RADIATION FROM A CIRCULAR BEND BETWEEN TWO DISCONTINUITIES IN DIELECTRIC SLAB WAVEGUIDES AND IN CIRCULAR, CYLINDRICAL DIELECTRIC WAVEGUIDES

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Radiation From a Circular Bend Between Two Discontinuities in Dielectric Slab Waveguides and in Circular, Cylindrical Dielectric Waveguides

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Abstract

Dielectric Waveguides radiate energy at bends. The radiated power and the radiation pattern have been studied theoretically for a dielectric slab waveguide and for a circular cylindrical waveguide. The slab configuration consists of two semi-infinite slabs joined by a circular bend. The cylindrical configuration consists of two straight circular cylinders joined by a circular bend. The results indicate that the discontinuities at the transitions between the straight sections and the bend cause the radiation to differ significantly from the radiation of a bend of the same size but being part of a uniform circular structure rather than being joined to two straight sections.

The radiated power for both the slab and circular cylinder configurations, is found to vary inversely as the square of the radius of curvature, and the radiation pattern is composed of many closely spaced narrow lobes. As the radius of curvature of the bend increases the lobes become more numerous and more closely spaced.

1. Introduction

The electromagnetic field of a dielectric waveguide propagates within as well as outside the waveguide structure and it is attenuated exponentially with distance away from the structure. The field is thus somewhat loosely bound when compared with that of metallic waveguide; and, as may be expected, some of the energy is lost at irregularities in the waveguide structure. In the particular case of a bend in the waveguide, energy will be radiated by the bend.

This analysis is a theoretical study of the radiated power and the radiation pattern of a dielectric slab waveguide with a circular bend and of a circular cylindrical dielectric waveguide with a circular bend.
2. Background

Radiation from dielectric waveguides with axes along a circular arc have been studied by Marcatili (1969), Chang and Barnes (1973), and Lewin (1974). These studies were limited to consideration of the circular part of a waveguide structure with no consideration given to the sections of waveguide joined by the circular bend. Furthermore the studies were limited to finding the radiated power with no attention given to the radiation pattern. This study was initiated in an effort to partially overcome these limitations.

3. Theory for Radiation from the Outer Surface of a Dielectric Slab with a Circular Bend

The configuration studied is shown in Fig. 1.

Fig. 1 Configuration of Dielectric Slab Waveguide
The waveguide is divided into three regions, 1, 2, and 3. Regions 1 and 3 are straight slabs and region 2 is the bend which is circular with outside radius, $r$. The origin of the coordinate system is on the axis of curvature. Coordinate $y$ is positive to the right and $z$ is positive upward. It is assumed there is no variation of the electromagnetic fields with the coordinate $x$. The point of observation, that is the point at which the radiated field will be found is point $P$ with coordinates $x$, $y$, and $z$. $P$ is a distance $R$ from the axis of curvature and the line from the axis to point $P$ makes an angle, $\theta$, with the $z$-axis. The radiated field will be expressed in terms of coordinates $R$ and $\theta$. Coordinates $R_3$ and $\theta_3$ and coordinates $R_1$ and $\theta_1$ give the position of point $P$ with respect to the upper and lower ends of the outer surface of the waveguide bend. For large values of $R$, $R_1$, and $R_3$ become nearly parallel with $R$ and angles $\theta_1$, $\theta_3$, and $\theta$ become nearly equal.

This analysis is concerned with the radiation from the bend in the slab waveguide assuming that the radius of curvature of the bend is large. It is assumed sufficiently large that the radiated power is small compared to the power carried by the waveguide. These assumptions permit the further assumption that the electromagnetic field of the slab waveguide propagates around the bend essentially unchanged in character.
The total magnetic vector potential due to all three regions is given by \( \vec{A} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 \):

\[
\vec{A} = \frac{-ik_0R}{4(\pi)^{1/2}k_0(k_0R)^{1/2}} \left[ e^{i\vec{k}_0 \cdot \vec{r} \sin \theta} f(\theta) - \frac{ib'k_0r\alpha}{e} e^{i\vec{k}_0 \cdot \vec{r} \sin(\theta+\alpha)} f(\theta+\alpha) \right]
\]

where

\[
f(\psi) = \frac{1}{b - \cos \psi} - \frac{1}{b' - \cos \psi}
\]

The total electric vector potential is given by \( \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \); thus

\[
\vec{F} = \frac{E_1(l+i)e^{-ik_0R}}{4(\pi)^{1/2}k_0(k_0R)^{1/2}} \left\{ \begin{array}{c}
\bar{a}_y \sin \alpha e^{i\vec{k}_0 \cdot \vec{r} \sin \theta} f(\theta) \\
\bar{a}_z \left[ e^{i\vec{k}_0 \cdot \vec{r} \sin \theta} f(\theta) - \cos \alpha e^{i\vec{k}_0 \cdot \vec{r} \sin(\theta+\alpha)} f(\theta+\alpha) \right] \end{array} \right\}
\]

The fields can be obtained from the expressions

\[
\vec{E} = -i\omega\mu\vec{A} - \nabla \times \vec{F}
\]

and

\[
\vec{H} = \frac{i}{\omega\mu} \nabla \times \vec{E}
\]

Retaining only the dominant terms \( \vec{E} \) and \( \vec{H} \) are given by [Maley, 1974]
\[ \vec{E} = -\vec{\alpha}_x M \begin{bmatrix} (Y - \sin \theta)e^{ik_o r \sin \theta} f(\theta) \\
- (Y - \sin(\theta + \alpha))e^{-ik_o \rho' a} e^{ik_o r \sin(\theta + \alpha)} f(\theta + \alpha) \\
- (Y - \sin(\theta + \alpha))e^{ik_o r \sin(\theta + \alpha)} f(\theta + \alpha) \end{bmatrix} \]

\[ \vec{H} = -\frac{M}{\eta_o} \begin{bmatrix} \vec{\alpha}_y \cos \theta e^{ik_o r \sin \theta} f(\theta) \\
- (Y - \sin(\theta + \alpha))e^{-ik_o \rho' a} e^{ik_o r \sin(\theta + \alpha)} f(\theta + \alpha) \\
- \vec{\alpha}_z \sin \theta e^{ik_o r \sin \theta} f(\theta) \end{bmatrix} \]

where

\[ M = \frac{(1 + i)e^{ik_o R}}{4(\pi)(k_o R)^{5/2}} \]

\[ Y = \frac{\eta_o H_{iz}}{E_i} \]

and

\[ \eta_o = \left( \frac{\mu_o}{\varepsilon_o} \right)^{1/2} \]

The Poynting vector, \( \vec{P} \), is found in App. 2; it is given by

\[ \vec{P} = |M|^2 \frac{1}{\eta_o} |G|^2 \left( \vec{\alpha}_y \sin \theta + \vec{\alpha}_z \cos \theta \right) \]

where

\[ |M|^2 = \frac{2|E_i|^2}{16\pi k_o R} \]

and

\[ |G|^2 \leq \left[ |Y|^2 - (Y + Y^*) \sin \theta + \sin^2 \theta \right] \frac{(b' - b)^2}{(b - \cos \theta)^4} \]

\[ + \left[ |Y|^2 - (Y + Y^*) \sin(\theta + \alpha) + \sin^2(\theta + \alpha) \right] \frac{(b' - b)^2}{[b - \cos(\theta + \alpha)]^4} \]
where the approximation \( f(\theta) \approx \frac{b' - b}{(b - \cos \theta)^2} \) (ref. App. 2)

has been used. This is valid for small \( b' - b \).

The total radiated power is found in App. 3. It is given by

\[
P_r = \frac{|E_1|^2 d^2 b^2}{64 \pi k_o r^2 n_o} \left[ |Y|^2 K_1(a,b) + (Y + Y^*) K_2(a,b) + K_3(a,b) \right]
\]

where \( K_1(a,b), K_2(a,b), K_3(a,b) \) are given by

\[
K_1(a,b) = \frac{\sin 2\alpha}{3(b^2-1)(b^2-\cos^2\alpha)} \left[ \frac{3b^2 + \cos^2\alpha}{(b^2-\cos^2\alpha)^2} + \frac{5b^2}{(b^2-1)(b^2-\cos^2\alpha)} + \frac{11b^2 + 4}{2(b^2-1)^2} \right]
\]

\[
+ \frac{b(2b^2 + 3)}{(b^2-1)^3 \sqrt{b^2-1}} \left( \frac{3\pi}{2} - \tan^{-1} \frac{\sqrt{b^2-1} \cot \alpha}{b} \right)
\]

\[
K_2(a,b) = \frac{2}{3} \left[ \frac{\cos \alpha (3b^2 + \cos^2\alpha)}{(b^2-\cos^2\alpha)^3} + \frac{3b^2 + 1}{(b^2-1)^3} \right]
\]

\[
K_3(a,b) = \frac{\sin 2\alpha}{3(b^2-\cos^2\alpha)} \left[ \frac{-3b^2 + \cos^2\alpha}{(b^2-\cos^2\alpha)^2} + \frac{b^2}{(b^2-1)(b^2-\cos^2\alpha)} \right]
\]

\[
+ \frac{b^2 + 2}{2(b^2-1)^2} + \frac{b}{(b^2-1)^2 \sqrt{b^2-1}} \left( \frac{3\pi}{2} - \tan^{-1} \frac{\sqrt{b^2-1} \cot \alpha}{b} \right) \right]
\]

In the derivation it is assumed that the phase velocity along the mid-plane of the bend is the same as in the straight sections. This leads to the relationship

\[
b' - b \approx - \frac{bd}{2r}.
\]
4. Theory for Radiation from the Inner Surface of a Dielectric Slab with a Circular Bend

The solution to this problem is very similar to the solution for the radiation from the outer surface of the same slab waveguide configuration. The configuration is shown in Fig. 2. The angles from the origin of coordinates and from the two ends of the inner surface of the circular bend are defined as negative quantities so as to make the definitions the same as used in the analysis of the radiation from the outer surface of the bend. This simplifies the adaptation of previously derived expressions to this problem.

With these definitions, the electromagnetic potentials and fields found in the previous section are valid for this problem [Maley, 1975a].

On the inside surface of the bend \( b' \) will be larger than \( b \). It can be found from \( b' = b \frac{r+d}{r} \). (Compare this with the expression found for the outer surface of the bend in the previous section.) Then \( b' - b \approx \frac{bd}{2r} \) where \( r \) is the inside radius of the bend.

This has the effect of changing the sign of the functions \( f(\theta) \) and \( f(\theta+\alpha) \) (as compared with their signs for radiation from the outer surface of the bend) in the expressions for the fields. In the expressions for the Poynting vector and for the total power radiated, the function \( b' - b \) only appears in a squared form; so there is no change from the expressions for radiation from the outer surface of the bend.
Fig. 2, Configuration of a dielectric slab waveguide
The expression for the Poynting vector for radiation from the outer surface of the waveguide is also valid for the inner surface if the inside radius of curvature is used for \( r \). It may be noted that the expression for the magnitude of the Poynting vector is

\[
|G|^2 = \left| |Y|^2 - (Y+Y^*) \sin \theta + \sin^2 \theta \right| \frac{(b'-b)^2}{(b-\cos \theta)^4}
\]

\[
+ \left| |Y|^2 - (Y+Y^*) \sin(\theta+\alpha) + \sin^2 (\theta+\alpha) \right| \frac{(b'-b)^2}{(b-\cos(\theta+\alpha))^4}
\]

+ rapidly oscillating terms having a zero average.

All of the terms in this expression are symmetric about \( \theta = -\frac{\alpha}{2} \) except the terms involving \( \sin \theta \) and \( \sin(\theta+\alpha) \). These are antisymmetric. Furthermore, in terms of the angle \( \theta' \) measured clockwise from the reference angle which in turn is the angle \( \theta = \frac{\alpha}{2} \) as indicated in Fig. 1.

\[
|G(-\theta' + \frac{\alpha}{2})|^2 = |G(\theta' - \frac{\alpha}{2})|^2 + 2(Y+Y^*) \sin(\theta' - \frac{\alpha}{2}) \frac{(b'-b)^2}{[b-\cos(\theta' - \frac{\alpha}{2})]^4}
\]

This shows that the radiation from the inside of the bend is somewhat smaller than that from the outside of the bend in directions symmetrically oriented with respect to \( \theta = -\frac{\alpha}{2} \).

Since \( (Y+Y^*) \approx 0 \) in some cases the Poynting vector is very nearly a symmetric function about the angle \( \theta = -\frac{\alpha}{2} \).
(Of course on the inside of the bend, different values of \( r \) & \( b' \) must be used. The different value of \( b' \) will have no effect and the difference in \( r \) is usually small.)

The total power radiated is given by

\[
P_r = R \int_{-\pi}^{-\alpha} p \cdot \left( \frac{a_y}{a} \sin \theta + \frac{a_z}{a} \cos \theta \right) d\theta
\]

which can be written

\[
P_r = \frac{|E|^2 d^2 b^2}{64\pi k_o \eta_o^2} \left[ |Y|^2 K'_1(a, b) + (Y+Y^*)K'_2(a, b) + K'_3(a, b) \right]
\]

where

\[
K'_1(a, b) = \frac{2b(2b^2+3)\pi}{(b^2-1)^{3/2}b^2-1} - K_1(a, b)
\]

\[
K'_2(a, b) = -K_2(a, b)
\]

and

\[
K'_3(a, b) = \frac{2b\pi}{(b^2-1)^{2/2}b^2-1} - K_3(a, b)
\]

It is apparent from the previous two sections that the power radiated from the bend in a slab is proportional, for each surface to \((b'-b)^2\). If it is assumed that the phase velocity along the center plane of the bend is the same as along the straight portion of the slab then \( b' \) is smaller than \( b \) on the outer surface and larger than \( b \) on the inner surface. There will then be radiation from both inner and outer surfaces.
If the phase velocity along the center plane of the bend is not equal to the phase velocity on the straight section then \( b' \) will change, in the same direction on both surfaces. Thus \( b'-b \) will increase on one of the surfaces and decrease on the other. (Assuming \( b'-b \) is still negative on the outer surface and positive on the inner.) By proper choice of the dielectric parameters of the dielectric in the bend it would be possible to make \( b'-b = 0 \) on one of the surfaces thereby eliminating radiation from that surface. It is not possible, however, to eliminate radiation from both surfaces at the same time.
5. Theory for Radiation from a Circular Bend in a Cylindrical Dielectric Waveguide

The procedure used to find the radiation from a bend in a dielectric slab is adaptable in a straightforward manner to a cylindrical fiber. Consider the fiber configuration shown in Fig. 3.

Assume a longitudinal current wave on the surface of the fiber having the form \( \vec{M}(\zeta, \phi, \phi_T) = \vec{a}_\zeta e^{-ik_o^0b_\zeta} \cos(n\phi' + \phi_T) \) where \( \zeta \) is distance along the center line of the fiber measured upward, \( \vec{a}_\zeta \) is a unit vector directed upward and tangent to the center line of the fiber, \( k_o = 2\pi/\lambda_o \) where \( \lambda_o \) is free space wavelength, \( b \) is the normalized wave number of the longitudinal current wave along the fiber surface and \( \phi' \) is the azimuthal coordinate used on the surface of the fiber. It is measured from the plane containing the fiber bend (x-z plane) as shown in sections B-B, C-C, and D-D of Fig. 3. \( \phi_T \) is a spatial phase reference of the propagating wave measured from the x-z plane just as is \( \phi' \). It is assumed that \( \phi_T \) is constant at all cross sections of the fiber; in other words the surface wave on the fiber is assumed to maintain an unvarying configuration, with respect to the x-z plane as it propagates around the bend.

The dimensionless potential, \( \vec{T} \), at point \( P \) due the longitudinal current wave is given by (the symbol \( T \) is used because this potential is due to a transverse field as will be discussed)
Fig. 3, Cylindrical fiber with a circular bend
\[
\bar{T}(\phi_T) = \int_\gamma k_o e^{-ik_o \rho} \bar{M}(\zeta, \phi, \phi_T) \, ds = \int_\gamma k_o e^{-ik_o \rho} -ik_o b \zeta \cos(n \phi + \phi_T) \, ds
\]

where \( \rho \) is the distance from the point of integration on the surface of the fiber to \( P \), and \( s \) is the surface of the fiber.

The evaluation of this integral is discussed in Appendix 4; the result is given below. It is assumed, in the evaluation, that \( R \) is sufficiently large that \( R, R_1, \rho, \) and \( R_2 \) can be considered parallel and that they can be considered equal in slowly varying factors of the expressions for potentials and fields.

\[
\bar{T}(\phi_T) = \frac{a_1^2 n}{4 R r} (\bar{a}_x L_x (\phi_T) + \bar{a}_z L_z (\phi_T)) = K(\bar{a}_x L_x (\phi_T) + \bar{a}_z L_z (\phi_T))
\]

where

\[
T_x(\phi_T) = \frac{-ik_o r b \alpha - ik_o R_2}{2e} e^{\sin \alpha [C(n+1, \phi_2, \phi_T, \theta_2) - C(n-1, \phi_2, \phi_T, \theta_2)]} \frac{b - \cos \theta_2}{b - \cos \theta_2}
\]

\[
T_z(\phi_T) = \frac{-ik_o r b \alpha - ik_o R_2}{2e} e^{\cos \alpha [C(n+1, \phi_2, \phi_T, \theta_2) - C(n-1, \phi_2, \phi_T, \theta_2)]} \frac{b - \cos \theta_2}{b - \cos \theta_2}
\]

\[
+ \frac{-ik_o R_1}{2e} \frac{[C(n+1, \phi, \phi_T, \theta) - C(n-1, \phi, \phi_T, \theta)]}{b - \cos \theta}
\]

\[
C(n, \phi, \phi_T, \theta) = \cos(n \phi + \phi_T) J_n(k_o a \sin \theta)
\]

and

\[
K = \frac{(k_o a)^2 i n}{8(k_o R)(k_o r)}
\]
Now assume a transverse current on the surface of the fiber having the form
\[ \mathbf{N}(\zeta, \phi', \phi_L) = \mathbf{a}_{\phi} e^{-ik_0 b \zeta} \cos(n \phi' + \phi_L) \]
where \( \mathbf{a}_{\phi} \) is the azimuthal unit vector at all points along the fiber. It is in the direction of increasing \( \phi' \). In terms of previously defined quantities it may be expressed,
\[ \mathbf{a}_{\phi} = \mathbf{a}_x \times \mathbf{a}_y \sin \phi' + \mathbf{a}_y \cos \phi'. \]
\( \phi_L \) is the spatial phase reference of the field. As in the case of the longitudinal current, it is assumed that the field configuration does not change as it propagates around the bend.

The dimensionless potential, \( \overline{L} \), at point \( P \) due to the transverse current is given by (The symbol \( L \) is used because this potential results from a longitudinally oriented field as will be discussed.)

\[
L(\phi_L) = \int_{s} \frac{k_0 e^{-ik_0 \rho}}{4\pi \rho} \mathbf{N}(\zeta, \phi', \phi_L) ds
\]

\[
= \int_{s} \frac{k_0 e^{-ik_0 \rho}}{4\pi \rho} [\mathbf{a}_x \times \mathbf{a}_y \sin \phi' + \mathbf{a}_y \cos \phi'] e^{-ik_0 b \zeta} \cos(n \phi' + \phi_L) ds
\]

The integral for \( \overline{L} \) can be evaluated in the same manner as was done for \( \overline{T} \) (ref. App. 4); the result is

\[
\overline{L}(\phi_L) = iK(\mathbf{a}_x L_x(\phi_L) + \mathbf{a}_y L_y(\phi_L) + \mathbf{a}_z L_z(\phi_L))
\]

where

\[
L_x(\phi_L) = \frac{-ik_0 \rho_1 - ik_0 R_2 \cos \alpha[S(n+2, \phi, \phi_L, \theta_2) - S(n-2, \phi, \phi_L, \theta_2)]}{(b - \cos \theta_2)}
\]

\[
- e^{ik_0 R_1 \frac{S(n+2, \phi, \phi_L, \theta) - S(n-2, \phi, \phi_L, \theta)}}{(b - \cos \theta)}
\]
\[ L_Y(\phi_L) = \frac{-ik_o \text{bra} - ik_o R^2(2\alpha(n, \phi_2, \phi_L, \theta_2) - C(n+2, \phi_2, \phi_L, \theta_2) - C(n-2, \phi_2, \phi_L, \theta_2)} \quad b - \cos \theta_2}{b - \cos \theta} \]

\[ L_Z(\phi_L) = \frac{-ik_o \text{bra} - ik_o R^2\sin \alpha[S(n+2, \phi_2, \phi_L, \theta_2) - S(n-2, \phi_2, \phi_L, \theta_2)] \quad b - \cos \theta_2}{b - \cos \theta} \]

\[ S(n, \phi, \phi_L, \theta) = \sin(n\phi + \phi_L)J_n(k_o a \sin \theta) \]

In spherical coordinates the potentials are

\[ \overline{T}(\phi_T) = K(\overline{a}_r T_r(\phi_T) + \overline{a}_\theta T_\theta(\phi_T) + \overline{a}_\phi T_\phi(\phi_T)) \]

where

\[ T_r(\phi_T) = T_x(\phi_T)\sin \theta \cos \phi + T_z(\phi_T)\cos \theta \]

\[ T_\theta(\phi_T) = T_x(\phi_T)\cos \theta \cos \phi - T_z(\phi_T)\sin \theta \]

\[ T_\phi(\phi_T) = -T_x(\phi_T)\sin \phi \]

and

\[ \overline{L}(\phi_L) = iK(\overline{a}_r L_r(\phi_L) + \overline{a}_\theta L_\theta(\phi_L) + \overline{a}_\phi L_\phi(\phi_L)) \]

where

\[ L_r(\phi_L) = L_x(\phi_L)\sin \theta \cos \phi + L_y(\phi_L)\sin \theta \sin \phi + L_z(\phi_L)\cos \theta \]

\[ L_\theta(\phi_L) = L_x(\phi_L)\cos \theta \cos \phi + L_y(\phi_L)\cos \theta \sin \phi - L_z(\phi_L)\sin \theta \]

\[ L_\phi(\phi_L) = -L_x(\phi_L)\sin \phi + L_y(\phi_L)\cos \phi \]

Assume that the actual electric and magnetic fields on the surface of the fiber are
\[ \bar{E} = E_L \bar{a}_\zeta e^{-ik_0 b \zeta} \cos(n\phi' + \phi_a) + E_T \bar{a}_\phi e^{-ik_0 b \zeta} \cos(n\phi' + \phi_b), \]
\[ \bar{H} = H_L \bar{a}_\zeta e^{-ik_0 b \zeta} \cos(n\phi' + \phi_c) + H_T \bar{a}_\phi e^{-ik_0 b \zeta} \cos(n\phi' + \phi_d). \]

If the fields are not of this form, they can be expressed as a sum of terms of this form. The electric, \( \bar{F} \), and magnetic, \( \bar{A} \), vector potentials resulting from these fields are easily found by recalling that the magnetic field is related to \( \bar{A} \) through the equivalent surface current \( \bar{n} \times \bar{H} \) where \( \bar{n} \) is the outward directed normal on the surface of the fiber, and the electric field is related to \( \bar{F} \) through the equivalent magnetic surface current, \( -\bar{n} \times \bar{E} \).

The potentials are

\[ \bar{F} = -\frac{E_T}{k_0} \bar{T}(\phi_b) + \frac{E_L}{k_0} \bar{T}(\phi_a) \]
\[ \bar{A} = \frac{H_T}{k_0} \bar{T}(\phi_d) - \frac{H_L}{k_0} \bar{T}(\phi_c) \]

The fields due to these potentials are given by

\[ \bar{E} = \frac{1}{i\omega \varepsilon} \nabla \times \nabla \times \bar{A} - \nabla \times \bar{F}, \]
\[ \bar{H} = \frac{i}{\omega \mu} \nabla \times \bar{E} \]

It is easily shown that

\[ \nabla \times \bar{T} = -ik_0 \bar{a}_r \times \bar{T}_t, \quad \nabla \times \nabla \times \bar{T} = k_0^2 \bar{T}_t \]
\[ \nabla \times \bar{L} = -ik_0 \bar{a}_r \times \bar{L}_t, \quad \nabla \times \nabla \times \bar{L} = k_0^2 \bar{L}_t \]
where the subscript \( t \) denotes the transverse part (\( \theta \& \phi \) components). Only the dominant terms have been retained in these evaluations. Using these results

\[
\bar{E} \approx \frac{H_t}{k_0 \omega c} k_0^2 t(\phi_d) - \frac{H_L}{k_0 \omega c} k_0^2 l(\phi_c) - ik_0 \frac{E_t}{k_0} \bar{a}_r \times \bar{T}_t(\phi_b) + ik_0 \frac{E_L}{k_0} \bar{a}_r \times \bar{L}_t(\phi_a)
\]

or

\[
\bar{E} \approx -i\eta H_t \bar{T}_t(\phi_d) + i\eta H_L \bar{L}_t(\phi_c) - iE_t \bar{a}_r \times \bar{T}_t(\phi_b) + iE_L \bar{a}_r \times \bar{L}_t(\phi_a)
\]

where \( \eta = (\mu/\epsilon)^{\frac{1}{2}} \). This may be written

\[
\bar{E} = \bar{a}_\theta E_\theta + \bar{a}_\phi E_\phi
\]

where

\[
E_\theta = K[-i\eta H_t T_\theta(\phi_d) - \eta H_L L_\theta(\phi_c) + iE_t T_\phi(\phi_b) + E_L L_\phi(\phi_a)]
\]

\[
E_\phi = K[-i\eta H_t T_\phi(\phi_d) - \eta H_L L_\phi(\phi_c) - iE_t T_\theta(\phi_b) - E_L L_\theta(\phi_a)]
\]

Then

\[
\bar{H} = \bar{a}_r \times \frac{\bar{E}}{\eta} = -\bar{a}_\theta \frac{E_\phi}{\eta} + \bar{a}_\phi \frac{E_\phi}{\eta}
\]

These fields have rapidly varying amplitude with variation of \( \theta \) in the \( x-z \) plane just as is true of the variation of the fields with \( \theta \) in the slab problem. However in this case there is rapid variation with \( \theta \) in the \( y-z \) plane also; actually there is rapid variation with \( \theta \) and \( \phi \). The expression for the Poynting vector \( \bar{P} = \frac{1}{2} \text{Re} \bar{E} \times \bar{H}^* \) where \( \bar{H}^* \) is the complex conjugate of \( \bar{H} \) is easily written. Its integral over a large spherical surface gives the power radiated by the bend. It does
not seem practical to integrate it analytically, however an upper bound can be found quite easily. The Poynting vector is

\[ P = \frac{i}{2} \bar{E} \times \bar{H}^* = \frac{i}{2} (\bar{a}_\theta E_\theta + \bar{a}_\phi E_\phi) \times (-\bar{a}_\theta \frac{E_\phi^*}{n} + \bar{a}_\phi \frac{E_\theta^*}{n}) \]

\[ = \frac{a_r}{2n} \left( |E_\theta|^2 + |E_\phi|^2 \right) \]

The total radiated power is

\[ P_R = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \bar{P} \cdot \bar{a}_r \sin \theta \, d\phi \, d\theta \]

which can be expressed as

\[ P_R = |K|^2 \frac{8\pi R^2 (Q_\theta^2 + Q_\phi^2) S}{2n (b^2 - 1)} \left( \frac{1}{b^2 - 1} \right) \frac{A_f}{r^2} \frac{a^2 (Q_\theta^2 + Q_\phi^2)}{n} \left( \frac{S}{16} \right) \]

where \( A_f \) is the cross-sectional area of the fiber,

\[ Q_\theta = [12n (|H_T| + |H_L|) + 4|E_T| + 10|E_L|] \]

\[ Q_\phi = [n (4|H_T| + 10|H_L|) + 12(|E_T| + |E_L|)] \]

and \( S \) is a dimensionless, real and positive constant with a magnitude less than one. It is given by

\[ S = \frac{b^2 - 1}{8\pi |K|^2 (Q_\theta^2 + Q_\phi^2)} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left( |E_\theta|^2 + |E_\phi|^2 \right) \sin \theta \, d\phi \, d\theta . \]
It is seen that the radiated power is proportional to several factors; the first is \((b^2-1)^{-1}\). \(b^2-1\) is positive and is small for loosely bound surface waves and larger for more tightly bound surface waves. Thus loosely bound surface waves radiate more strongly than tightly bound waves. The second factor, \(A_f/r^2\), is a geometric factor indicating radiated power is proportional to cross sectional area of the fiber and inversely proportional to the square of the radius of curvature of the bend. The third factor \(a^2 (Q^2_\theta + Q^2_\phi)/\eta\) is a somewhat involved function of the surface wave fields on the fiber; it is proportional to the power carried by the fiber. The function \(S\) is dimensionless and has a value \(0 < S < 1\). It is given by a rather complicated integral. It does not seem worthwhile to try evaluating the integral analytically, but, for a specific problem, it can be evaluated numerically without difficulty.

If it is desired to numerically evaluate the expression for \(P_r\), an expansion is given in App. 5.
6. Conclusions

The foregoing analyses of radiation from a bend in a slab waveguide and from a bend in a cylindrical fiber, the radiation has been expressed in terms of the surface fields. Nothing has been said about the fields inside the dielectric waveguide. This formulation was used intentionally with the objective of making the results more generally applicable. The actual dielectric waveguide may have any configuration of electrical parameters; it may even be inhomogeneous such as is true for a graded index fiber. If the surface fields of the waveguide are known, the above results are applicable. It should be obvious that the procedures used above can be further extended to find the radiation from any structure for which the surface fields are known or can be adequately approximated.

A limited comparison of these theoretical results with experimental results for radiation from the outer surface of a bend in a dielectric slab has been made [Haddad, 1975; Maley, 1975c]. With satisfactory agreement considering the fact that the experiment was conducted in the near radiation zone while the theoretical results were limited to the far zone.

The problem of radiation from a bend in a slab has also been treated in great detail by Kuester and Chang [1976] in a series of studies. Here again the agreement with the results of this report is remarkably good considering the fact that Kuester and Chang restricted their analysis to intermediate zone radiation rather than far zone radiation as was done in this study. Further comparison of these results is now in progress.
References


Maley, S.W., Radiation from the Inner Surface of a Bend Connecting Two Straight Sections of Dielectric Slab Waveguide, Abstracts of Papers, USNC-URSI Meeting, 3-5 June 1975(a), p. 10.

Maley, S.W., Radiation from a Discontinuity in a Planar, Surface-Wave Guiding Structure, Abstracts of Papers, USNC-URSI Meeting, 3-5 June 1975(b), p. 10.

Maley, S.W., Hussain Haddad, D.C. Chang, Microwave Model Study of Dielectric Optical Waveguides, Abstracts of Papers, USNC-URSI Meeting, 3-5 June 1975(c); p. 63.
APPENDIX 1

Evaluation of the Integrals

\[ k_0 \int_{-\infty}^{-\lambda} e^{-ik_0b\xi} H_0^{(2)}(k_0(\xi^2+\delta^2)^\frac{1}{4})f(\xi)d\xi \]

and

\[ \int_0^a e^{-ik_0\lambda\gamma} H_0^{(2)}(k_0((y-r\cos\gamma)^2+(z-r\sin\gamma)^2)^\frac{1}{4})f(\gamma)d\gamma \]

The evaluation of these integrals is complicated by the fact that \( y, z, \delta >> r \) and \( k_0r >> 1 \); so the exponential functions and the Hankel functions are rapidly varying functions of the variable of integration. The function \( f \) is considered to be a slowly varying function. An approximate evaluation is made possible by deforming the contour of integration [Maley, 1974]. The results are

\[ k_0 \int_{-\infty}^{-\lambda} \frac{e^{-ik_0(-b\lambda+(\lambda^2+\delta^2)^\frac{1}{4})}f(-\lambda)}{\left[\pi k_0(\lambda^2+\delta^2)^\frac{1}{2}\right]^\frac{1}{2}\left[b-\frac{\lambda}{(\lambda^2+\delta^2)^\frac{1}{2}}\right]} \]

and

\[ \int_0^a \frac{e^{-ik_0\lambda\gamma} H_0^{(2)}(k_0((y-r\cos\gamma)^2+(z-r\sin\gamma)^2)^\frac{1}{4})f(\gamma)d\gamma}{\left[(1+i)ie^{-ik_0R}-ik_0(\lambda\alpha-r\sin(\theta+\alpha))\right]} \]

\[ = \frac{e^{-ik_0R}-ik_0(\lambda\alpha-r\sin(\theta+\alpha))}{(\pi k_0 R)^\frac{1}{2}} \left\{ \frac{f(a)}{k_0(\lambda-r\cos(\theta+\alpha))} - \frac{f(0)}{k_0(\lambda-r\cos \theta)} \right\} \]

where

\[ R^2 = y^2 + z^2. \]
APPENDIX 2

Derivation of Expressions for the Poynting Vector for the Slab Problem

The Poynting vector is easily found from the expressions for \( \mathbf{E} \) and \( \mathbf{H} \) as given on p.9. It is

\[
\mathbf{P} = |M|^2 \frac{1}{2\pi\Omega} + \{\bar{a}_y|G|^2 \sin \theta + \bar{a}_z|G|^2 \cos \theta\}
\]

where

\[
G = (Y - \sin \theta)e^{-i k_r \sin \theta} f(\theta) - (Y - \sin(\theta + \alpha))e^{-i b'k_r \alpha \sin(\theta + \alpha)} f(\theta + \alpha)
\]

and

\[
|G|^2 = GG^* = [YY^* - (Y + Y^*) \sin \theta + \sin^2 \theta] f^2(\theta)
\]

\[
= -ib'k_r \alpha e^{-i k_r (\sin(\theta + \alpha) - \sin \theta)} f(\theta + \alpha) f(\theta)
\]

\[
- e^{-i k_r (\sin(\theta + \alpha) - \sin \theta)} f(\theta + \alpha) f(\theta)
\]

\[
+ [YY^* - (Y + Y^*) \sin(\theta + \alpha) + \sin^2(\theta + \alpha)] f^2(\theta + \alpha)
\]

where \( Y^* \) is the complex conjugate of \( Y \). Alternatively

\[
|G|^2 = [YY^* - (Y + Y^*) \sin \theta + \sin^2 \theta] f^2(\theta)
\]

\[
- \cos(k_r \sin(\theta + \alpha) - \sin \theta) - b' \alpha [2YY^* - (Y + Y^*) (\sin(\theta + \alpha) - \sin \theta)]
\]

\[
+ 2 \sin \theta \sin(\theta + \alpha) f(\theta) f(\theta + \alpha)
\]

\[
+ \sin(k_r \sin(\theta + \alpha) - \sin \theta) - b' \alpha \left[ \frac{Y - Y^*}{2} (\sin(\theta + \alpha) - \sin \theta) \right] f(\theta) f(\theta + \alpha)
\]

\[
+ [YY^* - (Y + Y^*) \sin(\theta + \alpha) + \sin^2(\theta + \alpha)] f^2(\theta + \alpha)
\]
When integrating $|G|^2$ to find the total radiated power, the two middle terms may be neglected because they are rapidly and almost periodically oscillating functions of $\theta$ and make a negligible contribution. Before finding total radiated power, it is desirable to assume $b'-b << 0$ and to use the approximation

$$ f(\theta) = \frac{1}{b-\cos \theta} - \frac{1}{b'-\cos \theta} = \frac{b'-b}{(b-\cos \theta)^2} - \frac{(b'-b)^2}{(b-\cos \theta)^3} + \cdots $$

Only the first term will be used in the integration for total power. Similarly

$$ f(\theta+\alpha) = \frac{1}{b-\cos(\theta+\alpha)} - \frac{1}{b'-\cos(\theta+\alpha)} = \frac{b'-b}{[b-\cos(\theta+\alpha)]^2} + \cdots $$

Thus $|G|^2$ will be approximated by

$$ |G|^2 = \left[ |Y|^2 - (Y+Y^*)\sin \theta + \sin^2 \theta \right] \frac{(b'-b)^2}{(b-\cos \theta)^4} $$

$$ + \left[ |Y|^2 - (Y+Y^*)\sin(\theta+\alpha) + \sin^2(\theta+\alpha) \right] \frac{(b'-b)^2}{[b-\cos(\theta+\alpha)]^4}. $$
APPENDIX 3

Derivation of Expressions for Total Radiated Power for the Slab Problem

The integral for total radiated power, $P_r$, per unit length in the x-direction, is

$$P_r = \int_{-\alpha}^{\pi} \mathbf{p} \cdot \mathbf{a} \, d\alpha$$

where

$$d\alpha = R_0 (a_y \sin \theta + a_z \cos \theta) d\theta .$$

Thus

$$P_r = \frac{|M|}{2 \eta_0} R \int_{-\alpha}^{\pi} |G| \, d\theta .$$

Using the above results

$$f^2(\theta) = \left[ \frac{1}{b \cos \theta} - \frac{1}{b' \cos \theta} \right]^2 \approx \frac{(b'-b)^2}{(b \cos \theta)^4}$$

and

$$f^2(\theta + \alpha) = \left[ \frac{1}{b \cos (\theta + \alpha)} - \frac{1}{b' \cos (\theta + \alpha)} \right]^2 \approx \frac{(b'-b)^2}{[b \cos (\theta + \alpha)]^4}$$

Thus the integrals that must be evaluated to find the radiated power are of the form
\[
\int \frac{d\theta}{(b - \cos \theta)^4} = \frac{\sin \theta}{3(b^2 - 1)(b - \cos \theta)^3} + \frac{5b \sin \theta}{6(b^2 - 1)^2(b - \cos \theta)^2} + \frac{(11b^2 + 4)\sin \theta}{6(b^2 - 1)^3(b - \cos \theta)}
\]
\[+ \frac{(6b^3 + 9b^2 + 2)}{6(b^2 - 1)^3\sqrt{b^2 - 1}} \tan^{-1}\frac{\sqrt{b^2 - 1} \tan \frac{\theta}{2}}{b - 1},\]

\[
\int \frac{\sin \theta \, d\theta}{(b - \cos \theta)^4} = \frac{-1}{3(b - \cos \theta)^3},
\]

and

\[
\int \frac{\sin^2 \theta \, d\theta}{(b - \cos \theta)^4} = \frac{1}{3} \left\{ \frac{-\sin \theta}{(b - \cos \theta)^3} + \frac{b \sin \theta}{2(b^2 - 1)(b - \cos \theta)^2} + \frac{(b^2 + 2)\sin \theta}{2(b^2 - 1)^2(b - \cos \theta)}
\]
\[+ \frac{6b}{2(b^2 - 1)^2\sqrt{b^2 - 1}} \tan^{-1}\frac{\sqrt{b^2 - 1} \tan \frac{\theta}{2}}{b - 1} \right\},\]

Using these integrals, the expression for \( P_r \) is as given on p. 10.
APPENDIX 4

Evaluation of the integral \( \mathbf{T} = \int_{s} e^{-ik_{o}\rho - ik_{o}b_{z}} e^{i\phi_{0} \cos(n\phi' + \phi_{T})} ds \)

Let the differential of area, \( ds \), be \( ds = ad\phi_{1}d\zeta \)

where \( \phi_{1} \) is as shown in sections B-B and C-C, Fig. 3 on the straight portion of the fiber and \( ds = (r + a \cos(\phi_{1} + \phi_{0}))d\phi_{1}d\gamma \)
on the curved portion of the fiber. \( \phi_{1} \) is as shown in section D-D of Fig. 3, \( \gamma \) is the angular position coordinate defined in Fig. 1 and

\[
\phi_{o} = \tan^{-1} \frac{\sin \theta \sin \phi}{\sin \theta \cos \phi \cos \gamma + \cos \theta \sin \gamma}
\]

is the azimuthal coordinate of the point \( P \) in a spherical coordinate system that moves along the axis of the fiber with \( ds \). It has its origin on the centerline of the fiber, with the \( \theta = 0 \) axis in the direction of \( \vec{a}_{z} \), and with the azimuthal coordinate measured from the \( x-z \) plane as sketched in section D-D of Fig. 3.

The variable \( \rho \) is given in terms of \( \phi_{1} \) and \( \gamma \) by

\[
\rho = R_{o} - a \cos \phi_{1} \sin \theta \]
on the lower straight portion of the fiber. \( R_{o} \) is the distance from the centerline of the fiber to \( P \); it is a function of \( \zeta \). It is given by

\[
\rho = R_{o} - a \cos \phi_{1} \sin \theta_{o}
\]
on the curved portion of the fiber. \( \theta_{o} = \cos^{-1}(\cos \theta \cos \gamma - \sin \theta \cos \phi \sin \gamma) \) is the polar angular coordinate of \( P \) in the spherical coordinate system described above in the definition of \( \phi_{o} \). Finally it is given
by \( \rho = R_o - a \cos \phi_1 \sin \theta_2 \) where \( \theta_2 = \cos^{-1}(\cos \theta \cos \alpha - 
abla \sin \theta \cos \phi \sin \alpha) \) on the upper straight portion of the fiber. \( \theta_2 \) is the polar angular coordinate in a spherical coordinate system with the \( \theta = 0 \) axis in the direction of \( \overline{a}_\zeta \) on the upper straight portion of the fiber and with the azimuthal coordinate measured from the \( x-z \) plane as shown in section C-C of Fig. 3.

The function \( \cos(n\phi' + \phi_T) \) expressed as a function of \( \phi_1 \) and \( \gamma \) is \( \cos(n\phi' + \phi_T) = \cos(n(\phi_1 + \phi) + \phi_T) \) on the lower straight portion of the fiber. It is \( \cos(n\phi' + \phi_T) = \cos(n(\phi_1 + \phi_3) + \phi_T) \) on the curved portion of the fiber; and it is \( \cos(n\phi' + \phi_T) = \cos(n(\phi_1 + \phi_2) + \phi_T) \) on the upper straight portion of the fiber. The unit vector \( \overline{a}_\zeta \) expressed in rectangular coordinates is \( \overline{a}_\zeta = \overline{a}_z \) on the lower straight portion of the fiber; it is \( \overline{a}_\zeta = \overline{a}_z \cos \gamma - \overline{a}_x \sin \gamma \) on the curved portion of the fiber; and it is \( \overline{a}_\zeta = \overline{a}_z \cos \alpha - \overline{a}_x \sin \alpha \) on the upper curved portion of the fiber.

The distance, \( R_o \), can be expressed as

\[
R_o = [(x - r)^2 + y^2 + (z - \xi)^2]^{\frac{1}{2}}
\]

for the lower straight portion of the fiber where \( -\infty < \xi < 0; \)

\[
R_o = [(x - r \cos \gamma)^2 + y^2 + (z - r \sin \gamma)^2]^{\frac{1}{2}}
\]

as on the curve where \( 0 \leq \gamma \leq \alpha \); and as

\[
R_o = [(x - r \cos \alpha + \xi \sin \alpha)^2 + y^2 + (z - r \sin \alpha - \xi \cos \alpha)^2]^{\frac{1}{2}}
\]

on the upper straight portion of the fiber where \( \xi \) is measured along the axis from the junction with the bend and \( 0 \leq \xi < \infty \). The differential \( d\xi \)
and the interval of integration becomes
\[ d\zeta = d\zeta, \quad -\infty < \zeta < 0, \]
on the lower straight portion of the fiber. It is
\[ d\zeta = r\,d\gamma, \quad 0 \leq \gamma \leq \alpha, \]
on the curved portion of the fiber, and it is
\[ d\zeta = d\xi, \quad 0 \leq \xi < \infty, \]
on the upper straight portion of the fiber.

The integral for \( \overline{\Phi} \) may be expressed as the sum of three integrals, one along the lower straight section, one along the bend and one along the upper straight section. Each of these integrals can then be evaluated. The integrations with respect to \( \phi_1 \) may be performed first using the integral
\[ \int_{-\pi}^{\pi} e^{i\omega \cos \phi_1} \cos n\phi_1 \, d\phi_1 = 2i^n \pi J_n(\omega). \]
Next the integration with respect to \( \xi \) may be performed using the approximate integrals (ref. App. 1).
\[ \int_{-\lambda}^{\lambda} e^{-i k_0 (b\xi + \sqrt{\xi^2 + \delta^2}) / \sqrt{\xi^2 + \delta^2}} f(\xi) \, d\xi \approx \frac{i e^{-ik_0 (-b\lambda + \sqrt{\lambda^2 + \delta^2}) / \sqrt{\lambda^2 + \delta^2}}}{k_0 \sqrt{\lambda^2 + \delta^2} (b - \lambda / \sqrt{\lambda^2 + \delta^2})} f(-\lambda), \]
and
\[ \int_{0}^{\alpha} e^{-i k_0 (\lambda \gamma - \delta \sin(\gamma + \theta))} f(\gamma) \, d\gamma \approx \frac{i e^{-ik_0 (\lambda \alpha - \delta \sin(\alpha + \theta)) / k_0 (\lambda - \delta \cos(\alpha + \theta))} f(\alpha)}{k_0 (\lambda - \delta \cos(\alpha + \theta))} - \frac{i k_0 \delta \sin \theta}{k_0 (\lambda - \delta \cos \theta)} \]
These approximations are highly accurate if the exponential varies much more rapidly than the remaining terms in the integrand as is the case in this problem. It should be noted that the results of integration over the surface of the lower straight section of the fiber can be adapted to the upper straight portion by a simple change of parameters. The results of the integration are given in the body of the report.
\[ P_r = \frac{|K|^2 R^2}{2\eta} \]

\[ \{ n^2 |H_T|^2 + |E_T|^2 \} \left[ 4 \cos^2 \alpha I_1(f_1(\phi), f_1(\phi), 2, 0, 2, 0) \right. \]

\[ + I_1(f_1(\phi), f_1(\phi), 0, 2, 0, 0) \]

\[ - 8 \cos \alpha \sin \alpha I_1(f_1(\phi), f_1(\phi), 1, 1, 0) \]

\[ + 4 I_0(f_0(\phi), f_0(\phi), 0, 2, 0, 0) \]

\[ + \left( n^2 |H_L|^2 + |E_L|^2 \right) \left[ \cos^2 \alpha I_1(f_5(\phi_c), f_5(\phi_c), 2, 0, 2, 0) \right. \]

\[ + \sin^2 \alpha I_1(f_5(\phi_c), f_5(\phi_c), 0, 2, 0, 0) \]

\[ - 2 \sin \alpha I_1(f_3(\phi_c), f_5(\phi_c), 1, 0) \]

\[ + 2 \cos \alpha (I_1(f_5(\phi_c), f_3(\phi_c), 2, 0, 1, 1) - I_1(f_5(\phi_c), f_5(\phi_c), 1, 1, 0)) \]

\[ + 4 I_0(f_4(\phi_c), f_4(\phi_c), 2, 0, 2, 0) + I_0(f_2(\phi_c), f_2(\phi_c), 2, 2, 0, 0, 2) \]

\[ + I_1(f_3(\phi_c), f_3(\phi_c), 2, 0, 0, 2) + 2 I_0(f_4(\phi_c), f_2(\phi_c), 2, 0, 1, 1) \]

\[ + (|E_T|^2 + n^2 |H_T|^2) \left[ 4 \cos^2 \alpha I_1(f_1(\phi_b), f_1(\phi_b), 0, 2, 0, 0) \right] \]

\[ + (|E_L|^2 + n^2 |H_L|^2) \left[ \cos^2 \alpha I_1(f_5(\phi_a), f_5(\phi_a), 0, 0, 0, 2) \right. \]

\[ - 2 \cos \alpha I_1(f_5(\phi_a), f_3(\phi_a), 0, 0, 1, 1) + I_0(f_4(\phi_a), f_4(\phi_a), 0, 0, 0, 2) \]

\[ - 2 I_0(f_4(\phi_a), f_2(\phi_a), 0, 0, 1, 1) + I_1(f_3(\phi_a), f_3(\phi_a), 0, 0, 2, 0) \]

\[ + I_0(f_2(\phi_a), f_2(\phi_a), 0, 0, 2, 0) \]
\[ + 2\eta \left[ I_m [H_T^* L_L] - I_m [E_L^* L_T] \right] [2 \cos^2 \alpha I_1 (f_3 (\phi_{e}), f_5 (\phi_{a}), 0, 1, 0, 1) \\
- 2 \sin \alpha I_1 (f_3 (\phi_{e}), f_5 (\phi_{a}), 0, 1, 0, 1) \]

\[ - 2 \sin \alpha \cos \alpha I_1 (f_3 (\phi_{e}), f_5 (\phi_{a}), 1, 0, 0, 1) \]

\[ + 2 \cos \alpha I_1 (f_1 (\phi_{d}), f_3 (\phi_{a}), 0, 1, 0, 1) - 2 I_0 (f_3 (\phi_{d}), f_4 (\phi_{a}), 0, 0, 1, 0) \]

\[ + 2 I_0 (f_3 (\phi_{d}), f_2 (\phi_{a}), 0, 0, 1, 0) \]

\[ + 2[\eta^2 I_m [H_T^* L_L] + I_m [E_L^* L_T]] [2 \cos^2 \alpha I_1 (f_3 (\phi_{e}), f_5 (\phi_{c}), 1, 1, 0, 0) \]

\[ - 2 \sin^2 \alpha I_1 (f_3 (\phi_{e}), f_5 (\phi_{c}), 1, 1, 0, 0) \]

\[ + 2 \sin \alpha \cos \alpha I_1 (f_3 (\phi_{e}), f_5 (\phi_{c}), 2, 0, 0, 0) \]

\[ - I_1 (f_1 (\phi_{d}), f_1 (\phi_{b}), 1, 0, 0, 2) + 2 \sin \alpha I_1 (f_3 (\phi_{e}), f_5 (\phi_{c}), 2, 0, 1, 1) \]

\[ + 2 \cos \alpha I_1 (f_1 (\phi_{d}), f_3 (\phi_{c}), 1, 1, 0, 1) \]

\[ + 2 I_0 (f_3 (\phi_{d}), f_2 (\phi_{c}), 1, 1, 0, 1) - 2 I_0 (f_3 (\phi_{d}), f_4 (\phi_{c}), 1, 1, 1, 0) \]

\[ - 2\eta \left[ I_m [H_T^* L_L] - I_m [E_L^* L_T] \right] [2 \sin^2 \alpha I_1 (f_5 (\phi_{c}), f_1 (\phi_{b}), 0, 1, 0, 1) \]

\[ - 2 \sin \alpha I_1 (f_3 (\phi_{c}), f_1 (\phi_{b}), 1, 0, 0, 2) \]

\[ - 2 \cos \alpha \sin \alpha I_1 (f_5 (\phi_{c}), f_1 (\phi_{b}), 1, 0, 1, 1) \]

\[ -2[I_m [E_T^* L_L^*] + \eta^2 [H_T^* L_L]] [2 \sin \alpha \cos \alpha I_1 (f_1 (\phi_{b}), f_5 (\phi_{a}), 0, 0, 0, 1) \]

\[ - 2 \sin \alpha I_1 (f_3 (\phi_{a}), f_5 (\phi_{a}), 0, 0, 1, 0) \]

\[ + 2 R[H_T^* E_T^*] - R[E_T^* E_T^*] [4 \cos^2 \alpha I_1 (f_3 (\phi_{d}), f_1 (\phi_{b}), 1, 0, 1, 1) \]

\[ - 4 \sin \alpha \cos \alpha I_1 (f_3 (\phi_{d}), f_1 (\phi_{b}), 0, 1, 0, 0) \]
\[ + 2\eta[R[H_L E_L^*] - R[E_L H_L^*]] \cos^2 \alpha \, I_1(f_5(\phi_c), f_5(\phi_a), 1, 0, 1, 1) \\
- \cos \alpha [I_1(f_5(\phi_a), f_3(\phi_c), 1, 0, 2, 0) - I_1(f_5(\phi_c), f_3(\phi_a), 1, 0, 0, 2) \\
+ I_1(f_5(\phi_c), f_5(\phi_a), 0, 0, 0, 2) + \sin I_1(f_3(\phi_a), f_5(\phi_c), 0, 1, 0, 0) \\
+ I_0(f_4(\phi_c), f_4(\phi_a), 1, 0, 1, 1) - I_0(f_4(\phi_a), f_2(\phi_c), 1, 0, 2, 0) \\
+ I_0(f_4(\phi_c), f_2(\phi_a), 1, 0, 0, 2) - I_1(f_3(\phi_c), f_3(\phi_a), 1, 0, 1, 1) \\
- I_0(f_2(\phi_c), f_2(\phi_a), 1, 0, 1, 1)] \]

where

\[ I_0(f(\phi_a), g(\phi_b), m, p, q, r) \]

\[ = \int_0^{2\pi} \int_0^{\pi} \frac{f(\phi_a)g(\phi_b)}{(b - \cos \theta)^2} \cos^m \theta \sin^p \theta \cos^q \phi \sin^r \phi \, d\phi d\theta, \]

\[ I_1(f(\phi_a), g(\phi_b), m, p, q, r) \]

\[ = \int_0^{2\pi} \int_0^{\pi} \frac{f(\phi_a)g(\phi_b)}{(b - \cos \theta_2)^2} \cos^m \theta \sin^p \theta \cos^q \phi \sin^r \phi \, d\phi d\theta \]

and

\[ f_0(\phi_T) = C(n+1, \phi, \phi_T, \theta) - C(n-1, \phi, \phi_T, \theta) \]

\[ f_1(\phi_T) = C(n+1, \phi_2, \phi_T, \theta_2) - C(n-1, \phi_2, \phi_T, \theta_2) \]

\[ f_2(\phi_L) = 2C(n, \phi, \phi_L, \theta) - C(n+2, \phi, \phi_L, \theta) - C(n-2, \phi, \phi_L, \theta) \]

\[ f_3(\phi_L) = 2C(n, \phi_2, \phi_L, \theta_2) - C(n+2, \phi_2, \phi_L, \theta_2) - C(n-2, \phi_2, \phi_L, \theta_2) \]

\[ f_4(\phi_L) = S(n+2, \phi, \phi_L, \theta) - S(n-2, \phi, \phi_L, \theta) \]

\[ f_5(\phi_L) = S(n+2, \phi_2, \phi_L, \theta_2) - S(n-2, \phi_2, \phi_L, \theta_2) \]