Growth of a Wave-group when the Group-velocity is Negative

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The following may be of interest in connection with the recent discussion on the flow of energy in such cases.

Let the energy of an element of a linearly arranged mechanical system be

$$\{(d^2y/dx\,dt)^2 + y^2\}dx/2$$

Such a system can be approximately realised by taking a bicycle chain, loading it so that the radius of gyration of each link has the same large value, and suspending it by equal threads attached to each link so that the chain is horizontal and the axes of the links vertical. By the principle of least action we immediately find the equation of motion to be $d^4y/dx^2dt^2 = y$. A simple harmonic wave is given by $y = \sin(pt - x/p)$. The group velocity is $-p^2$, and is negative. Let such a system, extending from $x = 0$ to $x = \infty$, be at rest in its position of equilibrium at time $t = 0$, and then let the point $x = 0$ be moved so that its position at any subsequent time is given by $y = 1 - \cos t$.

By application of the usual method via Fourier's integral, the motion of the system is found to be given by either of the equivalent formulae

$$y = \sum (-1)^n (t/x)^{n+1} J_{2n+2}(2\sqrt{tx})$$

or

$$y = 1 - \cos(t + x) - 1 + \sum (-1)^n (x/t)^n J_{2n}(2\sqrt{tx})$$

where the J's are Bessel's functions and the summations extend from $n = 0$ to $n = \infty$. There are some doubtful points in the reasoning, however, and the proof consists in showing (1) that y satisfies the differential equation, (2) from the second formula that $y = 1 - \cos t$ when $x = 0$, (3) from the first formula that y and dy/dt are both zero when $t = 0$, (4) from the first formula that when t is finite y is small for all large values of x. If, now, x is finite and t great, the second formula reduces to $y = -\cos(t + x)$, so that the motion now consists entirely of waves proceeding *towards* the source of the disturbance—a most remarkable result. If in the formulas for y we change the sign of x, the J functions are replaced by I functions. The resulting value of y does not satisfy (4), and cannot be accepted as a solution of the problem.