[
|\sin \Delta| = -\sin \Delta \quad \text{for } \sin \Delta < 0; \text{ right-handed}
\] (6.126)

where the handedness information is given by Eq. (6.15). The ratio between Eqs. (6.99) and (6.112), and the use of Eqs. (6.125) and (6.126) leads to

\[
\tan \Delta = \pm \cosec 2\theta \tan 2\beta
\] (6.127)

The plus and minus signs are for left-handed and right-handed elliptical polarization, respectively.

### 6.9.6 Summary of Essential Formulas

The formulas derived in the last few sections are often used for calculating the state of polarization and will be summarized here.

\[
\Delta = \phi_y - \phi_x
\]

\[
\Delta > 0, \quad \text{y component is lagging} \quad (6.7) \text{ and } (6.8)
\]

\[
\Delta < 0, \quad \text{y component is leading}
\]

\[
\tan \alpha = \frac{B}{A} \quad \left(0 \leq \alpha \leq \frac{\pi}{2}\right)
\] (6.91)
\[
\tan \beta = \frac{b}{a} = \epsilon \quad \left(0 \leq \beta \leq \frac{\pi}{4}\right) \quad (6.100) \; \text{and} \; (6.107)
\]

\[
\tan \theta = \left(-t + \sqrt{t^2 + \cos^2 \Delta}\right) / \cos \Delta \quad (0 \leq \theta < \pi) \quad (6.94)
\]

\[
t = \frac{A^2 - B^2}{2AB} \quad (6.95)
\]

\[
t = 1 / \tan 2\alpha \quad (6.98)
\]

\[
\tan 2\theta = \tan 2\alpha \cos \Delta \quad (0 \leq 2\theta < 2\pi) \quad (6.99)
\]

\[
\sin 2\beta = \sin 2\alpha |\sin \Delta| \quad (6.112)
\]

\[
a^2 + b^2 = A^2 + B^2 \quad (6.113)
\]

\[
ab = AB |\sin \Delta| \quad (6.117)
\]

\[
(a^2 - b^2) \cos 2\theta = A^2 - B^2 \quad (6.123)
\]

\[
(a^2 - b^2) \sin 2\theta = 2AB \cos \Delta \quad \text{(Prob. 6.12a)}
\]

\[
(a^2 - b^2) \cos 2\theta = (A^2 + B^2) \cos 2\alpha \quad \text{(Prob. 6.12b)}
\]

\[
\cos 2\alpha = \cos 2\theta \cos 2\beta \quad (6.125)
\]

\[
\sin \Delta > 0 \quad \text{and} \quad |\sin \Delta| = \sin \Delta; \; \text{left-handed}
\]

\[
\sin \Delta < 0 \quad \text{and} \quad |\sin \Delta| = -\sin \Delta; \; \text{right-handed} \quad (6.126)
\]

\[
\tan \Delta = \pm \csc 2\theta \tan 2\beta \quad (+ \text{is for left-handed and} \; - \text{for right-handed}) \quad (6.127)
\]

**Example 6.9** A linearly polarized wave is incident onto a retarder whose fast axis is along the x axis. The retardance \(\Delta\) is 38° and the amplitudes of the \(E_x\) and \(E_y\) components are 2.0 V/m and 3.1 V/m, respectively. Calculate the azimuth \(\theta\) and the ellipticity \(\epsilon\) of the emergent elliptically polarized wave. Also, determine the lengths \(a\) and \(b\) of the major and minor axes. Find the solution graphically as well as analytically.

**Solution**

For the given parameters,

\[
A = 2.0 \text{ V/m}
\]

\[
B = 3.1 \text{ V/m}
\]

\[
\Delta = 38^\circ
\]

\(\theta\) and \(\epsilon\) will be found.

\[
\tan \alpha = \frac{B}{A} = 1.55
\]

\[
\alpha = 57.2^\circ
\]

From Eq. (6.99), the angle \(\theta\) is obtained:

\[
\tan 2\theta = \tan 2\alpha \cos \Delta
\]

\[
= (-2.2)(0.788)
\]
Since $0 \leq \theta \leq \pi$, $\theta = 60^\circ$ is the answer.

From Eq. (6.112)

\[
\sin 2\beta = \sin 2\alpha | \sin \Delta |
\]

\[
= (0.910)(0.616)
\]

\[
= 0.560
\]

$\beta = 17.0^\circ$

$\epsilon = \tan \beta = 0.31$

Next, $a$ and $b$ are calculated from Eqs. (6.113) and (6.123):

\[
a^2 + b^2 = A^2 + B^2 = 13.61
\]

\[
a^2 - b^2 = \frac{1}{\cos 2\theta} (A^2 - B^2) = \frac{5.61}{0.5}
\]

\[
= 11.22
\]

\[
a = 3.52
\]

\[
b = 1.09
\]

The circle diagram is shown in Fig. 6.49a and the calculated results are summarized in Fig. 6.49b.

Example 6.10

The parameters of an elliptically polarized wave are $A = 10$ V/m, $B = 8$ V/m, $a = 12.40$ V/m, and $b = 3.22$ V/m.

(a) Find the azimuth $\theta$ and the retardance $\Delta$.

(b) For the given values of $A$ and $B$, what is the maximum ellipticity $\epsilon$ that can be obtained by manipulating the retardance?

(c) For the given values of $A$ and $B$, is it possible to obtain an ellipse with $\epsilon = 0.26$ and azimuth $\theta = 50^\circ$ by manipulating the retardance $\Delta$?

Solution

From $A$ and $B$, $\tan \alpha$ is obtained:

\[
A = 10 \text{ V/m}
\]

\[
B = 8 \text{ V/m}
\]

\[
\tan \alpha = 0.8
\]

From $a$ and $b$, $\tan \beta$ is obtained:

\[
a = 12.4 \text{ V/m}
\]

\[
b = 3.20 \text{ V/m}
\]

\[
\tan \beta = 0.26
\]
Figure 6.49 Solutions of Example 6.9 obtained graphically as well as analytically. (a) Graphical solution. (b) Summary of calculated results.
(a) \( \theta \) and \( \Delta \) are computed as follows. From Eq. (6.123), \( \theta \) is calculated:

\[
(a^2 - b^2) \cos 2\theta = A^2 - B^2
\]

\[
\cos 2\theta = \frac{36}{143.52} = 0.25
\]

\[
\theta = 37.8^\circ
\]

From Eq. (6.117), \( \Delta \) can be found:

\[
ab = AB|\sin \Delta|
\]

\[
|\sin \Delta| = \frac{ab}{AB} = 0.496
\]

\[
\Delta = \pm 29.7^\circ
\]

(b) From Eq. (6.112), the value of \( \Delta \) that maximizes \( 2\beta \) for a given value of \( \alpha \) is \( \Delta = 90^\circ \). The maximum value of \( \epsilon \) is

\[
\epsilon = \tan \beta = \tan \alpha = 0.8
\]

(c) Let us see if Eq. (6.125) is satisfied:

\[
\cos 2\alpha = \cos 2\theta \cos 2\beta
\]

with \( \tan \alpha = 0.8 \),

\[
\alpha = 38.7^\circ
\]

and with \( \tan \beta = 0.26 \)

\[
\beta = 14.6^\circ
\]

\[
\theta = 50^\circ
\]

\[
0.218 = -1.74 \times 0.873
\]

\[
0.218 \neq -1.52
\]

For given \( A \) and \( B \), the value of \( \epsilon \) and \( \theta \) are mutually related, and one cannot arbitrarily pick the two values. \( \square \)

**Example 6.11** A right-handed elliptically polarized wave with \( a = \sqrt{3} \) V/m and \( b = 1 \) V/m and \( \theta = 22.5^\circ \) is incident onto a \( \lambda/4 \) plate with its fast axis oriented at \( \Theta = 45^\circ \) with respect to the X axis. Find the state of polarization of the wave emergent from the \( \lambda/4 \) plate.

The circle diagram was used to solve the same question in Section 6.2.5, Fig. 6.10.

**Solution**

The following steps will be taken:

1. Calculate \( A \), \( B \), and \( \Delta \) of the incident wave.
2. Find $B'/A'$, and $\Delta'$ of the emergent wave from the $\lambda/4$ plate by means of the Jones matrix.
3. Convert $B'/A'$, and $\Delta'$ into $a'$, $b'$, and $\theta'$.

**Step 1.** The given parameters are

$$
\begin{align*}
a &= \sqrt{3} \text{ V/m} \\
b &= 1 \text{ V/m} \\
\theta &= 22.5^\circ \\
\tan \beta &= b/a = \frac{1}{\sqrt{3}} \\
\beta &= 30^\circ
\end{align*}
$$

From Eq. (6.125), $\alpha$ is calculated:

$$
\begin{align*}
\cos 2\alpha &= \cos 2\theta \cos 2\beta \\
&= 0.354 \\
\alpha &= 34.6^\circ
\end{align*}
$$

Next, the retardance $\Delta$ will be found using Eq. (6.127):

$$
\begin{align*}
\tan \Delta &= \pm \csc 2\theta \tan 2\beta \\
&= \pm 2.45
\end{align*}
$$

where the $+$ sign is for left-handed and the $-$ sign is for right-handed. Since the problem specifies right-handed, $\tan \Delta = +2.45$ is eliminated.

The two possibilities for $\Delta$ are

$$
\Delta = -67.8^\circ \text{ or } +112.2^\circ
$$

From Eq. (6.126), the correct choice of $\Delta$ is

$$
\Delta = -67.8^\circ
$$

From Eq. (6.113), $A$ and $B$ are found:

$$
\begin{align*}
a^2 + b^2 &= A^2 + B^2 \\
&= A^2(1 + \tan^2 \alpha) \\
A^2 &= \frac{1 + 3}{1 + \tan^2 34.6^\circ} \\
A &= 1.65 \\
B &= 1.14
\end{align*}
$$
Step 2. The Jones matrix of the $\lambda/4$ plate whose fast axis azimuth angle $\Theta$ is $45^\circ$ is used to calculate emergent wave.

\[
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix}
= \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & -j \\
-j & 1
\end{bmatrix} \begin{bmatrix}
1.65 \\
1.14e^{j67.8^\circ}
\end{bmatrix}
\]

\[=
\frac{1}{\sqrt{2}} \begin{bmatrix}
1 & e^{-j90^\circ} \\
e^{-j90^\circ} & 1
\end{bmatrix} \begin{bmatrix}
1.65 \\
1.14e^{j67.8^\circ}
\end{bmatrix}
\]

\[=
\frac{1}{\sqrt{2}} \begin{bmatrix}
1.65 + 1.14e^{-j157.8^\circ} \\
1.14e^{j67.8^\circ}
\end{bmatrix}
\]

\[= \frac{1}{\sqrt{2}} \left[ \sqrt{[1.65 + 1.14\cos(-157.8^\circ)]^2 + [1.14\sin(-157.8^\circ)]^2} \right] e^{j\theta_1}
\]

\[= \frac{1}{\sqrt{2}} \left[ \sqrt{(0.595)^2 + (-0.43)^2} \right] e^{j\phi_1}
\]

\[= \frac{1}{\sqrt{2}} \left[ 0.73 \ e^{-j35.5^\circ} \right] = \frac{e^{-j35.5^\circ}}{\sqrt{2}} \begin{bmatrix}
0.733 \\
2.74 e^{-j81.0^\circ}
\end{bmatrix}
\]

$\Delta' = 45.5^\circ$

$\tan \alpha = \frac{2.74}{0.733} = 3.74$

$\alpha' = 75.0^\circ$

Step 3. From Eq. (6.99), $\theta$ is found:

$\tan 2\theta' = \tan 2\alpha' \cos \Delta'$

$= (-0.566)(0.72) = -0.407$

$2\theta' = -22.2^\circ$ or $157.8^\circ$

Since $0 \leq \theta' \leq 180^\circ$, and $0 \leq 2\theta' \leq 360^\circ$, the negative value is rejected and $\theta' = 79^\circ$.

From Eq. (6.112), $\beta'$ is calculated:

$\sin 2\beta' = \sin 2\alpha' \sin \Delta'$

$= 0.5 \times |-0.707|$

$= 0.354$

$\beta' = 10.36^\circ$

$\epsilon = \tan \beta' = 0.18$

$\sin \Delta' = -0.707 < 0$

The emergent wave is right-handed.

$a'^2 + b'^2 = a^2(1 + \epsilon^2)$
From Eq. (6.113), the above equation is rewritten

\[ A^2 + B^2 = a'^2(1 + \epsilon^2) \]

and conservation of energy gives

\[ A^2 + B^2 = a^2 + b^2 = a'^2 + b'^2 \]
\[ a'^2 = \frac{\sqrt{3}^2 + 1}{1 + 0.18^2} = 3.87 \]
\[ a' = 1.96 \]
\[ b' = 0.35 \]

Compare with the answer shown in Fig. 6.10 using a circle diagram.

PROBLEMS

6.1 A linearly polarized wave with \( A = \sqrt{3} \) V/m, and \( B = 1 \) V/m is incident onto a \( \lambda/4 \) plate with its fast axis along the \( x \) axis. \( A \) is the amplitude of the \( E_x \) component wave, and \( B \) is the amplitude of the \( E_y \) component wave. Obtain the emergent elliptically polarized wave by using the two different conventions of \( E_- = e^{j\omega t - jkz} \) and \( E_+ = e^{-j\omega t + jkz} \) representing a forward wave, and demonstrate that both results are the same.

6.2 A linearly polarized wave with azimuth \( \theta_1 = 63.4^\circ \) is incident onto a retarder with \( \Delta = 315^\circ \) whose fast axis is oriented along the \( x \) axis. Graphically determine the azimuth angle \( \theta_2 \), which is the angle between the major axis of the emergent ellipse and the \( x \) axis, and the ellipticity \( \epsilon \) of the ellipse.

6.3 In Example 6.1, the direction of polarization of the incident light was fixed and the fast axis of the \( \lambda/4 \) plate (retarder with \( \Delta = 90^\circ \)) was rotated. The results were drawn in Fig. 6.8. Draw the results (analogous to Fig. 6.8) for

Figure P6.3 A linearly polarized wave is incident onto a quarter-waveplate with its fast axis oriented horizontally.
the case when the direction of the fast axis is fixed horizontally, and the
direction of the incident linear polarization is rotated as shown in Fig. P6.3
at $\theta = 0^\circ$, $22.5^\circ$, $45^\circ$, $67.5^\circ$, $90^\circ$, $112.5^\circ$, $135^\circ$, $157.5^\circ$, and $180^\circ$.

6.4 Decompose graphically an elliptically polarized wave into component waves
that are parallel and perpendicular to the major or minor axis, and verify that
the phase difference between the component waves is $90^\circ$.

6.5 Obtain graphically the state of polarization with the same configuration as that
shown in Fig. 6.10, but with the opposite handedness of rotation of the incident
wave, that is, left-handed.

6.6 A linearly polarized light wave is incident normal to a pair of polarizers $P_1$
and $P_2$ whose transmission axes are oriented at $\theta_1$ and $\theta_2$ (Fig. P6.6). Assume
$k_1 = 1$ and $k_2 = 0$ for both polarizers.
(a) The light is incident from $P_1$ to $P_2$. What is the orientation $\theta$ of the linearly
polarized eigenvector?
(b) What is the orientation $\theta$ of the linearly polarized eigenvector when the light
is incident from $P_2$ to $P_1$?

6.7 The horseshoe crab’s eyes are known to be polarization sensitive. It is believed
that this sensitivity is used as a means of navigating in sunlight, and the principle
involved is that of the polarization of sunlight by scattering from particles in
the water. Referring to the configuration in Fig. P6.7, how can the horseshoe
crab orient itself along a north–south line in the early morning? What is the
direction of polarization that the horseshoe crab sees when facing south in the
early morning?

6.8 Linearly polarized laser light ($\lambda = 0.63$ $\mu$m) is transmitted through a quartz
crystal along its optical axis. Due to Rayleigh scattering from minute irregular-
ities in the crystal, one can observe a trace of the laser beam from the side of
the crystal. One may even notice a spatial modulation of the intensity along the

**Figure P6.6** Direction of the eigenvector (looking from the source).
trace, as illustrated in Fig. P6.8. If one assumes that the spatial modulation is due to the rotary power, what is the period of the modulation? The rotary power of quartz is 19.5 deg/mm at $\lambda = 0.63 \, \mu m$ and at a temperature of 20°C.

6.9 Figure 6.11 shows a diagram of a $\lambda/4$ plate. If one assumes $d_1 > d_2$, is the birefringence of the crystal in the figure positive or negative?

6.10 Devise a scheme to determine the directions of the fast and slow axes of a retarder.

6.11 The ellipse shown in Fig. P6.11 was made with $B/A = 1$ and $\Theta = 0$, just like the ellipses shown in Fig. 6.4. Prove that the retardance $\Delta$ is identical to the angle $\angle ABC = 2\beta$ on the ellipse.

6.12 Prove the following equalities:

(a) $(a^2 - b^2) \sin 2\theta = 2AB \cos \Delta$.

(b) $(a^2 - b^2) \cos 2\theta = (A^2 + B^2) \cos 2\alpha$. 
Figure P6.11  Prove that $2\beta$ is identical to the retardance $\Delta$ when $B/A = 1$ and $\theta = 0$.

REFERENCES
