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ANALYTIC DETERMINATION OF THE TRANSIENT RESPONSE OF A THIN-WIRE ANTENNA BASED UPON A SEM REPRESENTATION

Part I: Unloaded Antenna

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ABSTRACT

Based upon an analytical expression for the frequency-domain current induced on a thin-wire antenna, a theory for determining the SEM representation of the transient response of an unloaded transmitting as well as a receiving antenna is presented. An analytical expression for the SEM poles are derived, and the explicit expressions for the associated coefficients, natural mode current and coupling coefficient, are uncovered. A theoretical explanation to some of the physical process, related to the SEM representation of the transient current on a thin-wire antenna, is also provided. Excellent agreement with the numerical works for both the transmitting and receiving antennas are obtained.
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1. Introduction

Transient responses of cylindrical dipole antennas and scatterers have been of interest for many years. Recently, the area has received a considerable attention because of problems involving the investigation of electromagnetic-pulse (EMP) effects on thin wire structures. Determination of the transient responses of these structures have been studied by many authors, both for an infinitely-long cylindrical structure [1-4, 29-31] and structures of finite length [5-10] (For a more detailed bibliography, see [10]). Most of these attempts however have been numerical in nature; they either involve time harmonic analysis coupled with Fourier inversion [5,8,9] or direct numerical solutions of space-time-domain integral equation [6,7].

More recently, after Baum [11] introduced the singular expansion method (SEM) as an effective tool for treating general transient scattering problems, several papers [12-18] have appeared in the literature dealing with cylindrical antenna structures and the associated (exterior) natural frequencies, natural mode current, and coupling coefficients. Among them, Marin [13] has found a modal representation of the induced current on a perfectly conducting finite body, and shown that this representation is a meromorphic function of frequency, \( \omega \), and has only poles (i.e., natural frequencies) but no branch cut in the complex \( \omega \)-plane; Tesche [14] has numerically calculated the SEM poles and related coefficients for thin-wire antennas, and concluded that as a numerical technique the method to be more efficient than the conventional direct time-domain approaches.
However, the lack of analytical results sometimes obscures the physical insights of a representation which itself is closely associated with the ringing phenomenon of a current on a finite scattering object exposed to a transient electromagnetic environment. For example, little is known analytically about the arrival time of various components even though the structure itself has a distinct geometry where the current wave flows mainly along the axis of the antenna and bounces back and forth between the two ends of the antenna. In this work, based upon a frequency-domain expression for the current induced on a thin-wire antenna, we shall try to formulate a rather simple analytical SEM representation for the transient current in order to provide a theoretical explanation to some of the physical processes related to this representation.

Analytical expression in the real frequency domain for the current induced on a perfectly conducting thin-wire dipole antenna of radius, $a$, and half-length, $h$, has been previously obtained by Shen, et al. [20] in terms of the current existing on an infinite antenna, plus the multitude reflections from the two ends. Quite recently, it has been shown by Chang, et al. [27] and Rispin and Chang [21] that a modification of Shen's formula can yield a satisfactory result for a very broad frequency range under the condition, $\Omega = 2\ln(\frac{2h}{a}) \gg \ln(2kh)$ where $k = \omega(\mu\varepsilon)^{\frac{1}{2}}$ is the free-space wave number. Equipped with these results, an analytical expression for the SEM poles of an unloaded thin-wire antenna as well as the explicit expression for the current distribution of each mode is derived in Section 1. In Section 2, the SEM representation of the transient current induced on a transmitting antenna, driven by a delta function voltage generator, is determined.
In section 3, the problem is extended to that of a receiving thin wire antenna exposed to a transient plane-wave illumination. Explicit expression for the coupling coefficient is presented and physics of the "early" time \( t < \frac{2h}{c} \) response is discussed. Finally, results computed from the present theory are compared with those of the numerically evaluated "exact" solutions, and with the approximate results of Marin and Liu [18] which are obtained by use of the so-called "Hallen's first-order approximate expression" for the natural frequencies.
2. Resonance Condition; Natural Frequencies and Natural Mode Currents

Before we can derive the resonance condition for the natural frequencies of a thin-wire antenna, we first need to recognize that a resonance can be established on such an antenna only when a source-free current wave propagating in one direction bounces back from the end of the thin-wire and interacts constructively with its counterpart propagating in the opposite direction.

Consider now a finite antenna of length \( L (= 2h) \) and radius \( a \), as shown in Figure 1-a. Based upon the expression derived for the current on an infinitely long antenna \([20, 21]\), the modal current (i.e., free current distribution on a sourceless finite antenna) which consists of those waves bouncing back and forth between the two ends of the antenna, can be written as

\[
g(\omega; z) = AI_{\infty}(\omega; z) + BI_{\infty}(\omega; L-z) \quad ; \quad 0 \leq z \leq L
\]  

(1)

where \( A \) and \( B \) are the amplitudes of a forward current wave travelling in the + \( z \) direction away from the end of the wire at \( z = 0 \) and the reflected wave travelling in the \( z \)-direction away from the end at \( z = L \). \( I_{\infty}(\omega; z-z') \), in general, is the current at \( z \) on an infinitely-long antenna emanating from a source at \( z' \). Various approximate expressions for \( I_{\infty} \), in the real frequency domain are given in Appendix A. Those expressions are derived subject to a so-called thin wire condition, \( 2\ln(2 \frac{Z-z'}{a}) \gg |\ln(2 \frac{\omega}{c}|z-z'|)| \).

To obtain the resonance condition of a given mode, we need to require the amplitudes \( A \) and \( B \) in the modal current expression of (1) to satisfy the known boundary condition at each end of the wire \([20, 21]\):

\[
\frac{A}{BI_{\infty}(\omega; L)} = - R(\omega), \quad \text{at } z = 0
\]

(2)
Fig. 1-a A transmitting cylindrical antenna

Fig. 1-b A receiving cylindrical antenna illuminated by an incident plane wave
\[ \frac{B}{AI_\infty(\omega;L)} = -R(\omega), \quad \text{at } z = L \quad (3) \]

where \( R(\omega) \) is the reflection coefficient of an incident current wave onto the end of a semi-infinite antenna. The approximate forms of \( R(\omega) \) are given in Appendix A. By eliminating \( A \) and \( B \) from (2) and (3), we have the resonance condition

\[ \Delta(\omega_s) = 0 \quad ; \quad s = 0, 1, 2, 3, \ldots \quad (4) \]

where

\[ \Delta(\omega) = 1 - R^2(\omega) I^2_\infty(\omega;L) \quad (5) \]

Roots of (5), in the complex \( \omega \)-plane, are the eigenvalues of a resonant thin-wire structure.

In SEM, the eigenvalues are usually referred to as the natural or resonance frequencies. To obtain more information about the location and the nature of these natural frequencies, we rewrite the equation (4) as

\[ R(\omega_s)I_\infty(\omega_s;L) = e^{is\pi} ; \quad s = 1, 2, 3, \ldots \quad (6) \]

From the expressions for \( R \) and \( I_\infty \) in Appendix A we note that since the magnitude of \( RI_\infty \) is typically less than unity for a real frequency, the roots of (6) can exist only in the lower half of the complex \( \omega \)-plane. This is consistent with the time causality principle (for \( e^{-i\omega t} \)), which does not allow the exponentially growing currents. One can also show that the roots at \( \omega_s \) and \( -\omega_s^* \) always form a pair that satisfies (6); as we will see later this implies that the time domain current is real.

By substituting the expression for \( B \) [from (3)] into (1) and using the normalization \( A=1 \), the current expression can be given in general as
\[ g(\omega;z) = I_{\infty}(\omega;z) - R(\omega)I_{\infty}(\omega;L)I_{\infty}(\omega;L-z) \]  

(7)

Eigenfunctions of the problem, which are the current distributions at resonance, are now given by the expression in (7) at \( \omega = \omega_s; s = 1, 2, 3, \ldots \). From (6), and (7) we can therefore obtain an explicit form for the natural mode current as

\[ G_s(z) = I_{\infty}(\omega_s;z) \pm I_{\infty}(\omega_s;L-z) \]  

(8)

where \( \pm \) sign is chosen when \( s \) is odd or even, respectively. We can show without any difficulty that the zero-order approximation (i.e., as \( \frac{a}{L} \to 0 \)) of (8), which, aside from a normalization factor, corresponds basically to a transmission line problem, is now given as

\[
G_{s}(\text{tra})(z) = \left(\frac{2\pi}{\eta_0 \Omega}\right) \left[ e^{i\omega_s z} s + e^{i\omega_s (2h-z)} \right] \\
= \left(\frac{4\pi}{\eta_0 \Omega}\right) e^{i\omega_s h} \begin{cases} 
\cos[\omega_s (z-h)], & s \text{ odd} \\
\sin[\omega_s (z-h)], & s \text{ even}
\end{cases}
\]  

(9)

By inserting the approximate expressions given in Appendix A [(A.5) for \( I_{\infty}(\omega;L) \) and (A.6) for \( R(\omega) \)] into the modal equation (4) and using an iterative scheme, we have calculated the natural frequencies for resonant modes corresponding to the first-layer in the modal spectral.

In Figure 2, we have shown the location of the first natural frequency in the complex \( \omega \)-plane, for various antenna diameters, and compared with Tesche's numerical result [14], Hallen's first and second order results [18, 19, 22] and the result obtained from Weinstein's first order approximation [23]. The latter is a first order perturbation solution of the modal equation (6) (see Appendix B). As the Figure shows, there is an excellent agreement between the result obtained from equation (4) and the numerical
Fig. 2 The location of the first natural frequency of a thin-wire antenna in the complex frequency plane as antenna diameter, $d = 2a$, varies.
1. Tesche's numerical results
2. Results from eq. (4)
3. Weinstein's first-order approximation, eq. (B.6)
4. Hallén's first-order approximation
5. Hallén's second-order approximation
result of Tesche, for $\frac{2a}{L} \leq 10^{-2}$. As we increase the diameter of the wire antenna beyond $\frac{2a}{L} = 10^{-2}$, the accuracy of our result decreases. However, it can be seen from the Figure that as $\frac{2a}{L}$ increases, our overall result remains relatively much closer to the numerical result, especially for the thicker antennas (i.e., $\frac{2a}{L} > 10^{-2}$), than that of Hallen's second order approximation. Also it is obvious from Figure 2 that Weinstein's result is relatively better than that of Hallen's first order approximation; nevertheless both approximations should be used only for very thin wire antennas. Not surprisingly, as $\frac{2a}{L} \to 0$, all approximate results converge to a resonance frequency of $\frac{\omega L}{c} = \pi$, which is the transmission line solution for the first natural frequency of an infinitely-thin antenna. The current distribution of the first natural mode $[G_1(z)$ in eq. (8)] on an antenna with $\frac{2a}{L} = 10^{-2}$, along with corresponding numerical solution of Tesche [14] and the zero order approximation [eq. (9)] are shown in Figure 3. The results are normalized to the value of natural mode current at the center of the antenna. The agreement between our result [i.e., $\frac{G_1(z)}{G_1(h)}$] and Tesche's solution is excellent, especially for the real part which is the dominant part of the resonant current. The zero order approximation which is given by a simple $\cos[\omega_s(z-h)]$ term in equation (9) does not agree well with the numerical result, as expected. Similar agreements, as these shown in Figures 2 and 3 are also found for the higher-order modes of the first layer. However, we note that the formulation of the modal equation as it stands does not contain roots of any higher-order layers. This is because the expressions for $R$ and $I_\infty$ which were used in (4) for calculating the natural modes
Fig. 3 First normalized natural mode current for a thin-wire antenna with $2a/L = 0.01$. 
are derived for a real frequency and can be analytically continued only to frequencies close to the positive real axis [21]. Consequently, any resonance which is brought about by a large exponentially-growing wave in \( z \) (i.e., \( \omega \) near the negative imaginary axis) will be automatically eliminated. These resonances however are quite inconsequential in the calculation of transient response [14].

Finally, it should be noted that the modal equation (4) has also a root at \( \omega_0 = 0 \). This root, however, corresponds to a static charge distribution along the dipole antenna, and is of no importance in the present problem.
3. Transient Response of a Transmitting Antenna

3.1. Analytical Expression for the Antenna Current

Let us consider a transmitting antenna, as shown in Figure 1-a, driven by a delta-function voltage generator \( V_g \) which is located at \( z = z' \), has an amplitude of \( V_0 \) and a step-function response of \( H(t-t') \), i.e.,

\[
V_g(t) = V_0 H(t-t'), \text{ at } z = z'
\]  

(10)

with \( H(t) \) being the Heaviside unit step-function. By an analogy to transmission-line theory, we first postulate that the total current distribution on the antenna, in the real frequency domain, can be written down approximately as

\[
\tilde{I}^{(0)}(\omega; z, z') = \tilde{I}_0(\omega)g(\omega; z_>)g(\omega; L-z_<)
\]

(11)

where \( z_>(z_<) \) is the larger (smaller) value of the observation point \( z \) and source point \( z' \); \( g(\omega; z) \) and \( g(\omega; L-z) \), as given in (7), are the appropriate current expressions on the antenna that satisfy the end condition at \( z = L \) and \( z = 0 \), respectively. We note that the current continuity condition at the source point, where \( z = z' \), was invoked in the expression we constructed. As shown in Appendix C, \( \tilde{I}_0 \) which represents the amplitude of the antenna current, can be determined from the voltage condition in (10) in a manner similar to the treatment of a transmission-line:

\[
\tilde{I}_0(\omega) = \frac{\tilde{V}_g(\omega)}{\Delta(\omega)\tilde{I}_\infty(\omega; L)}
\]

(12)

where \( \Delta(\omega) \) is given by (9), and \( \tilde{V}_g(\omega) \) is the Fourier transform of \( V_g(t) \) in (14), i.e.,
\[
\tilde{V}_g(\omega) = \int_{-\infty}^{\infty} V_g(t)e^{i\omega t} dt = \lim_{\delta_0 \to 0} \int_{-\infty}^{\infty} e^{-\delta_0 t} e^{i\omega t} dt = -\frac{V_0}{i\omega} e^{i\omega t'}
\]

The question arises as to whether one can indeed construct an expression for antenna current strictly from a transmission-line analogy. After all, it is well-known that in transmission-line theory, the current on the line as derived from the telegraphist equation satisfies an one-dimensional wave equation. However for thin-wire antennas, it is the longitudinal component of the vector potential, rather than the current, that satisfies the same wave equation. Thus the constructed expression has to be examined more carefully before we can use it with confidence.

To this end, we can first write down the general solution for current distribution on a thin-wire antenna of finite length as derived in [20,21]:

\[
\tilde{I}_T(\omega;z,z') = \tilde{V}_g(\omega)[I_\infty(\omega;z-z') + C_+(\omega;z')I_\infty(\omega;z) + C_-(\omega;z')I_\infty(\omega;L-z)]
\]

Physically the first term in the bracket represents the outgoing current wave away from the source, while the second and third terms are the reflected waves away from the \( z = 0 \) and \( z = L \) ends of the antenna, respectively. Amplitudes of the reflected currents \( C_+ \) and \( C_- \) can be found by matching with the solution obtained from a Wiener-Hopf procedure for a semi-infinite thin-wire [20,21]:

\[
C_+(\omega;z') = -\frac{R(\omega)}{\Delta(\omega)} g(\omega;z')
\]

\[
C_-(\omega,z') = -\frac{R(\omega)}{\Delta(\omega)} g(\omega;L-z')
\]
where $\Delta(\omega)$ and $g(\omega; z')$ are given by (5) and (7), respectively. After (15) and (16) are substituted into (14), we can write the latter equation in terms of the expression we constructed in (11) and a reminder. After some algebraic manipulation, we can show that

$$\tilde{I}_T(\omega; z, z') = \tilde{V}_g(\omega) \left[ \frac{1}{\Delta(\omega)I_\infty(\omega; L)} g(\omega; z)g(\omega; L-z_\epsilon) + \tilde{e}(\omega; z_\epsilon, z_\epsilon) \right]$$

(17)

$$= \tilde{I}_T^{(0)}(\omega; z, z') + \tilde{V}_g(\omega)\tilde{e}(\omega; z_\epsilon, z_\epsilon)$$

(18)

where

$$\tilde{e}(\omega; z_\epsilon, z_\epsilon) = I_\infty(\omega; z_\epsilon - z_\epsilon) - \frac{1}{I_\infty(\omega; L)} I_\infty(\omega; z_\epsilon)I_\infty(\omega; L-z_\epsilon)$$

(19)

Since $I_\infty(\omega; z-z')$ as discussed in Appendix A, is essentially a propagating wave with a slow varying amplitude away from the source at $z = z'$ thus $\tilde{e}$ in general is negligibly small except at location of the voltage source. Moreover, because $\tilde{e}$ does not possess any isolated singularities in the complex $\omega$-plane, poles of $\tilde{I}_T^{(0)}$ as obtained from the approximate model still adequately describe the natural resonances of a thin-wire antenna. We further note that Marin in [13] has shown that a perfectly conducting structure of finite size in a lossless space possesses only isolated singularities (i.e., natural frequencies) in the finite $\omega$-domain. It is evident that the appearance of a logarithmic branch cut in our expression is a result of constructing the total current from the current on an infinitely-long antenna (i.e., $I_\infty$), and should be ignored in our later analysis. A rather similar but mathematically more traceable situation happens in formulating the current on a lossy finite transmission line (which has only isolated singularities), based on a lossy infinite
line (which has a branch cut singularity). A transient analysis of this latter problem is given in the Appendix D, in which we can show that the appearance of a branch-cut singularity eventually presents no net effect in computing the transient response.

To obtain a real-time response for the antenna current in (18), we need to perform the Fourier inverse transform along the real axis of the complex $\omega$-plane, with an indentation upward around $\omega = 0$, as shown in Figure 4. Then

$$I_T(t,t';z,z') = \frac{1}{2\pi} \int_{\Gamma_0} \tilde{I}_T(\omega; z,z') e^{-i\omega t} d\omega \quad (20.1)$$

$$= \frac{1}{2\pi} \int_{\Gamma_0} \tilde{I}^{(0)}_T(\omega; z,z') e^{-i\omega t} d\omega + \int_{\Gamma_0} \tilde{V}_g(\omega) \tilde{e}(\omega; z,z') e^{-i\omega t} d\omega \quad (20.2)$$

Because the integrand in (20.1) as well as those in (20.2) behave like $\exp[H_i(\omega)\frac{Z_\ge - Z_\le}{C} - t + t')]$ for $I_{\text{in}}(\omega) > 0$, a deformation of the contour $\Gamma_0$, for $t - t' < \frac{Z_\ge - Z_\le}{C}$, into the upper half $\omega$-plane yields

$$I_T(t,t';z,z') = 0, \text{ for } t - t' < \frac{Z_\ge - Z_\le}{C} .$$

For $t - t' > \frac{Z_\ge - Z_\le}{C}$, however, in order to obtain a proper SEM representation of the time response $I_T(t,t';z,z')$, we deform the contour of integration $\Gamma$ into the lower half $\omega$-plane. In this process, residues at the poles of the integrand [zeroes of $\Lambda(\omega)$] are captured; whereas the contribution from the pole, $\omega = 0$, of $\tilde{V}_g(\omega)$ has no consequence.

With some simplifications, finally we have:

$$I_T(t,t';z,z') = 2V_0 \left[ \text{Re} \sum_{s=1}^{\infty} \frac{R(\omega_s)}{\omega_s \cdot \Delta'(\omega_s)} G_S(z) G_S(z') e^{-i\omega_s(t-t')} \right] H(t-t' - \frac{Z_\ge - Z_\le}{C})$$

$$\text{where } G_S(z) \text{ is the natural mode current given in (8), and } \Delta'(\omega_s) \equiv \left. \frac{\partial \Lambda(\omega)}{\partial \omega} \right|_{\omega = \omega_s} . \quad (21)$$
Fig. 4 Deformation of contour in the complex $\omega$-plane
Expression given in (21) is analogy to the matrix form given in [14]. Unlike [14] however, is the addition of the unit-step function resulting physically from the time casualty requirement.

3.2 Results and Comparisons

The transient responses of the current at \( z = \frac{L}{2} \) and \( z = \frac{L}{4} \) for a dipole center driven by a delta-gap generator whose output voltage is a step function in time and has strength \( V_0 \), are shown in Figures 5 and 6. Also included for comparisons are the corresponding numerically "exact" results of Liu and Mei [7] and approximate results of Marin and Liu [18]. The currents are normalized and the "antenna parameter" is \( \Omega = 10 \) (i.e., \( \frac{L}{a} \approx 148.4 \)). In calculating the transient results (solid lines in Fig. 5-6) the first ten modes in the summation (21) have been used; however, there are no contributions from the even modes \( (s=2,4,6,8,10) \) for current at the mid-point \( (z=\frac{L}{2}) \). Figure 5 shows the time history of the driving point \( (z=\frac{L}{2}) \) current, while Figure 6 demonstrates the current at \( z = \frac{L}{4} \).

In both cases the agreement between our results and those obtained numerically by solving a space-time domain integral equation, is indeed excellent. Also as we can see, a definite over-all improvement over the results of Marin and Liu [18] is obtained. Discrepancy in the result of [18] comes partly from the fact that the fundamental resonance mode (i.e., first natural frequency) in [18], as obtained by the Hallen's first-order approximation has smaller real and larger imaginary parts than those obtained by equation (8) and the numerical work of Tesche (see Fig. 2). In addition, the natural mode current given in [18] is only a "zero order" approximation which neglects the effects of the finite radius, \( a \), and the wire ends.
Fig. 5 Time-domain response of the driving-point current on a transmitting antenna of finite length $L$, center-driven by a delta-function generator with output voltage $V_0H(t)$. Also included is the corresponding response of an infinite cylindrical antenna as obtained from eq. (E.10)
Fig. 6 Time-domain response of the current at $z = \frac{L}{4}$ on a transmitting antenna of finite length $L$, center-driven by a delta-function generator with output voltage $V_0 H(t)$. 
To get more insight into the physical process which results in the
damping and the oscillations of the current responses, we have also included
in Figure 5, in dashed-line, the corresponding driving-point current response of
an infinite cylindrical antenna [for derivation see Appendix E]. It can
be seen that for $0 \leq \frac{Ct}{L} < 1$, the transient response of the finite
antenna is almost entirely given by that of the infinite one, as expected
(see Appendix D). The drop in the current in this time interval is due
to radiation losses in the vicinity of the feed point. For $\frac{Ct}{L} > 1$ however,
the infinite antenna predictions are no longer applicable; this is because
at $\frac{Ct}{L} = 1$ the current waves reflected from the $z = 0$ and $z = 2h = L$ ends
of the finite dipole antenna reach the observation point at $z = h$ and
cause the total current to drop to the negative values. At $\frac{Ct}{L} \approx 2$
return of the second set of reflections from the two ends increases the
current to the positive values. The repeat of the process at $\frac{Ct}{L} \approx 3, 4, 5, \ldots$
causes the damped oscillations in the current. One can also see in the
same figure that the transient response of the finite antenna is more
attenuated than the corresponding response of the infinite antenna.
Radiation losses due to the reflection processes at the ends of the finite
antenna are responsible for the extra attenuation. If there were not any
losses at the ends, the corresponding points, at $\frac{Ct}{L} = 3$, on the two curves
in Figure 5 would coincide. However as one clearly sees, there is an
amplitude difference between the two curves at $\frac{Ct}{L} = 3$, and this difference
becomes larger (at $\frac{Ct}{L} = 5, 7, \ldots$) as time increases, because of the multiple
reflections and consequently the radiation losses at the ends.

A similar discussion as above could be used to explain the transient
behavior of the current at $z = \frac{L}{4}$, in Figure 6.
4. Transient Response of A Receiving Antenna

4.1 Analytical Expression for the Antenna Current

A thin wire receiving antenna, exposed to a plane-wave $E_0$ with a step-function time response $H(t - \frac{X}{C} \sin \theta - \frac{Z}{C} \cos \theta)$, is shown in Figure 1-b. The incident tangential electric field (in the $\omega$-domain) along the wire is then given by

$$E^{inc}(\omega, z) = \frac{E_0}{i \omega} \sin \theta \exp(i\omega z \cos \theta/c)$$  \hspace{1cm} (22)

Transient response of the receiving antenna can be now determined from the expression given in (21), by "switching" sequentially voltage sources of amplitude $V_0 = E_0 \sin \theta dz'$ at $z = z'$ and $t' = z' \cos \theta/c$. Therefore, the integration of (21) with respect to $z'$ over the entire length of the antenna yields the following expression for the transient response of a receiving antenna

$$I_R(t; z, \theta) = 2E_0 \sin \theta \left[ \text{Re} \sum_{s=1}^{\infty} \frac{R(s)}{\omega_s \Delta'(\omega_s)} G_s(z)W_s(t; z, \theta)e^{-i\omega_s t} \right]$$  \hspace{1cm} (23)

where

$$W_s(t; z, \theta) = \int_0^L G_s(z')e^{i\omega_s t'} H(t-t' - \frac{Z_s - Z_c}{C}) dz'$$  \hspace{1cm} (24)

which can be shown to be of the form (see Figure 1-b)

$$W_s(t; z, \theta) = H(t - \frac{Z_c}{C} \cos \theta) \int_{Z_1}^{Z_2} G_s(z')e^{i\omega_s \cos \theta z'/c} dz'$$  \hspace{1cm} (25)

where

$$z_1 = \left( \frac{Z - ct}{1 - \cos \theta} \right) [H(t - \frac{Z}{C} \cos \theta) - H(t - \frac{Z}{C})]$$  \hspace{1cm} (26.1)

$$z_2 = \left( \frac{Z + ct}{1 + \cos \theta} \right) [H(t - \frac{Z}{C} \cos \theta) - H(t - t_0)] + L H(t - t_0)$$  \hspace{1cm} (26.2)
wherein
\[ t_0 = \left[ L \left( 1 + \cos \theta \right) - z \right]/c \]

Physically, as shown in Figure 1-b, this means one should not expect any current at the observation point \( z \) until the plane wave arrives at \( t = \frac{Z}{c} \cos \theta \). This of course is consistent with the time casual requirement. After that, the portion of sources from \( z \) to \( z_1 \), which arrived earlier, travels forwardly at the speed of \( c \) along the antenna and will contribute to the current at \( z \). Likewise, the portion of sources from \( z \) to \( z_2 \), which arrives at a time somewhat later, will travel along the negative \( z \)-direction and then contribute to the current at \( z \) (Figure 1-b). Thus, not until the observation time \( t \) is greater than the larger value of \( \frac{Z}{c} \) and \( t_0 \), will the expression of \( \mathbf{W}_S(t;z,\theta) \) be equivalent to that of the coupling coefficient \( \beta_s \) as defined by [13,14,16],

\[
\beta_s(\theta) = \int_0^L G_s(z') e^{i \omega_s \cos \theta z'/c} dz'.
\]  (27)

One should not expect the contribution of sources over the entire length of the antenna immediately upon the arrival of the first wave front and consequently, the full ringing phenomenon associated with the natural modes, until after the wave has ample time to bounce back from the two ends of the antenna.

The integral in (25) and consequently, the coupling coefficient \( \beta_s \) are evaluated analytically in Appendix F.
4.2 Results and Comparisons

Since \( W_s(t;z; \theta) = \beta_s(\theta) \) for \( t \geq \max(\frac{Z}{c}, t_0) \), thus in calculating the "late time" response from (26), the coupling coefficient \( \beta_s \) which determines how much of the natural mode current is excited by the incident field, will be an important factor. Because for a given incident angle \( \theta \), \( \beta_s \) is a function of \( \omega_s \) only, and independent of \( t \) and \( z \), so that once it is evaluated for the natural modes (i.e., \( \omega_s, s=1,2,3,\ldots \)) it can be used in (23) for calculating the current responses at different times and locations along the wire. In Figure 7 we have shown the real and imaginary parts of \( \beta_s \) for the first natural mode \( (s=1) \), together with Tesch's numerical results [14]. Also depicted in this Figure is the "zero order" result obtained by using the transmission line approximation \((G_s^{(tra)} \) in (9)) for \( G_s(z') \) in (27). The agreement between our result and that of Tesche's numerical with regard to the real part is excellent. And because the imaginary part is very small, it turns out that the discrepancy in the imaginary part is quite acceptable.

The first five odd modes \( (s=1,3,5,7,9) \) in the summation of (23) were used to calculate the time response of the induced current at \( z = \frac{L}{2} \) on the antenna, at two different angles of incidence of the plane wave. Comparisons between these results and the corresponding numerical results of Liu and Mei [7] and approximate results of Marin and Liu [18] are shown in Figures 8 and 9 for \( \theta = 30^\circ \) and \( 90^\circ \), respectively. The currents are normalized and the "antenna parameter" is \( \Omega = 10 \). Again, excellent agreements with the numerical works of Liu and Mei, and a definite improvement over the results of Marin and Liu, for both angles of incidence,
Fig. 7 Normalized coupling coefficient as a function of incident angle, \( \theta \), for first natural frequency of a thin-wire antenna with \( 2a/L = 0.01 \).
Fig. 8: Time-domain response of the mid-point current on a receiving antenna of finite length $L$, illuminated by a uniform plane wave with step-function time response. The plane wave is incident at angle $\theta = 30^\circ$, and has an electric field strength $E_0$. 

\[ \frac{\Omega \eta_0 I}{E_0 L \sin \theta} \]

- **$\theta = 30^\circ$**
- **$z = L/2$**

\[ \Omega = 10 \]

- **NUMERICAL**
- **MARIN AND LIU**
- **FROM Eq. (23)**
Fig. 9 Time-domain response of the mid-point current on a receiving antenna of finite length L, illuminated by a uniform plane wave with step-function time response. The plane wave is incident at angle $\theta = 90^\circ$, and has an electric field strength $E_0$. 

$\Omega \frac{\eta_0 I}{E_0 L \sin \theta}$  

$\theta = 90^\circ$  

$z = L/2$  

$\Omega = 10$  

- NUMERICAL  

- MARIN AND LIU  

- FROM Eq. (23)
are obtained. As expected, for $\theta = 30^\circ$ in Figure 8 there is no current at $z = \frac{L}{2}$ until $\frac{ct}{L} = 0.433$, at when the plane wave arrives. For $\theta = 90^\circ$ in Figure 9 however, the current response starts at $t=0$ and it then increases almost linearly to its maximum at $\frac{ct}{L} \approx 0.5$ when the reflected currents from the ends reach the observation point and cause the total current to drop. Finally, it can be observed from Figures 8 and 9 that for $\theta = 90^\circ$ the overall amplitude of the response is larger than that of $\theta = 30^\circ$ incidence. This is expected because for the former case the incident field is more strongly coupled into the fundamental mode current, as Figure 7 reveals.
5. Conclusion

The analytical expression in the frequency-domain for the current, which had been previously obtained in [21], was used to find a SEM representation for the transient current on an unloaded thin-wire dipole antenna.

In Section 2, a simple modal equation for the natural frequencies, in terms of the current existing on a infinite antenna and the reflection coefficient $R(\omega)$, was derived. Excellent agreements between our calculated results for the first natural frequency and the numerical results were obtained for $\frac{2a}{L} \leq 10^{-2}$. For $10^{-2} < \frac{2a}{L} \leq 5 \times 10^{-2}$ our results were acceptable, and showed a definite overall improvement over the Hallen's second order approximate results. The expression for the natural mode current was also explicitly expressed in this section in terms of the current existing on an infinitely long antenna; it was shown that this rather simple expression gives very accurate results for the natural mode currents.

In Section 3, the $\omega$-domain current induced on a transmitting antenna driven by a delta-function voltage generator was approximately expressed in terms of the product of two modal functions due to the source and observation points. It was shown that this expression contains all the "necessary information" in the complex $\omega$-domain, for a transformation to time-domain. The relatively simple SEM representation found for a transmitting antenna was used in Section 4 to obtain the corresponding representation for a receiving thin-wire antenna exposed to an incident-pulsed wave field. The coupling coefficient was also defined and its integral was evaluated analytically in Appendix F. Finally, for step-function response of the current, excellent agreements with the numerical works
and a definite improvement, particularly in the early times, over the approximate works of Marin and Liu [18] for both the transmitting and receiving antennas were demonstrated.

The time-domain SEM representations (21) and (23) were obtained for an excitation waveform of a time unit step-function only. However, the method is also applicable to cases of other excitation waveforms. In fact when the excitation \( \tilde{V}_g(\omega) \) in (14) [or \( \tilde{E}^{\text{inc}}(\omega,z) \) in (22)] has poles' singularities other than the one at \( \omega = 0 \), the corresponding transient response [in (20)] can be determined by simply adding the residues' contributions at the poles of \( \tilde{V}_g(\omega) \) [or \( \tilde{E}^{\text{inc}}(\omega,z) \)] to those at the SEM poles. It should also be mentioned here that for any pole of the excitation other than \( \omega = 0 \), the function \( \tilde{e}(\omega;z_>,z_<) \) in (20.2) could have a residue contribution to the total response. This contribution however would be negligible for all practical purposes.

The result obtained in Section 3 can also be applied easily to obtain the transient far-field response of a transmitting antenna. The \( \theta \)-component of the electric far-field in the frequency-domain can be written as [25]

\[
\tilde{E}_\theta(\omega;r,\theta) \approx \frac{-i \omega}{4\pi c} \sin \theta \frac{e^{-i \omega r}}{r} \int_0^L \tilde{I}(\omega;z,z') e^{-i \omega z \cos \theta} dz
\]  

where \( \tilde{I} \) is the current distribution on the antenna, and \((r,\theta,\phi)\) refers to a spherical coordinate system coincident with the cylindrical coordinate system implied in Figure 1-a. Now by substituting \( \tilde{I}(\omega;z,z') \) by \( \tilde{I}_T(\omega;z,z') \) given in (12), and following the Fourier inverse transformation, we will finally arrive at the following SEM representation for the transient electric far-field response.
\[ \tilde{E}_\theta(t;r,\theta) = \left( \frac{V_0 \eta_0}{2\pi c} \right) \frac{\sin \theta}{r} \left[ \text{Re} \sum_{s=1}^{\infty} \frac{R(\omega_s)}{i\omega_s \Delta^e(\omega_s)} G_s(z') \psi_s(t,r,\theta) e^{-i\omega_s(t-\frac{r}{c})} \right] \]

(29)

where

\[ \psi_s(t;r,\theta) = \int_0^L G_s(z) e^{-i\omega_s \cos \theta \frac{z}{c}} H\left(t - \frac{r}{c} + \frac{z}{c} \cos \theta - \frac{|z-z'|}{c}\right) dz \]  

(30)

\( \psi_s(t;r,\theta) \) can be evaluated analytically, according to the technique used for evaluation of \( W_s \) in Section 4 and Appendix F.

Finally, it should be mentioned that the method used in this work can be extended to formulate a SEM representation for the transient current induced on a loaded thin-wire antenna. This problem is now under study and the results will be presented in a future report.
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APPENDIX A

Approximate Expressions for \( I_\infty(w;z) \) and \( R(\omega) \)

A very good approximate expression for \( I_\infty \), in the real frequency domain, under the thin wire condition, \( 2 \ln (2 \frac{|z|}{a}) \gg |\ln (2 \frac{\omega}{c}|z|)| \), is given by [21]

\[
I_\infty(\omega;z) \approx \frac{1}{\eta_0} e^{\frac{i \omega}{c}|z|} \{ \ln[U(\omega;z) - i \pi] - \ln[U(\omega;z) + i \pi] \}
\]  \( (A.1) \)

where \( \eta_0 \) and \( c \) are the free space intrinsic impedance and light velocity, respectively; and

\[
U(\omega;z) = \Gamma(\omega) + \lambda(-i \frac{\omega}{c}|z|)
\]  \( (A.2) \)

wherein

\[
\lambda(x) = \ln(x) + e^{X_1(x)} + \gamma
\]  \( (A.3) \)

and

\[
\Gamma(\omega) = -2 \ln(-i \frac{\omega}{c} e^{-\frac{\gamma}{2}}) \quad \gamma = 0.57721... \quad (A.4)
\]

The function \( E_1 \), appearing in (A.3) is the exponential integral of the first kind defined in equation 5.1.1 of [24].

A simpler approximate form of \( I_\infty \), which stems from a Taylor series expansion of (A.1) subject to the thin wire condition, is given by

\[
I_\infty(\omega;z) = \frac{2\pi}{\eta_0} \frac{e^{\frac{i \omega}{c}|z|}}{U(\omega;z)}
\]  \( (A.5) \)

It has been found in [21] that (A.5) has a more desirable behavior than (A.1), for \( \frac{\omega}{c}a \ll 1 \) and \( z \approx 0 \).

The desired expression for the reflection coefficient, \( R(\omega) \), is derived in [21] and has the following approximate forms in the \( \omega \)-domain.
\[ R(\omega) = \begin{cases} \frac{\eta_0}{2\pi} \frac{\Gamma(\omega)}{|A_0|[1+B_0]} , & \text{Re}(\omega) \geq 0 \\ \frac{\eta_0}{2\pi} \frac{\Gamma(\omega)}{|A_0|[1-B_0+C_0]} , & \text{Re}(\omega) \leq 0 \end{cases} \]  

(A.6)

where \( \Gamma(\omega) \) is given in (A.4), and

\[ A_0 = \frac{\Gamma(\omega)}{[i\pi J_0(ka)H_1^0(ka)]^{\frac{1}{2}}} \]  

(A.7)

\[ B_0 = \frac{1}{2\pi} \text{Re} \left\{ i\{\ln[\Gamma(\omega) - g_0 - i \frac{3\pi}{2}] - \ln[\Gamma(\omega) - g_0 + i \frac{3\pi}{2}] \} \right\} \]

\[ - \text{Re} \left[ \frac{1}{\Gamma(\omega) + \ln(2)} \right] \]  

(A.8)

wherein

\[ g_0 = 33.88 \left( \frac{\omega}{c} \right)^2 \exp\left[ - \frac{3.26}{\left( \frac{\omega}{c} \right)} \right] \]  

(A.9)

and finally

\[ C_0 = \frac{1 - |A_0|^2(1-B_0^2)}{|A_0|^2(1+B_0^2)} \]  

(A.10)

A simpler formula for \( R(\omega) \) can be obtained when \( \left| \frac{\omega}{c} \right| \) is very small. For \( \left| \frac{\omega}{c} \right| \ll 1 \), it can be readily shown from (A.7 - A.8) that \( A_0 \approx 1.0 \) and \( B_0 \approx 0 \); this reduces (A.6) to

\[ R(\omega) \approx \frac{\eta_0}{2\pi} \Gamma(\omega) \]  

(A.11)

It is noteworthy that the approximation (A.11) for \( R(\omega) \) used with the simple form of \( I_{\infty} \) in (A.5) will force the natural mode current in (8) [or the total current in (18.1)] to zero, at the ends of a finite thin wire antenna.
Finally, the comparison of (A.6) and (A.11) with the numerical evaluation of the exact expression obtained from the use of the Wiener-Hopf technique has shown the accuracy of these approximate forms [21].
APPENDIX B

A First-Order Perturbation Solution of the Modal Equation

An approximate solution for the natural frequency $\omega_s$, for a very thin wire antenna, is given in this Appendix. It was shown in equation (6) that

$$R(\omega_s)I_\infty(\omega_s; L) = e^{i\pi s} ; \quad s = 1, 2, 3,... \quad (B.1)$$

Now by using the approximate expressions (A.5) for $I_\infty$ and (A.11) for $R$, equation (B.1) may be written as

$$\exp\left[i\frac{\omega_s L}{c} + \ln \phi(\omega_s; L)\right] = \exp(i\pi s) ; \quad s = 1, 2, 3,... \quad (B.2)$$

where

$$\phi(\omega_s; L) = \frac{\Gamma(\omega_s)}{U(\omega_s; L)} = \left[1 + \frac{\lambda(-i2 \frac{\omega_s}{c} L)}{\Gamma(\omega_s)}\right]^{-1} \quad (B.3)$$

where $U, \Gamma$ and $\lambda$ are given in Appendix A. For very thin wire antennas,

$$\left|\frac{\omega_s}{c}\right|a \ll 1 \quad \text{and} \quad \left|\Gamma(\omega_s)\right| \gg \left|\lambda(-i2 \frac{\omega_s}{c} L)\right|.$$ 

Subsequently, by keeping the first two terms in the Taylor series expansion of (B.3), we have

$$\phi(\omega_s; L) \approx 1 - \frac{\lambda(-2 \frac{\omega_s}{c} L)}{\Gamma(\omega_s)}$$

$$\approx 1 - \varepsilon_0(\omega_s) ; \quad \left|\varepsilon_0\right| \ll 1. \quad \quad (B.4)$$

Since $\ln(1 - \varepsilon_0) \approx -\varepsilon_0$ for $\left|\varepsilon_0\right| \ll 1$, then insertion of (B.4) into (B.2) now yields

$$\omega_s = \frac{c}{L} \left[s\pi - i\varepsilon_0(\omega_s)\right] ; \quad s = 1, 2, 3,... \quad (B.5)$$
which may be solved iteratively to obtain \( \omega_s \). The zero-order solution is \( \omega_s^0 = \left( \frac{c \pi}{L} \right) s \) which, in turn, can be substituted for \( \omega_s \) in the right hand side of (B.5) to yield the first-order perturbed solution

\[
\omega_s^1 \approx \frac{c}{L} \left[ s \pi - i \varepsilon_0 \left( \frac{c \pi}{L} \right) s \right]; \quad s = 1, 2, 3, \ldots \quad (B.6)
\]

where from (B.4), (A.3) and (A.4)

\[
\varepsilon_0 \left( \frac{c \pi}{L} \right) s = \frac{\gamma + \ln(-i2s\pi) + e^{-i2s\pi} E_1(-i2s\pi)}{-2\pi n(-i2s\pi e^{\gamma \frac{a}{L}})} \quad (B.7)
\]

The solution in B.6 is identical to the first-order approximate result obtained by Weinstein [23].
APPENDIX C

An Approximate "Differential Equation" for Current on Thin Wire Antennas

As it is well known, current on a transmission line satisfies a one-dimensional wave equation. For thin wire antennas however, it is not the current but the vector potential, \( A_z \), which satisfies the same equation. For a thin wire antenna, the two physical qualities are related to each other in the following manner.

\[
A_z(s) = \frac{\mu_0}{4\pi} \int_0^L I(z') \frac{e^{ik_0r_1}}{r_1} \, dz'
\]  

(\text{C.1})

where \( k_0 = \omega \sqrt{\mu_0 \varepsilon_0} \) and \( r_1 = [(z-z')^2 + a^2]^\frac{1}{2} \). However, because of the similarities between the current distribution on a transmission line and that on a thin-wire antenna, as radius \( a \to 0 \), it is logical to look for an approximate "differential equation" for the current on a thin wire antenna.

We start with the current on an infinitely long wire antenna; from (A.5) we have:

\[
I_\infty(\omega;z) = \frac{i\omega |z|}{\Gamma(\omega) + \lambda(-i2\omega \frac{|z|}{c})}
\]

(\text{C.2})

For thin wire antenna, we found \( |\Gamma^2| > 1 \) for a typical frequency \( \omega \). Now, since the phase term \( \exp(i\frac{\omega}{c}|z|) \) is the only rapidly varying term, we have, after differentiation with respect to \( z \), the following approximate expression:
\[
\frac{\partial I_\infty(\omega; z)}{\partial z} = i \frac{\omega}{c} I_\infty(\omega; z)[H(z) - H(-z)] - \frac{\omega i |z|}{c} \cdot \frac{\lambda'(i2\omega |z|)}{\Gamma^2(\omega)} \left[ 1 - \frac{2\lambda}{\Gamma} + \cdots \right]
\]

\[
= i \frac{\omega}{c} I_\infty(\omega; z)[H(z) = H(-z)] + O\left(\frac{1}{|\Gamma|^2}\right).
\]

(C.3)

In (C.3), \(H(z)\) is the Heaviside unit step-function. By differentiating (C.3) with respect to \(z\) and neglecting again terms of the order \(\Gamma^{-2}\) or higher, we get

\[
\frac{\partial^2}{\partial z^2} I_\infty(z) = -\left(\frac{\omega}{c}\right)^2 I_\infty(\omega; z) + i2 \frac{\omega}{c} I_\infty(\omega; 0) \delta(z)
\]

or

\[
\left[ \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \right] I_\infty(\omega; z) = i2 \frac{\omega}{c} I_\infty(\omega; 0) \delta(z)
\]

(C.4)

This is the approximate "differential equation" for the current on an infinite antenna. However, (C.4) can easily be extended to the total current on a finite length antenna. In fact, by using equation (C.4) and (14), the corresponding approximate "differential equation" for the current on a finite thin wire antenna is

\[
\left[ \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \right] \tilde{I}_0(\omega; z, z') = i2 \frac{\omega}{c} I_\infty(\omega; 0) \tilde{V}_g(\omega) \delta(z - z')
\]

(C.5)

where the delta-function voltage generator \(\tilde{V}_g\) is located at \(z = z'\).

Our next step is to use this "differential equation" for determining the amplitude \(\tilde{I}_0\) in the product expression for the antenna current as given in (11). Substitution of (11) into (C.5) yields

\[
\tilde{I}_0(\omega) \left[ \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \right] g(\omega; z_\rightarrow) g(\omega; L-z_\leftarrow) = i2 \frac{\omega}{c} I_\infty(\omega; 0) \tilde{V}_g(\omega) \delta(z - z')
\]

(C.6)
But since
\[
\frac{\partial}{\partial z}[g(\omega;z')g(\omega,L-z')]= \begin{cases} 
g(\omega;L-z')\frac{\partial}{\partial z}g(\omega;z) & z > z' 
g(\omega,z')\frac{\partial}{\partial z}g(\omega;L-z) & z < z'
\end{cases}
\]
by integrating (C.6) with respect to \( z \) from \( z' - 0 \) to \( z' + 0 \), we obtain
\[
\tilde{I}_0(\omega)W(\omega;z') = i \frac{2}{\omega} \frac{\Delta(\omega)}{C} \tilde{I}_\infty(\omega;0)\tilde{V}_\omega(\omega)
\]  
(C.7)

where the "Wronskian" \( W \) is given by
\[
W(\omega;z') = g(\omega;L-z')\frac{\partial}{\partial z'}g(\omega;z') - g(\omega;z')\frac{\partial}{\partial z'}g(\omega;L-z') - \Delta(\omega)[I_\infty(\omega;L-z')\frac{\partial}{\partial z'}I_\infty(\omega;z') - I_\infty(\omega;z')\frac{\partial}{\partial z'}I_\infty(\omega;L-z')]
\]  
(C.8)

Because the right-hand side of (C.7) was derived by neglecting \( O(\frac{1}{|\Gamma|^2}) \) terms, the left-hand side must also be approximated by the same order.

It is not difficult to show that
\[
W(\omega,z') = i 2\frac{\Delta(\omega)}{C}I_\infty(\omega;L)\left[I_\infty(\omega;0) + O(\frac{1}{|\Gamma|^2})\right].
\]  
(C.9)

Hence, from (C.7) and (C.9) we finally conclude that
\[
\tilde{I}_0(\omega) = \frac{1}{\Delta(\omega)I_\infty(\omega;L)}
\]  
(C.10)

The advantage of using the approximate "differential equation" would be appreciated in determining the corresponding constant \( \tilde{I}_0(\omega) \) of a more complicated problem involving multiple sources and/or multiple loads, for which sometimes the factorization method of Section 3.1 (i.e., steps from (14) to (17)) is cumbersome. The corresponding "differential equation" for such problems can be readily written from (C.5) as
\[
\left[ \frac{\partial^2}{\partial z^2} + \left( \frac{\omega}{c} \right)^2 \right] \tilde{\zeta}^{(0)}(\omega;z,z') = i2\left( \frac{\omega}{c} \right) I(\omega;0) [\tilde{V}_1(\omega) \delta(z-z_1) + \tilde{V}_2(\omega) \delta(z-z_2) \\
+ \ldots \tilde{V}_n(\omega) \delta(z-z_n)]
\]  \tag{C.11}
\]

where \( \tilde{V}_n \) is the voltage (voltage-drop) due to the \( n \)th generator (load impedance \( Z_n(\omega) \)), and \( \tilde{\zeta}^{(0)} \) is the corresponding approximate expression for the current induced on the antenna.
APPENDIX D

Transient Response Formulation of a Lossy Transmission Line

In this Appendix, we briefly review the transient analysis of finite length lossy transmission lines by two different methods. In the first case a SEM method, which employs the residue theorem for natural frequencies of the line, is applied; and in the second case, by employing branch cut contribution due to the current on the corresponding infinitely long line, a Series Method is used to establish multiple reflections on the finite transmission line. Our purpose is to show that the appearance of branch cut singularity in the frequency-domain current expression of an infinitely long transmission line will produce no net effect on a finite line.

Consider the lossy transmission line of Figure D-1, which is driven by a delta-function voltage generator, \( V_g(t) = V_0 H(t-t') \), located at \( z = z' \). Defining the Fourier transform as

\[
\tilde{I}(\omega; z, z') = \int_{-\infty}^{\infty} I(t, t'; z, z') e^{i\omega(t-t')} \, dt
\]

(D.1)

the so-called telegraphist's equations, can then be written as [25]

\[
\begin{cases}
\frac{d\tilde{V}}{dz} = (i\omega \varepsilon_0 - g_0)\tilde{V} + \frac{iV_0}{\omega} \delta(z-z') \\
\frac{d\tilde{V}}{dz} = (i\omega \varepsilon_0 - r_0)\tilde{I} + \frac{iV_0}{\omega} \delta(z-z')
\end{cases}
\]

\[
\begin{align*}
\tilde{V}(\omega; 0, z') &= Z_\varepsilon I(\omega; 0, z') \\
\tilde{V}(\omega; 2h, z') &= Z_\varepsilon I(\omega; 2h, z')
\end{align*}
\]

Substitution for \( \tilde{V} \), from the second equation in the first one, yields the following wave equation for the current
Fig. D-1 A lossy transmission line of finite length $2h$.

Fig. D-2 The complex $\omega$-plane for a lossy infinitely-long transmission line.
\[
\left( -\frac{d^2}{dz^2} - \kappa^2 \right) \tilde{I} = - c_o V_o \left( \frac{(\omega + i\xi_o)}{\omega} \right) \delta (z-z') \quad (D.3)
\]

where
\[
K(\omega) = \frac{i}{c} b(\omega) \; ; \quad b(\omega) = [(\omega + i\xi_o)(\omega + i\sigma_o)]^{1/2}, \quad \text{Re}[b(\omega)] \geq 0
\]

wherein
\[
\xi_o = \frac{g_o}{c_o}, \quad \sigma_o = \frac{r_o}{\xi_o}, \quad c = \frac{1}{\sqrt{\xi_o c_0}}
\]

and \(c_o, \xi_o, g_o\) and \(r_o\) are the capacitance, inductance, conductance and resistance per unit length of the line, respectively. Solution of (D.3), subject to the boundary conditions given in (D.2), may be written as
\[
\tilde{I}(\omega; z, z') = I_o(\omega) \left[ e^{\frac{K(2h-z_z)}{2\kappa h}} - \Gamma_{\kappa}(\omega) e^{2\kappa h} \right] \left( e^{\frac{Kz_z}{2\kappa h}} - \Gamma_{\kappa}(\omega) e^{-2\kappa h} \right)
\]

where \(z_z \equiv \max (z, z')\) and \(\Gamma_{\kappa}\) is the reflection coefficient.

\[
\Gamma_{\kappa}(\omega) = \frac{Z_{\kappa} - Z_{\kappa}(\omega)}{Z_{\kappa} + Z_{\kappa}(\omega)} \; ; \quad Z_{\kappa}(\omega) = R_o \left( \frac{\omega + i\sigma_o}{\omega + i\xi_o} \right)^{1/2}, \quad R_o = \sqrt{\frac{r_o c_0}{c_o}}.
\]

We now insert (D.4) into (D.3) and integrate equation (D.3) with respect to \(z\), from \(z' - 0\) to \(z' + 0\), to obtain
\[
I_o(\omega) = \left( \frac{IV_o}{2\omega Z_o(\omega)} \right) \left[ e^{-2\kappa h} \right] \left[ 1 - \Gamma_{\kappa}(\omega) e^{4\kappa h} \right] \quad (D.5)
\]

We now notice that \(b(\omega)\) and consequently \(K(\omega), Z_o(\omega)\) and \(\Gamma_{\kappa}(\omega)\) have a finite branch cut in the complex \(\omega\)-plane which is defined by \(\text{Re}[b(\omega)] > 0\) (since we require that \(I_m(K) \geq 0\)) and shown in Figure D-2 for \(\sigma_o > \xi_o\). However, this singularity of \(b(\omega)\) does not introduce any branch cut singularity in \(\tilde{I}\), given by (D.4). In fact, one can see that as
\[ b(\omega) \rightarrow -b(\omega) \Rightarrow (K \rightarrow K, \ Z_0 \rightarrow -Z_0 \ \text{and} \ \Gamma_2 \rightarrow \frac{1}{\Gamma_2}) \Rightarrow \tilde{I} \rightarrow -\tilde{I}, \ i.e., \ \tilde{I} \]
is an even function of \( b(\omega) \). Therefore, \( \tilde{I}(\omega;z,z') \) for a finite transmission line is a meromorphic function of \( \omega \), and has only isolated singularities at the poles \( \omega_s^+ \) and \( \omega_s^- \), which are the solution of

\[ 1 - \Gamma_2(\omega_s) e^{4K(\omega_s)h} = 0, \quad s = 1, 2, 3, \ldots \]

We now proceed to find the transient response for the special case when \( \Gamma_2 = 1 \). In this case

\[ I(t,t';z,z') = \frac{1}{2\pi} \int_{\Gamma_0} \tilde{I}(\omega;z,z') e^{-i\omega(t-t')} d\omega \quad (D.6) \]

where

\[ \tilde{I}(\omega;z,z') = \left( \frac{iv_0}{\omega Z_0(\omega)} \right) \frac{\sinh(Kz_s)\sinh[K(2h-z_s)]}{\sinh(2hK)} \quad (D.7) \]

(I) - SEM:

For \( t - t' \geq \frac{Z_{>}-Z_{<}}{c} \), we deform the contour in the lower half of the \( \omega \)-plane (Figure 4). In this process, residues at the poles of \( \tilde{I}(\omega;z,z') \) are captured; then

\[ I(t,t';z,z') = \frac{v_0 c}{h} \left\{ \sum_{s=1}^{\infty} \frac{b(\omega_s)}{\omega_s Z(\omega_s)[i\omega_s - \frac{\xi_{>0} + \xi_{>0}}{2}]} \cdot \frac{\sinh(K_s z_{>})\sinh[K_s(2h-z_{<})]}{\cosh(2K_s h)} \right. \]

\[ \left. -i\omega_s e^{i\omega_s(t-t')} \right\} e^{H(t-t' - \frac{Z_{>}-Z_{<}}{c})} \quad (D.8) \]

where \( K_s = K(\omega_s) = \frac{i}{c} b(\omega_s) \).
(II) - Series Method:

Taylor series expansion of the denominator in (D.5) with \( \Gamma_\ell = 1 \), gives

\[
1 - e^{-4K_\ell h} = 1 + e^{4K_\ell h} + e^{8K_\ell h} + \ldots .
\]

From (D.6), we then have

\[
I(t,t',z,z') = \frac{iV}{4\pi} \int_{\Gamma_o} \frac{1}{\omega Z_\omega(\omega)} \left[ e^{KL_0} - e^{KL_1} - e^{KL_2} + e^{KL_3} \right] \left[ 1 + e^{4K_\ell h} + e^{8K_\ell h} + \ldots \right] e^{-i\omega(t-t')} d\omega \tag{D.9}
\]

where

\[
L_0 = z_+ - z_<, \quad L_1 = z_+ - z_+; \quad L_2 = (2h - z_+) + (2h - z_+)\quad \text{and} \quad L_3 = (2h - z_+) + (2h + z_<). \]

Each term in (D.9) of course corresponds to the frequency-domain solution of the current on an infinitely long transmission line and indeed possesses branch cut singularities associated with \( b(\omega) \). If we now integrate individual reflection currents separately, we have by deforming the contour \( \Gamma_o \) in the lower half of the complex \( \omega \)-plane and around the finite branch cut (Figure D-2), a typical term which behaves like:

\[
\int_{\Gamma_o} \frac{1}{\omega Z_\omega(\omega)} e^{K(\omega)x} e^{-i\omega(t-t')} d\omega = H(t-t', x) \int_{\Gamma_o} \left[ e^{K(-ip)x} - e^{K(-ip)x} \right] e^{-p(t-t')} dp
\]

\[
= H(t-t', x) F(t, t'; x) \tag{D.10}
\]

where the subscript "0" and "\pi" denote the argument of \( b(\omega) \). In (D.10), \( F(t, t', x) \) is the (normalized) current, at \( z - z' = x \), on an infinitely long transmission line. In terms of \( F \), (D.9) can now be written as

\[
I(t, t', z, z') = \frac{iV}{4\pi} \sum_{m=0}^{\infty} \left[ H_0 F(t, t'; L_0 + 4mh) - H_1 F(t, t'; L_1 + 4mh) \right.
\]

\[
- H_2 F(t, t'; L_2 + 4mh) + H_3 F(t, t'; L_3 + 4mh) \]  \tag{D.11}
where \( H_n = H(t - t' - \frac{L_n + 4 mh}{c}) \), \( n = 0, 1, 2, 3 \), is the unit step-function.

One may note that for \( \frac{L_0}{c} \leq (t - t') \leq \min(\frac{L_1}{c}, \frac{L_2}{c}) \), total current, \( I \), is completely given by \( F(t, t'; L_0) \) which is the response of an infinitely long transmission line. It should also be noted that the integral in (D.10) can be evaluated exactly in terms of the modified Bessel function, for \( g_0 = 0 \) (i.e., \( \xi_0 = 0 \)) [26].

Solutions (D.8) and (D.11), obtained by two different methods must be equivalent, since both are the solutions of the same differential equation. However, (D.8) is given in terms of the residues' contribution at the poles of the meromorphic function \( \tilde{I}(\omega; z, z') \), whereas (D.11) is given entirely in terms of the branch-cut contribution of the current \( \tilde{F}(\omega; z, z') \) on a lossy infinitely long transmission line. For early times, first term in (D.11) gives the total current, while in (D.8) many terms are needed for an accurate result. For late times however, the summation in (D.8) converges very fast and is more efficient than that in (D.11).

Finally, it is worthy to note that the series method could also be used for the transient response of thin cylindrical antennas. For this latter problem however, the process is tedious and the final result is cumbersome. Therefore for the antenna problems, one should particularly appreciate the simple SEM representations of (21) and (23).
APPENDIX E

Late-Time Response of an Infinite Cylindrical Antenna

In this Appendix, we derive an expression for the late-time transient current induced on an infinite thin-cylindrical antenna by a delta-function voltage generator.

Consider an infinite antenna of radius $a$ extended from $-\infty$ to $+\infty$ in the $z$-direction. For the delta-function generator excitation:

$V_g(t) = V_o H(t-t')$ located at $z = 0$, the transient current response may be written as

$$I_{\text{inf}}(t,t';z) = \frac{1}{2\pi} \int_{\Gamma_0} \tilde{V}_g(\omega)I_\infty(\omega,z)e^{-i\omega t}d\omega$$

$$= \frac{iV_o}{\eta_0} \int_{\Gamma_0} e^{-i\omega(t-t'-|z|/c)} \frac{d\omega}{\omega} U(\omega;z)$$

(E.1)

where $U(\omega;z)$ is defined by (A.2) and the contour $\Gamma_0$ is shown in Figure E-1. In (E.1), the expression (A.5) for $I_\infty(\omega,z)$ has been used.

In order to evaluate the integral in (E.1), we now deform the contour of integration $\Gamma_0$, in the lower half of the complex $\omega$-plane and around the branch cut, which is due to the logarithmic singularity in $U(\omega,z)$. Deformation is performed only when $t-t' > \frac{|z|}{c}$ to yield [with a change in variable to $\alpha = i(\beta e^Y)\omega$],

$$I_{\text{inf}}(t,t';z) = \frac{iV_o}{2\pi\eta_0} H(t-t'-\frac{|z|}{c}) \int_0^\infty f(\alpha)e^{-\alpha T_0} \frac{d\alpha}{\alpha}$$

(E.2)

where the contribution from the pole at $\omega = 0$ has no consequence;

$T_0 = e^{-\gamma c} \left( t-t'-\frac{|z|}{c} \right)$ and
Fig. E-1 Deformation of contour in the complex $\omega$-plane for an infinite cylindrical antenna
\[ f(\alpha) = \left[ -\ln(\alpha) - i\pi + \frac{\lambda_+}{2} \right]^{-1} + \left[ -\ln(\alpha) + i\pi + \frac{\lambda_-}{2} \right]^{-1} \]  \hspace{1cm} (E.3)

where from (A.3)

\[ \lambda_\pm = \lambda(2\alpha \frac{|z|}{c} e^{\pm i\pi}) \hspace{0.5cm} \lambda(x) = \ln(x) + e^{x}E_1(x) + \gamma \]  \hspace{1cm} (E.4)

For \( \alpha \) close to zero, the integrand in (E.2) behaves like \( \frac{1}{\alpha(\ln^2\alpha + \pi^2)} \), and is infinite at \( \alpha = 0 \). For the purpose of numerical computation it is then advantageous to introduce a new change of variable, \( \alpha = \exp[\pi\tan(\pi y)] \); subsequently we have

\[ I_{inf}(t, t'; z) = \left( \frac{iV\pi^2}{2n_0} \right)H(t-t' - \frac{|z|}{c}) \int_{-\frac{i\pi y}{2}}^{\frac{i\pi y}{2}} f(e^{\pi\tan(\pi y)}\exp[-T_0 e^{\pi\tan(\pi y)}][1+\tan^2(\pi y)])\, dy \]  \hspace{1cm} (E.5)

The integral has now a finite integrand and also a finite range of integration.

To evaluate the integral in (E.2) asymptotically for large \( T_0 \), we first break the integral into two parts:

\[ \int_0^\infty f(\alpha)e^{-\alpha T_0} \frac{d\alpha}{\alpha} = -i2\pi \int_0^{\infty} \frac{e^{-\alpha T_0}}{\ln^2(\alpha) + \pi^2} \cdot \frac{d\alpha}{\alpha} + \int_0^{\infty} \left[ f(\alpha) + \frac{12\pi}{\ln^2(\alpha) + \pi^2} e^{-\alpha T_0} \right] \frac{d\alpha}{\alpha} \]  \hspace{1cm} (E.6)

We should first note that for \( z = 0 \): \( \lambda_\pm = 0 \) and the second integral on the right side of (E.6) vanishes. Furthermore for \( T_0 \gtrsim 1 \), the main contribution to (E.6), for \( z \) not too large, clearly comes from the first integral on the right side, which we now proceed to evaluate asymptotically. Since for \( T_0 \gtrsim 1 \), the main contribution to the integral comes from \( \alpha \) close to zero, thus

\[ \int_0^{\infty} \frac{e^{-\alpha T_0}}{\ln^2(\alpha) + \pi^2} \frac{d\alpha}{\alpha} \sim \int_0^{\infty} \frac{e^{-\alpha T_0}}{\alpha \ln^2(\alpha)} \frac{d\alpha}{\alpha} = -T_0 \int_0^{\infty} \frac{e^{-\alpha T_0}}{\alpha \ln \alpha} \frac{d\alpha}{\alpha} \]  \hspace{1cm} (E.7)
Now by a new change of variable, \( \beta = \alpha T_0 \), we have
\[
\int_0^\infty f(\alpha) e^{-\alpha T_0} \frac{d\alpha}{\alpha} \sim \frac{i2\pi}{\ln \alpha} \int_0^\infty \frac{e^{-\alpha T_0}}{-\ln \alpha} d\alpha = \frac{i2\pi}{\ln T_0} \int_0^\infty \left(1 - \frac{\ln \beta/\ln T_0}{\ln T_0}\right)^{-1} e^{-\beta} d\beta \quad (E.8)
\]

Keeping only the first two terms in the expansion of \( (1 - \ln \beta/\ln T_0)^{-1} \) for \( T_0 >> 1 \), then yields
\[
\int_0^\infty f(\alpha) e^{-\alpha T_0} \frac{d\alpha}{\alpha} \sim \frac{i2\pi}{\ln T_0} \int_0^\infty \left(1 + \frac{\ln \beta}{\ln T_0}\right) e^{-\beta} d\beta = \frac{i2\pi}{\ln T_0} (1 - \frac{\gamma}{\ln T_0}) \quad (E.9)
\]

where in obtaining (E.9), the identity
\[
\int_0^\infty e^{-\beta} \ln \beta d\beta = -\gamma \quad ; \quad \gamma = 0.57721 \ldots
\]
has been used [28]. Finally, by inserting (E.9) in (E.2), we conclude that
\[
I_{\text{inf}}(t,t';z) \sim (\frac{\pi V}{\eta_0}) \frac{1}{\ln T_0} (1 - \frac{\gamma}{\ln T_0}) \quad , \quad \text{for} \quad T_0 >> 1 \quad (E.10)
\]

In Figure E-2, the driving point current (i.e., \( z = 0 \)) as obtained from (E-10) is compared with the "exact" numerical results of Einarsson [3]. As expected for "late times" (\( \frac{ct}{a} > 100 \)) the agreement is excellent, while for "early times" (\( \frac{ct}{a} \leq 100 \)), the agreement is very good for \( 6 < \frac{ct}{a} \leq 100 \). However for \( \frac{ct}{a} \leq 6 \), which (for a typical wire radius, say \( a = 1 \) cm) corresponds to very early times, the formula in (E.10) fails to predict the correct behavior of the current response.

In Figure 5 of Section 3, the expression (E.10) has been used to plot the transient response of an infinite-antenna as a function of \( \frac{ct}{L} \) and for \( \frac{ct}{L} > 0.05 \). It should be noted that for the antenna parameter, \( \Omega = 10 \), used in that Figure, \( \frac{ct}{L} > 0.05 \) corresponds to the range, \( \frac{ct}{a} > 7.4 \), where the expression in (E.10) can be accurately used.
Fig. E-2 Time-domain response of the driving-point current on an infinite cylindrical antenna, center-driven by a delta-function generator with output voltage $V_0 H(t)$; $V = 1$ V, $H$.
Appendix F

Approximate Expression for $W_s(t;z,\theta)$

Expression (25) for $W_s(t;z,\theta)$ may be rewritten as

$$W_s(t;z,\theta) = H(t - \frac{Z}{c} \cos \theta) \left[\alpha_s(z_1, z_2; \theta) \pm e^{i\frac{\omega_s}{c} L \cos \theta} \alpha_s(L - z_2, L - L_1; \pi - \theta)\right]$$  \hspace{1cm} (F.1)

where $+$ or $-$ sign is chosen when $s$ is odd or even, respectively and

$$\alpha_s(z_1, z_2; \theta) = \int_{z_1}^{z_2} I_\infty(\omega_s; z') e^{i \frac{\omega_s}{c} z' \cos \theta} dz'$$  \hspace{1cm} (F.2)

To evaluate (F.2) analytically, we first split the integral such that

$$\alpha_s(z_1, z_2; \theta) = \alpha_s(z_1, \infty; \theta) - \alpha_s(z_2, \infty; \theta)$$  \hspace{1cm} (F.3)

where

$$\alpha_s(z, \infty; \theta) = \int_z^\infty I_\infty(\omega_s; z') e^{i \frac{\omega_s}{c} z' \cos \theta} dz' ; \quad 0 \leq z \leq L$$

$$= \frac{2\pi}{\eta_0} \int_{\Gamma(\omega_s) + \gamma + \ln(-i2\frac{\omega_s}{c} z')}^\infty \frac{\exp[i \frac{\omega_s}{c} (1 + \cos \theta) z']}{\Gamma(\omega_s) + \gamma + \ln(-i2\frac{\omega_s}{c} z')} dz'$$  \hspace{1cm} (F.4)

where the expression (A.5) (in Appendix A) for $I_\infty$ has been used, and $\Gamma$ is given by (A.4).

In [27], a method for approximate evaluation of the integral

$$\int_z^\infty \frac{\exp[i \frac{\omega}{c}(1 + \cos \theta) z']}{\Gamma(\omega) + \gamma + \ln(-i2\frac{\omega}{c} z')} dz' ; \quad (\frac{\omega}{c} z \text{ not too small})$$
for $z$ not overly large is presented. To evaluate the integral in (F.4) one may now employ that method. Details of the evaluation, which follows the steps given in [27], are not presented here; the final result however may be stated as

$$\alpha_s(z, \omega; \theta) = \left( \frac{i2\pi}{\eta_0} \right) \frac{iv_s}{\mu_s(1 + \cos \theta)} \frac{1}{Q_0(\theta) + \lambda \ln(-iv_s) + e^{-iv_s} E_1(-iv_s) + P(\theta)}$$

(F.5)

where

$$v_s = \left( \frac{\omega}{c} \right) (1 + \cos \theta), \quad Q_0(\theta) = \Gamma(\omega_s) + \gamma - 2\lambda \ln(\cos \theta/2)$$

and

$$P(\theta) = \cot^2 \left( \frac{\theta}{2} \right) \left[ e^{-iv_s} E_1(-iv_s) - e^{-i\frac{\omega}{c} z} E_1(-i2\frac{\omega}{c} z) \right]$$

(F.6)

To obtain the correct behavior of $P(\theta)$ for the grazing incidence as $\theta \to 0$ we note that for $\theta \sim 0$

$$e^{-i(v_s-2\frac{\omega}{c} z)} + i2\frac{\omega}{c} z \sin^2 \left( \frac{\theta}{2} \right) \sim 1 + i2\frac{\omega}{c} z \sin^2 \left( \frac{\theta}{2} \right).$$

Then as $\theta \to 0$

$$P(0) = e^{-i2\frac{\omega}{c} z} \left[ (1 + i2\frac{\omega}{c} z) E_1(-i2\frac{\omega}{c} z) + \lim_{\theta \to 0} \frac{E_1(-iv_s) - E_1(-i2\frac{\omega}{c} z)}{\sin^2 \left( \frac{\theta}{2} \right)} \right]$$

(F.7)

But by using the series expansion for the exponential integral $E_1(x)$ [24], one could show that

$$\lim_{a_0 \to 1} \left[ E_1(a_0 x) - E(x) \right] = e^{-x}$$

Hence, (F.7) becomes

$$P(0) = 1 + (i2\frac{\omega}{c} z)e^{-i2\frac{\omega}{c} z} E_1(-i2\frac{\omega}{c} z)$$

(F.8)
The coupling coefficient (27) can now be expressed as

$$\beta_s(\theta) = \alpha_s(0, L; \theta) \pm e^{i \omega_s \frac{L \cos \theta}{c}} \alpha_s(0, L; \pi - \theta)$$

$$= [\alpha_s(0, 0; \theta) - \alpha_s(L, \infty; \theta)] \pm e^{i \omega_s \frac{L \cos \theta}{c}} [\alpha_s(0, \infty; \pi - \theta) - \alpha_s(L, \infty; \pi - \theta)]$$

\[\text{(F.9)}\]

where + or - sign is chosen when \( s \) is odd or even, respectively; and \( \alpha_s(z, \infty, \theta) \) is given in (F.5).

To check the accuracy of the approximate result in (F.5), the real and imaginary parts of \( \alpha_s(0, L; \theta) \) obtained by using (F.3) and (F.5) were compared with the numerical integration of (F.2), as \( \theta \) varies between 5° and 90°. The results which are presented in Table F.1 are calculated for the first natural frequency of an antenna with \( \frac{2a}{L} = 10^{-2} \). As can be seen, the agreement between the approximate evaluation and the numerical integration of the integral in (F.2) is excellent.
TABLE F.1

COMPARISON OF THE APPROXIMATE EVALUATION OF $\alpha_s(0,L;\theta)$, USING Eqs. (F.3) AND (F.5), WITH THE NUMERICAL INTEGRATION OF (F.2); $\frac{2a}{L}=10^{-2}$, AND THE RESULTS ARE COMPUTED FOR THE FIRST NATURAL FREQUENCY, $\frac{\omega_1L}{c} = 0.924 - 0.078$.

<table>
<thead>
<tr>
<th>$\theta$ (Degree)</th>
<th>Approximate Evaluation of $\alpha_s(0,L;\theta)$</th>
<th>Numerical Integration of (F.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$-1.74464 \times 10^{-4}$ - $17.92195 \times 10^{-5}$</td>
<td>$-1.73662 \times 10^{-4}$ - $17.88341 \times 10^{-5}$</td>
</tr>
<tr>
<td>15</td>
<td>$-2.08466 \times 10^{-4}$ - $15.24884 \times 10^{-5}$</td>
<td>$-2.07638 \times 10^{-4}$ - $15.0998 \times 10^{-5}$</td>
</tr>
<tr>
<td>30</td>
<td>$-3.06556 \times 10^{-4}$ + $15.82485 \times 10^{-5}$</td>
<td>$-3.05644 \times 10^{-4}$ + $15.86488 \times 10^{-5}$</td>
</tr>
<tr>
<td>45</td>
<td>$-4.00451 \times 10^{-4}$ + $12.93472 \times 10^{-4}$</td>
<td>$-3.99411 \times 10^{-4}$ + $12.93903 \times 10^{-4}$</td>
</tr>
<tr>
<td>60</td>
<td>$-3.70258 \times 10^{-4}$ + $16.58222 \times 10^{-4}$</td>
<td>$-3.69052 \times 10^{-4}$ + $16.58756 \times 10^{-4}$</td>
</tr>
<tr>
<td>75</td>
<td>$-9.44499 \times 10^{-5}$ + $11.04968 \times 10^{-3}$</td>
<td>$-9.30052 \times 10^{-5}$ + $11.00505 \times 10^{-3}$</td>
</tr>
<tr>
<td>90</td>
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<td>$+4.34265 \times 10^{-4}$ + $11.27383 \times 10^{-3}$</td>
</tr>
</tbody>
</table>