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IMAGING AND COUPLING OF TWO PARALLEL SLAB WAVEGUIDES
by
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PARALLEL SLAB WAVEGUIDES

by

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Imaging And Coupling Of Two Parallel Slab Waveguides.
Thesis directed by Assistant Professor Edward F. Kuester

The object of this thesis was to study the imaging and coupling properties of two adjacent placed multimode dielectric slab waveguides. Under the condition of degenerate coupling (only the modes in two guides with close propagation constants couple strongly), we can treat the input field as a combination of symmetrical system modes and antisymmetrical system modes which lead to a quite simple mathematical modal to study the imaging and coupling properties of those two slab waveguides.

A very useful parameter, called $Z_c$, the total power coupling distance which is a function of the guides separation distance was introduced (at which the power couples totally from one guide to the other). This parameter could be used as a guideline for the design of waveguide switches and directional couplers in integrated optical circuits.

This abstract is approved as to form and content.

Signed
Faculty Member in Charge of Thesis
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CHAPTER I

INTRODUCTION

The coupling phenomena between two dielectric optical waveguides have been both observed and discussed extensively during the last 10 years. Many important optical devices, such as the directional coupler [1], [2], the light switch [3], [4], the modulator [5] and branching waveguides [6] are based on this principle. For example, in the directional-coupler switch case [3], this device consists of two parallel metal-gap optical stripline waveguides forming a passive directional coupler with an electro-optic pad at the edge of each guide. Initially light is focused onto one of the waveguides and the amount of light coupled into the adjacent channel can be controlled electro-optically. This scheme not only permits direct amplitude modulation of the light propagating in one channel but allows light to be switched from one channel to the other. A schematic diagram of the electro-optic directional-coupler (EDC) switch is shown in Fig. 1.

The theoretical treatments of this topic have long been discussed by many authors. However; in general, we can group them into three categories; namely, the finite element method [7], coupled mode theory [8] and the geometrical optics (G.O.) method [9]. Both the finite element method and coupled mode theory work quite well in predicting the coupling between single mode
Figure 1. Schematic diagram of the ECD switch. Since the effective refractive index in the region underneath the metal cladding is lower than the unclad region, light is confined to epitaxial layer underneath the gap in the metal. By reverse biasing the Schottky barrier contact at the side of one of the waveguides, light can be switched from one channel to another (after Reference 3).

waveguides or guides with a small number of propagating modes. But when used for multimode couplers, the computations become very lengthy and costly in terms of computer time. In the G.O. cases, one approximates the large number of discrete propagating modes as a "continuous spectrum" of modes. This "continuous spectrum" is then treated as a cone of "rays." Intuitively, G.O. is the right candidate for multimode coupling computations, however, as the propagating length becomes greater, more and more rays might come in and contribute, eventually this method suffers the same kind of drawbacks as the first two.

Recently a hybrid method for computing the beam propagating in multimode dielectric waveguide was developed by Chang and Kuester
Based upon an assumption that the bulk of optical power is carried by low-order modes where the corresponding "rays" are paraxial, a simple theory for computing beam propagation in a multimode dielectric waveguide was developed. Based on this method, a simple way to calculate the coupling between two identical parallel slab waveguides will be constructed in this thesis. The propagation on the slab is described in terms of a combination of system modes; that is, symmetrical modes and anti-symmetrical modes. Since these two groups of modes possess different propagation constants, (hence different imaging properties), eventually power couples back and forth between the two guides. In this thesis we will discuss how the power couples between these two guides and their imaging properties as well. A special case of Gaussian beam coupling will be studied in detail. We will show that in the weakly coupling case (waveguides separation is sufficient largely), total power transfer is possible.
CHAPTER II
GREEN'S FUNCTION AND IMAGING

A parallel metallic waveguide is shown in Fig. 2. The Green's function of this structure is a well-known result:

\[ G(x,x';z) = \frac{2}{a} \sum_{m=1}^{\infty} \sin \frac{m \pi x}{a} \sin \frac{m \pi x'}{a} e^{-i \beta_m z}; \quad z > 0 \quad (2.1) \]

where \( \beta_m = (k^2 - m^2 \pi^2 / a^2)^{1/2} \).

In the paraxial approximation, (that is, when most propagation takes place nearly in the z-direction) we can expand \( \beta_m \) in a binomial expansion, \( \beta_m = k - m^2 \pi^2 / 2ka^2 \), putting this back into (2.1), we have

\[ G_0(x,x';z) = \frac{2}{a} e^{-ikz} \sum_{m=1}^{\infty} \sin \frac{m \pi x}{a} \sin \frac{m \pi x'}{a} e^{izm^2 \pi^2 / 2ka^2}. \quad (2.2) \]

---

Figure 2. A parallel-plate metallic waveguide.
$G_0$ is the Green's function under the paraxial approximation. Using the following relation

$$\sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} = \frac{1}{4} \sum_{m=-\infty}^{\infty} \left[ e^{-im\pi(x-x')/a} - e^{-im\pi(x+x')/a} \right]$$

(2.2) can be rewritten as

$$G_0(x,x';z) = \frac{1}{2a} e^{-ikz} \sum_{m=-\infty}^{\infty} e^{im\pi\frac{2}{2\pi} z/ka^2}$$

$$\left[ e^{-im\pi(x-x')/a} - e^{-im\pi(x+x')/a} \right]. \quad (2.3)$$

Now we define

$$Z_{11} = \frac{4ka^2}{\pi} \quad (2.4)$$

putting this back into (2.3), we have

$$G_0(x,x';z) = \frac{1}{2a} e^{-ikz} \sum_{m=-\infty}^{\infty} e^{-im\pi\frac{2}{2\pi} z/z_{11}}$$

$$\left[ e^{-im\pi(x-x')/a} - e^{-im\pi(x+x')/a} \right]. \quad (2.5)$$

An interesting result of $G_0$ is that $e^{ikz}G_0$ is a periodic function of $z$:

$$G_0(x,x';z+z_{11}) = G_0(x,x';z)e^{ikz_{11}}. \quad (2.6)$$

In particular, (2.1) implies that $G(x,x';0)$ is equal to $\delta(x-x')$ for $0 < (x,x') < a$, we have
From (2.7) we can see that the input field is replicated at every $z = nz_{11}$ plane, $n = 1, 2, 3, \ldots$. These planes are the so-called Fourier or Fresnel images plane which had been discussed extensively by J. M. Cowley and A. F. Moodie [11], Rivlin and Shul'dyaev [12] and some others [13], [14].

It's also interesting to look at the image properties at other values of $z$; so define

$$z_{pq} = \frac{q}{p} z_{11}$$

where $p$ and $q$ are some integers. Now consider the sum

$$Q_{pq}(x) = \sum_{m=-\infty}^{\infty} e^{-im\pi/a+2\pi m^2 z_{pq}/z_{11}}.$$  \hspace{1cm} (2.8)

Let $m = p\ell + r$, where $r$ runs from 0 to $p-1$. Equation (8) now can be expressed as

$$Q_{pq}(x) = \sum_{r=0}^{p-1} e^{-im\pi x/a+2\pi m^2 r/p} \sum_{\ell=-\infty}^{\infty} e^{-im\pi \ell x/a}. \hspace{1cm} (2.9)$$

Using the formula

$$\sum_{m=-\infty}^{\infty} e^{-2\pi m x / \lambda} = \lambda \sum_{m=-\infty}^{\infty} \delta(x-m\lambda), \hspace{1cm} (2.10)$$

equation (9) can be written in the following form
\[ Q_{pq}(x) = \frac{2a}{p} \sum_{r=0}^{p-1} e^{i\pi rx/a} + 2\pi ipr^2/q \sum_{n=-\infty}^{\infty} \delta(x - \frac{2na}{p}). \quad (2.11) \]

Using this relationship, (5) can now be written as:

\[ G_0(x,x',z)_{pq} = e^{-ikz} \sum_{n=-\infty}^{\infty} C_n(p,q)[\delta(x-x') - \frac{2na}{p}] - \delta(x+x' - \frac{2na}{p}) \]

(2.12)

with the coefficients \( C_n \) given by

\[ C_n(p,q) = \frac{1}{p} \sum_{r=0}^{p-1} e^{2\pi i r(q+n)/p}. \quad (2.13) \]

This coefficient possesses many interesting properties. Kuester [24] has done some detailed study on this. Here we only draw some basic conclusions.

When \( p = 1 \), only one of the delta functions, \( \delta(x-x') \) falls within the guide region \( 0 < (x,x') < a \), so we attain a single Fourier image at \( z = qz_{11} \) \((q=1,2,3,...)\), this agrees with what we have said in (7). When \( p = 2 \) \((z=z_{21})\) and the input function \( E_y(x,0) \) is not necessarily symmetric with respect to the center of the guide \( x = a/2 \), we have \( E_y(x,z_{21}) = E_y(a-x,0) \). Figure 3 shows a typical input function and its image. When \( p = 3 \), \( E_y(x,z_{31}) \) can be written as follows:

![Figure 3](image-url)

Figure 3. Image at \( z_{21} \) plane.
Figure 4. Images at $z_{31}$ plane.

\[
E_y(x,z_{31}) = \frac{i}{\sqrt{3}} E_y(x,0) + \frac{e^{i\pi/6}}{\sqrt{3}} \left\{ \begin{array}{ll}
-\frac{2a}{3} - x, 0 \leq x \leq \frac{2a}{3} \\
E_y(x - \frac{2a}{3}, 0), \quad \frac{2a}{3} < x < a \\
E_y(x + \frac{2a}{3}, 0), \quad 0 < x < \frac{a}{3} \\
-\frac{4a}{3} - x, 0 \leq x \leq \frac{a}{3} \end{array} \right. 
\]

(2.14)

So, here we have three images, the amplitude of each of which is reduced by $1/\sqrt{3}$. Figure 4 shows each of the components. When \( p = 4(z\equiv z_{41}) \), there is both an erect and an inverted image

\[
E_y(x,z_{41}) = \left[ e^{i\pi/4} E_y(x,0) + e^{i3\pi/4} E_y(a-x,0) \right]/\sqrt{2} \quad . \quad (2.15)
\]

Figure 5 shows the two components.

Figure 5. Images at $z_{41}$ plane.
When the excitation is symmetrical with respect to the center of the guide, $z_{21}$, $z_{41}$, $z_{81}$ all reproduce the original input function. These results agree with Rivlin and Shul'dyaev's [12]. In their paper, they used the intermode phase difference to find out the imaging distance. Under the paraxial approximation, the phase difference between the first order mode and $m$th order mode can be approximate as,

$$
\phi_{m,1} = (\beta_m - \beta_1)z = -2\pi \frac{nz}{4ka} \frac{m^2}{(m^2 - 1)} = -2\pi \frac{z}{z_{11}} (m^2 - 1)
$$

So at $z = qz_{11}$ ($q = 1, 2, \ldots$), $\phi_{m,1}$ is a multiple of $2\pi$, the cophasal condition is fulfilled. If all the input modes are odd modes, that is $m = 1, 3, 5 \ldots$, the reconstructed images occur as 8 times as often. If all the input modes are even modes, these images occur 4 times as often.

Equation (12) was obtained in the case of a parallel plane metallic waveguide. Now, what about the case of a dielectric slab waveguide: is (12) still valid? or what kind modification do we have to make to make it valid? In order to answer this question, we compare these two cases: (1) In the slab waveguide case, when the incident angle is greater than the critical angle, total reflection occurs at the boundaries, that is to say, the incident power is totally reflected. (2) In the metallic waveguide case, because the transmitting medium is a perfect conductor, the incident power is also totally reflected. As far as the reflected wave concerned, it makes no difference whether the wave is reflected by a perfect conductor or by a dielectric boundary if
Figure 6. Goos–Hänchen effect of the dielectric slab waveguide.

both boundaries render a reflected wave with identical phase.
Experiments by Goos and Hänchen [15] have demonstrated that a beam of light is laterally shifted when totally reflected at a dielectric boundary. It might be easier to illustrate this Goos–Hänchen effect in a graphical manner. Figure 6 states that when treating the problem of a dielectric waveguide, as long as the total reflection abides, we can treat it as if we had a parallel plane waveguide with the fictitious metallic boundaries.

In order to find out this $a_e$; consider the step-index dielectric slab waveguide shown as above. The slab has thickness $a$ and refractive index $n_1$. The cladding index is $n_2$, and both media are assumed to be nonmagnetic. The TE modes for this waveguide have the field distribution [25]
\[
E_y = \begin{cases} 
-ik_0az & x < 0 \\
\frac{k_0}{\sqrt{a^2 - n_2^2}} x e^{-ik_0az} & 0 < x < a \\
-ik_0az & x > a \\
A \sin[k_0 \sqrt{\frac{2}{n_1-a^2}} x + \phi(a)] & 0 < x < a \\
A \sin[k_0 \sqrt{\frac{2}{n_1-a^2}} a + \phi(a)] & x > a
\end{cases}
\] (2.16)

where \(A\) is an arbitrary constant amplitude, \(k_0\) is the wavenumber of free space, \(k_0 \alpha\) is the propagation constant of the mode, and \(\phi(a)\) is the phase shift associated with the Goos–Hänchen effect:

\[
\phi(a) = \sin^{-1} \left[ k_0 a \sqrt{\frac{2}{n_1-a^2}} / \sqrt{\alpha^2 / V} \right]
\] (2.17)

where \(V\) is the so-called normalized frequency:

\[
V = k_0 a \sqrt{\frac{2}{n_1 - n_2^2}}
\] (2.18)

Note that \(V \gg 1\) for a highly multimode guide.

The characteristic equation which determines the eigenvalues \(\alpha\) is obtained by requiring \(H_z\) to be continuous at \(x = a\):

\[
\sin[k_0 \sqrt{\frac{2}{n_1-a^2}} a + 2\phi(a)] = 0
\] (2.19)

The paraxial approximation to these modes (\(\alpha = n_1\)) is found as in [26] by reckoning \(\phi(a)\) to be small. From (2.17), (2.19) we then obtain approximately
\[
\sin[k_0n_1^2 - \alpha^2 a(1+2/V)] = 0
\]

or

\[
\alpha_m = \sqrt{n_1^2 - \frac{m^2 \pi^2}{k_0^2 a^2 (1+2/V)^2}}
\] (2.20)

i.e. the propagation constants for a parallel-plate waveguide of slightly larger width \(a_e = a(1+2/V)\).

We conclude this chapter by pointing out that this method allows us to calculate the field in the guide at any \(z_{pq}\) (any arbitrary \(z\) can be approached as closely as desired by a proper choice of \(p\) and \(q\)). But as \(p\) grows larger and larger, this method becomes no more efficient than the modal approach. However, in some special cases, efficient computation of field at any \(z\) can be carried out using only a relatively small number of images.

Chapter 4 will discuss one such case.
CHAPTER III

COUPLING BETWEEN TWO MULTIMODE PARALLEL SLAB WAVEGUIDES

In this chapter we discuss the coupling between two multimode parallel slab waveguides. We follow the derivation of Chang and Kuester [16]. The coupling between two guides occurs when the fields of a surface-wave mode of one guide penetrate into the other guide. Figure 7 depicts the cross section of two parallel slab guides immersed in a common substrate of refractive index $n_3$. Each of the two guides has a refractive index profile $(n_3^2 + \Delta n_{1,2}^2)$ where $\Delta n_{1,2}^2$ are usually small but may be inhomogeneous in the transverse direction. Without the presence of another guide, we first assume that guide supports only a single surface-wave mode of transverse coordinate dependence given by $\tilde{E}_{1,2}^+$ and $\tilde{H}_{1,2}^+$, and propagation constants $\beta_1$ and $\beta_2$. Notice that the field components considered here are for single modes.

\[
\begin{align*}
\epsilon &= n_3^2 + \Delta n_1^2 \\
\epsilon &= n_3^2 \\
\epsilon &= n_3^2 + \Delta n_2^2
\end{align*}
\]

Figure 7. Cross-section view of two parallel slab waveguides.
When the distance between the two guides is large enough; that is, the evanescent mode field of one guide is small compared with the field of the other guide in its vicinity, the combined system of these two guides now can be considered as a perturbation of two individual isolated guides; $E^+ = m_1 E_t^+ + m_2 E_z^+$, $H^+ = m_1 H_t^+ + m_2 H_z^+$, where $m_1$, $m_2$ still need to be determined. The fields $E^+$, $H^+$ can be separated into a transverse part and longitudinal part,

$$E^+ = E_t^+ + E_z^+ a_z$$

$$H^+ = H_t^+ + H_z^+ a_z$$  \hspace{1cm} (3.1)

Gabriel and Brodwin have obtained a variational formula for the propagation constant $\Gamma$ of a uniform lossy inhomogeneous anisotropic waveguide [17]. When specialized to the isotropic case, it becomes (see also [23])

$$\Gamma = \frac{\omega [\epsilon E^+ \cdot E^- - \mu H^+ \cdot H^-] ds + j [\overline{E_t^+ \cdot \nabla \times H^- + \overline{H_t^+ \cdot \nabla \times E^-}] ds}{\int [E_t^+ \cdot H_t^- - E_z^+ \cdot H_z^-] \cdot a_z ds}$$  \hspace{1cm} (3.3)

Here the integrations ds are over the entire cross section. $E^-$ and $H^-$ are the "transpose fields" of $E^+$ and $H^+$. They correspond to a mode traveling in the $-z$ direction, and they are related to $E^+$ and $H^+$ by

$$E_t^- = E_t^+ , \quad E_z^- = -E_z^+ , \quad H_t^- = -H_t^+ , \quad H_z^- = H_z^+$$

We apply the stationary conditions $\partial \Gamma / \partial m_1 = 0$ and $\partial \Gamma / \partial m_2 = 0$ [22]. After some steps and simplifications, we have
\[ \Gamma = \beta_{av} \pm \Delta \Gamma \] (3.4)

where

\[ \beta_{av} \approx \pm (\beta_1 + \beta_2) \quad \text{and} \quad \Delta \Gamma \approx \sqrt{\Delta^2 + \delta^2} \] (3.5)

\[ \Delta = \frac{1}{2} (\beta_1 - \beta_2) \quad \text{and} \quad \delta^2 = \frac{C_1 C_2}{P_1 P_2} \]

\[ C_{1,2} = \omega e \int_0^2 \Delta n_{1,2} \frac{E^+ E^-}{\varepsilon_1 \varepsilon_2} \, ds \] (3.6)

\[ p_{1,2} = 2 \int [E^+_{t1,2} \times H^+_{t1,2}] \cdot a_z \, ds \]

The surface integrations of \( P \) are over the infinite cross section in the transverse plane; however, the integrations of \( C \) are obviously finite surface integrals.

The ratio \( q = m_2/m_1 \) is also obtainable by applying \( \partial \Gamma/\partial m_1 = 0 \) and \( \partial \Gamma/\partial m_2 = 0 \). The result is

\[ q^2 = \frac{P_1}{P_2} \frac{1}{\delta} \left[ \frac{-\Delta \pm \sqrt{\Delta^2 + \delta^2}}{\delta} \right] \] (3.7)

Equation (3.4) is an interesting result: it says that when the coupling exists between two guides, the system mode propagation constant holds two values; one \( \beta_{av} + \Delta \Gamma \), the other \( \beta_{av} - \Delta \Gamma \). Now the system mode representation can be rewritten as

\[ E_{s\pm} = m_1 (E^+_{1\pm} + q_{1\pm} E^-_{2\pm}) \] (3.8)
Chang and Kuester [16] have shown that the maximum power transfer of the mode in guide 1 coupled into the mode in guide 2 at the location $z_M = (2p+1)\pi/2\Gamma$, $p = 0, 1, 2...$ is

$$|S_{12}| \approx \frac{|\delta_{12}|^2}{|\Delta_{12}^2 + \delta_{12}^2|}.$$  \hspace{1cm} (3.9)

It is obvious that significant power is possible only if $|\Delta_{12}| << |\delta_{12}|$, and total power transfer occurs only if $\Delta_{12} = 0, (\beta_1 = \beta_2)$. This is the well-known degenerate coupling result [8], [19]. Under this condition, $\beta_1 = \beta_2$, we have $\Gamma_\pm = \beta_{av} \pm C/P$, and $q_\pm = \pm 1$. This means that one of the system modes is symmetrical, $E_\pm = E_1^+ + E_2^+$, while the other is antisymmetrical, $E_a^+ = E_1^+ - E_2^+$.

Now we can generalize our outlook to coupling of a finite number, $2N$, of modes, assuming we have two identical guides and each guide has $N$ modes,

$$E^+ = \sum_{i=1}^{2N} m_i E_i^+, \quad H^+ = \sum_{i=1}^{2N} m_i H_i^+.$$ \hspace{1cm} (3.10)

Chang and Kuester [16] have obtained a matrix version of (3.4) for this case:

$$\{[C] - \Gamma[I]\}[m] = 0$$ \hspace{1cm} (3.11)

Here $[C]$ is the coupling matrix, and $[I]$ is the identity matrix. In order to find out the nontrivial solutions for the amplitude vector $[m] = [m_1, m_2, \ldots, m_{2N}]^T$, and system mode eigenvalues $\Gamma$, the determinant in (3.11) has to be equal to zero.
\[ ||C - \tau[I]|| = 0 \]

We write each component in the coupling matrix \([C]\) as \(C_{ij}\), where \(i\) or \(j\) between 1 and \(N\) denotes the \(ith\) and \(jth\) mode in guide 1, whereas \(i\) or \(j\) between \(N + 1\) and \(2N\) denotes the \((i-N)^{th}\) or \((j-N)^{th}\) mode in guide 2. Coupling between two modes on the same guide can be neglected as second-order \([16]\); the remaining matrix elements can be written as

\[
C_{m+N,n}^{1} = \frac{C_{mn}^{1}}{P_{lm}} ; \quad C_{m,n+N}^{2} = \frac{C_{mn}^{2}}{P_{2n}}
\]

where \(C_{mn}^{1,2}\) are given by expressions similar to those for \(C_{1,2}\) in (3.6), and \(P_{lm}, P_{2n}\) denote respectively the integrals \(P_{1}\) and \(P_{2}\) for the \(m^{th}\) or \(n^{th}\) mode as given in (3.6) also.

Ter-Martirosyan [18] has shown that if

\[ C_{ij} \ll |\beta_i - \beta_j| \]  

(3.12)

for all \(i\) and \(j\) such that \(\beta_i \neq \beta_j\), then the system modes are made up only of those modes which are degenerate in the absence of coupling. The problem will then reduce to a set of \(N\) pairs of modes each coupling independently of all the others.

Our interest here is on the coupling between two parallel slab waveguides. Figure 8 shows the structure of these two guides. The widths of these two guides are \(a_1\) and \(a_2\) respectively. The distance between these two guides is \(d\). Guide 1, 2 and the substrate have the permittivities \(\varepsilon_1\), \(\varepsilon_2\) and \(\varepsilon_3\) respectively.
Figure 8. Geometry of two parallel slab waveguides.
For simplicity, we only consider coupling between odd TE modes. This same result applies to coupling between other TE modes also. The fields are:

\[
E_{y sm} = A_{sm} \sin p_{sm} x_{sm} \\
H_{z sm} = \frac{jp_{sm}}{w_{0}} A_{sm} \sin p_{sm} x_{sm} & |x_s| < \frac{a_s}{2} \tag{3.13a}
\]

\[
H_{x sm} = -\frac{\beta_{sm}}{w_{0}} A_{sm} \sin p_{sm} x_{sm}
\]

inside the slabs, where \( p^2_s = k_0^2 \varepsilon_s - \beta^2_s \). The subscript \( s = 1, 2 \) denotes whether they refer to guide 1 or guide 2, while \( m \) indicates the mode number (see Chapter II). Also

\[
E_{y sm} = A_{sm} \sin \frac{p_{sm} a_s}{2} r_{sm} \left( |x_s| - \frac{a_s}{2} \right)
\]

\[
H_{z sm} = \frac{x_s}{|x_s|} - j r_{sm} \frac{p_{sm} a_s}{2} e^{-r_{sm} \left( |x_s| - \frac{a_s}{2} \right)} & |x_s| > \frac{a_s}{2}
\]

\[
H_{x sm} = -\frac{\beta_{sm}}{w_{0}} A_{sm} \sin \frac{p_{sm} a_s}{2} e^{-r_{sm} \left( |x_s| - \frac{a_s}{2} \right)} \tag{3.13b}
\]

outside the guides, where \( r^2_s = \beta^2_s - k_0^2 \varepsilon_s \). \( \beta_{sm} \) in the slab waveguide case is obtained from a well-known characteristic equation [25]:

\[
\tan \frac{p_{sm} a_s}{2} = -\frac{p_{sm}}{r_{sm}}
\]
Using (3.13a), (3.13b) we now can calculate the power carried by each mode $P_{sm}$, as well as the coupling power terms $C_{mn}^{s}$

\[
P_{sm} = -2 \int_{-\infty}^{\infty} E_{ysm} H_{xsm} \text{d}x
\]

\[
= -2 \left( \int_{-a_s/2}^{a_s/2} E_{ysm} H_{xsm} \text{d}x + \int_{-a_s/2}^{a_s/2} E_{ysm} H_{xsm} \text{d}x + \int_{a_s/2}^{\infty} E_{ysm} H_{xsm} \text{d}x \right)
\]

\[
= 2 A^2 \frac{\beta_{sm}}{w_0} \left( \frac{a_s}{2} + \frac{1}{r_{sm}} \right)
\]

(3.14)

\[
C_{mn}^{1} = w_{eo} \int_{-a_1/2}^{a_1/2} (\varepsilon_1 - \varepsilon_3)E_{y1m}E_{y2n} \text{d}x_1
\]

\[
= A_{1m} A_{2n} \frac{p_{lm} p_{2n}}{w_0} \frac{\varepsilon_1 - \varepsilon_3}{\varepsilon_2 - \varepsilon_3} \left[ (r_{1m} - r_{2n} e^{-r_{2n} a_1}) - (r_{1m} + r_{2n}) \right]
\]

(3.15)

\[
C_{mn}^{2} = w_{eo} \int_{-a_2/2}^{a_2/2} (\varepsilon_2 - \varepsilon_3)E_{y1m}E_{y2n} \text{d}x_2
\]

\[
= A_{1m} A_{2n} \frac{p_{lm} p_{2n}}{w_0} \frac{\varepsilon_2 - \varepsilon_3}{\varepsilon_1 - \varepsilon_3} \left[ (r_{2n} - r_{1m} e^{-r_{1m} a_2}) - (r_{1m} + r_{2n}) \right]
\]

(3.16)

Now we can write down the coupling coefficient between the $m$th mode in guide 1 to the $n$th mode in guide 2:
\[ \delta_{mn}^2 = C_{m+n, n}^m, C_{m+n, n}^n = \frac{C_{mn}^1 C_{mn}^2}{P_{1m} P_{2n}} = Q_{mn} \]

\[
= \frac{\left[ (r_{1m} - r_{2n}) e^{-(r_{2m} + r_{1m} + r_{2m})} \right] \left[ (r_{2n} - r_{1m}) e^{-(r_{1m} + r_{2m})} \right]}{[k_0^2 (e_2 - e_3) + \beta_{2n}^2 - \beta_{1m}^2] [k_0^2 (e_2 - e_3) + \beta_{1m}^2 - \beta_{2n}^2]} \]

where

\[
Q_{mn} = \frac{2^2 \frac{-(r_{1m} + r_{2n}) d}{p_{1m} p_{2n} e}}{4 \beta_{1m} \beta_{2n} (\frac{a_1}{2} + \frac{1}{r_{1m}}) (\frac{a_2}{2} + \frac{1}{r_{2n}})} .
\]

In the paraxial approximation [see Eq. (2.20)],

\[
\frac{a_1}{2} + \frac{1}{r_{1m}} = \frac{a_1}{2} \left[ 1 + \frac{2}{V_1} \left( \frac{1}{2} \right) \right] = \frac{a_1}{2} + \frac{1}{V_1} \equiv \frac{a_{e1}}{2}
\]

\[
\frac{a_2}{2} + \frac{1}{r_{2n}} = \frac{a_2}{2} \left[ 1 + \frac{2}{V_2} \left( \frac{1}{2} \right) \right] = \frac{a_2}{2} + \frac{1}{V_2} \equiv \frac{a_{e2}}{2}
\]

where

\[
V_1 = k_o a_1 \sqrt{n_1 - n_3} ; \quad V_2 = k_o a_2 \sqrt{n_2 - n_3}
\]

also

\[
p_{1m} = \frac{m \pi}{a_{e1}} ; \quad p_{2n} = \frac{n \pi}{a_{e2}}
\]

\[
\beta_{1m} \approx k_1 - \frac{m \pi}{2 k_1 a_{e1}^2} ; \quad \beta_{2n} \approx k_2 - \frac{n \pi}{2 k_2 a_{e2}^2}
\]

\[
r_{1m} = \frac{v_1}{a_1} \left[ 1 - \frac{m \pi}{2 V_1^2} \frac{a_1^2}{a_{e1}^2} \right] ; \quad r_{2n} = \frac{v_2}{a_2} \left[ 1 - \frac{n \pi}{2 V_2^2} \frac{a_2^2}{a_{e2}^2} \right]
\]
Putting these back into (3.17) and retaining only the lowest order terms, we have

\[ \delta_{mn}^2 \approx \frac{2^2 e^{a_1}}{a_2} \left( \frac{v_1 + v_2}{a_1 a_2} \right) \]

\[ \frac{\left( r_{1m} - r_{2n} \right)}{r_{1m} + r_{2n}} \left( r_{2n} - r_{1m} \right) \frac{e^{-r_{1m}a_2} - e^{-(r_{1m} + r_{2n})}}{e^{-r_{2n}a_1} - e^{-(r_{1m} + r_{2n})}} \]

\[ \left[ k_o^2 (e_1 - e_3) + \beta_2^2 - \beta_1^2 \right] \left[ k_o^2 (e_2 - e_3) + \beta_2^2 - \beta_2^2 \right] \] (3.19)

Also, from (3.4)

\[ \Delta_{mn} = 1/2 \left( \beta_{mn} - \beta_n \right) = 1/2 (k_1 - k_2) + \frac{n^2 \pi^2}{4k_a^2 e^2} - \frac{m^2 \pi^2}{4k_1^2 a_e^2} \] . (3.20)

In the case of identical guides, \( k_1 = k_2 = k \), \( a_1 = a_2 = a \), \( \varepsilon_1 = \varepsilon_2 = \varepsilon \). Also in the paraxial approximation (3.18), we have

\[ |r_{1m} - r_{2n}| = \frac{\pi^2 a_2}{2V} \left( m^2 - n^2 \right) \ll \frac{2V}{a} = \left| r_{1m} + r_{2n} \right| \] .

Now (3.19) can be reduced to

\[ \delta_{mn} = \frac{2mn \pi^2 e^{-Vd/a}}{k_a^2 V a_e^2} \]

Also (3.20) becomes

\[ |\Delta_{mn}| \approx \frac{\pi^2 n^2 m^2}{4ka_e^2} \]

Comparing \( \delta_{mn} \) and \( |\Delta_{mn}| \), is the same as comparing the following two factors: \( 2mn e^{-Vd/a} a_e/Va_e \) and \( |n^2 - m^2|/4 \).
Recall that in the paraxial approximation, we assume that most power is carried by the lower order modes, so \( m \) and \( |n - m^2| \) won't differ by much. However \( e^{-Vd/a/V} \ll 1/4 \), and the result is that unless \( m = n \) (in which case \( |\Delta_{mn}| = 0 \)), we will have \( |\Delta_{mn}| \gg \delta_{mn} \), which satisfies condition (3.12). The problem becomes that of the coupling of \( N \) pairs of degenerate modes, where each pair has its system mode propagation constants \( \Gamma_{mn} \).

We have in this case \( \beta_{1m} = \beta_{2m} = \beta_m \). Putting (3.18) back into (3.19), we have

\[
\delta_{mn} = \frac{2^2 \pi e^{-Vd/a}}{kVd/a e^2} \cdot \frac{a}{a_e^2}.
\] (3.21)

Having obtained this, we can put this result back into (3.4), thus obtaining the propagation constants \( \Gamma_{m+} \) and \( \Gamma_{m-} \) for symmetrical and antisymmetrical system modes,

\[
\Gamma_{m+} = \beta_m + \delta_{mn} = k - \frac{2 \pi}{2ka_e} \left( 1 - \frac{4 e^{-Vd/a}}{V a_e} \right)
\] (3.22)

\[
\Gamma_{m-} = \beta_m - \delta_{mn} = k - \frac{2 \pi}{2ka_e} \left( 1 + \frac{4 e^{-Vd/a}}{V a_e} \right)
\] (3.23)

Now we redefine the coordinate system for the convenience of later discussion. The new coordinate is shown in Fig. 9.

Suppose that now we have a field distribution \( E_{y0} (x_1, z=0) \) at \( z = 0 \) in guide #1, while a zero distribution is present in guide #2. The symmetrical system modes are excited as if by the initial fields
Figure 9. New coordinate system of two parallel slab waveguides.
\[ \frac{1}{2} E_{y_0}(x_1,0) \text{ in guide 1} \]
\[ \frac{1}{2} E_{y_0}(x_2,0) \text{ in guide 2} \]

while the antisymmetrical system modes are excited by the fields

\[ \frac{1}{2} E_{y_0}(x_1,0) \text{ in guide 1} \]
\[ -\frac{1}{2} E_{y_0}(x_2,0) \text{ in guide 2} \]

Combining the knowledge of (2.2) and (3.22), (3.23), we obtained the Green's functions of the symmetrical and antisymmetrical system modes in guide 1 and guide 2 are,

\[
G_{11}^s(x_1,x';z) = \frac{1}{4a_e} e^{-ikz} \sum_{m=-\infty}^{\infty} \left\{ e^{-\frac{im \pi z}{2}} \left( 1 - \frac{4e^{-Vd/a}}{V} \right) \frac{a}{a_e} \right\} e^{-i \frac{m \pi}{a} (x_1-x')} e^{-i \frac{m \pi}{a} (x_1+x')} \]

(3.24)

\[
G_{11}^a(x_1,x';z) = \frac{1}{4a_e} e^{-ikz} \sum_{m=-\infty}^{\infty} \left\{ e^{-\frac{im \pi z}{2}} \left( 1 + \frac{4e^{-Vd/a}}{V} \right) \frac{a}{a_e} \right\} e^{-i \frac{m \pi}{a} (x_1-x')} e^{-i \frac{m \pi}{a} (x_1+x')} \]

(3.25)

\[
G_{21}^s(x_2,x';z) = \frac{1}{4a_e} e^{-ikz} \sum_{m=-\infty}^{\infty} \left\{ e^{-\frac{im \pi z}{2}} \left( 1 + \frac{4e^{-Vd/a}}{V} \right) \frac{a}{a_e} \right\} e^{-i \frac{m \pi}{a} (x_2-x')} e^{-i \frac{m \pi}{a} (x_2+x')} \]

(3.26)
\[ G_{21}(x_2, x_1'; z) = \frac{1}{4a^2} e^{-ikz} \sum_{m=-\infty}^{\infty} \left[ e^{im\pi(z_1 - x_2)} - e^{im\pi(x_1 + x_1')} \right] \]

\[
= \frac{i2m\pi z}{z_1 z_{11}} \cos \frac{m^2 \pi z}{2z_c} \]

\[ G_{11}(x_1, x_1'; z) = G_{11}^s + G_{11}^a = \frac{1}{2a^2} e^{-ikz} \sum_{m=-\infty}^{\infty} \left[ e^{-im\pi(x_1 - x_1')/a^2} - e^{-im\pi(x_1 + x_1')/a^2} \right] \]

\[
= \frac{i2m\pi z}{z_1 z_{11}} \cos \frac{m^2 \pi z}{2z_c} \]

The Green's function in guide 1 can be written as:

\[ G_{11}(x_1, x_1'; z) = \frac{1}{2a^2} e^{-ikz} \sum_{m=-\infty}^{\infty} \left[ e^{-im\pi(x_1 - x_1')/a^2} - e^{-im\pi(x_1 + x_1')/a^2} \right] \]

\[ = \frac{i2m\pi z}{z_1 z_{11}} \cos \frac{m^2 \pi z}{2z_c} \]

in guide 2,

\[ G_{21}(x_2, x_1'; z) = G_{21}^s + G_{21}^a = \frac{1}{2a^2} e^{-ikz} \sum_{m=-\infty}^{\infty} \left[ e^{-im\pi(x_3 - x_1')/a^2} - e^{-im\pi(x_2 + x_1')/a^2} \right] \]

\[ = \frac{i2m\pi z}{z_1 z_{11}} \cos \frac{m^2 \pi z}{2z_c} \]

where,

\[ z_{11} = \frac{4ka^2}{\pi}, \quad z_c = \frac{z_{11} V a^2}{16} \]

The field at any z can be found as

\[ E_y(x_1, z) = \int_0^{a^2} E_y(x_1', 0) G_{11}(x_1, x_1'; z) dx_1' \] in guide 1

\[ E_y(x_2, z) = \int_0^{a^2} E_y(x_1', 0) G_{21}(x_2, x_1'; z) dx_1' \] in guide 2

\[ E_y(x_2, z) = \int_0^{a^2} E_y(x_1', 0) G_{21}(x_2, x_1'; z) dx_1' \] in guide 2
From (3.28), (3.29) we see that when the initial field is symmetrically excited with respect to the center of the guide \( l \); that is, if only odd modes are excited, total power transfer occurs at \( z = z_c, 3z_c, 5z_c, \ldots \), etc. Also, when \( m = 1 \), the coupling length for this mode is \( L = (\pi I_{1+} - I_{1-}) = z_c \). This is not a surprising result. It should be obvious also that \( z_c / 9 \) is the coupling length for the \( m = 3 \) mode, and \( z_c / 25 \) is the coupling length for the \( m = 5 \) mode.

It will be beneficial at this point to define two new parameters; namely, \( z_{11}^s \), \( z_{11}^a \) which are the imaging distances for symmetrical and antisymmetrical modes respectively. Similar to (2.3), (2.4), we have

\[
\frac{2 \pi}{z_{11}^s} = \frac{2 \pi}{2k a_e^2} \left( 1 - \frac{4 \ e^{-Vd/a}}{V} \frac{a}{a_e} \right) \tag{3.31}
\]

or,

\[
z_{11}^s = \frac{z_{11}^1}{1 - \frac{4 \ e^{-Vd/a}}{V} \frac{a}{a_e}} \tag{3.32}
\]

Also,

\[
z_{11}^a = \frac{z_{11}^1}{1 + \frac{4 \ e^{-Vd/a}}{V} \frac{a}{a_e}} \tag{3.33}
\]

These two expressions will be needed in the later discussion.
CHAPTER IV

PROPAGATION OF GAUSSIAN BEAMS

In this chapter we shall study the imaging properties of two coupled slab waveguides when one is excited with a Gaussian beam. Before going into the study of coupling case, we need to discuss the propagation of Gaussian beam in a single guide first. Here we follow the discussion of Chang and Kuester [10].

The field $E_y(x,z)$ for $z > 0$ in a metallic waveguide can be expressed in terms of the field $E_y(x,0)$, the field at the input plane ($z = 0$), and the Green's function,

$$E_y(x,z) = \int_0^a E_y(x',0) G(x,x';z) dx' \quad (4.1)$$

Our interest here is in the Gaussian beam

$$E_y(x,0) = e^{-(x-x_0)^2/2w_0^2} \quad (4.2)$$

where $x_0$ is the center of the Gaussian beam, and $0 < x_0 < a$; $w_0$ is the beam waist parameter. Also assume that $w_0 << x_0$ and $w_0 >> a - x_0$; that is, the "tails" of this beam are negligible at the walls of the guide. Finally, we assume that the beam is well collimated, $kw_0 >> 1$.

The field at any point within the guide under the paraxial approximation can now be rewritten as:
\[ E_y(x,z) = \int_{-\infty}^{\infty} e^{-\frac{(x'-x_0)^2}{2\omega_0^2}} G_0(x,x';z) \, dx' \quad (4.3) \]

Now we introduce a new function, called the Jacobian theta function, defined by Whittaker and Watson [27] as

\[ \vartheta_3(z|\tau) = \sum_{m=-\infty}^{\infty} e^{2i\pi m^2 + 2i\pi mz} \quad (4.4) \]

This function has some very interesting properties which we summarize in Appendix A.

Compare this Jacobian theta function with (2.3), the paraxial Green's function. We see that these two functions are quite similar to each other. We can now put \( G_0 \) in the form of \( \vartheta_3 \) functions.

\[ G_0(x,x';z) = \frac{1}{2a} e^{-ikz} \left[ \vartheta_3 \left( \frac{\pi(x-x')}{2a}, \frac{2z}{z_{11}} \right) - \vartheta_3 \left( \frac{\pi(x+x')}{2a}, \frac{2z}{z_{11}} \right) \right] \quad (4.5) \]

Putting this back into (4.3), then

\[ E_y(x,z) = \frac{e^{-ikz}}{2a} \int_{-\infty}^{\infty} e^{-\frac{(x'-x_0)^2}{2\omega_0^2}} \left[ \vartheta_3 \left( \frac{\pi(x-x')}{2a}, \frac{2z}{z_{11}} \right) \right] \, dx' \quad (4.6) \]

with the help of (A.11) we obtain
\[
E_y(x, z) = \frac{w_0}{a} \sqrt{\frac{\pi}{2}} e^{-ikz} \left[ J_3 \left( \frac{\pi(x-x_0)}{2a} \right) \left( \frac{2z}{z_{11}} + \frac{i\pi w_0^2}{2a^2} \right) - \frac{\pi(x+x_0)}{2a} \left( \frac{2z}{z_{11}} + \frac{i\pi w_0^2}{2a^2} \right) \right]. \tag{4.7}
\]

Let us consider the focusing relative to \(z_{11}\) first. Let \(z = qz_{11} + \Delta z\), where \(-z_{11}/2 < \Delta z < z_{11}/2\) and \(q\) is an integer. From (A.3) and (A.8), we have

\[
E_y(x, z) = \frac{w_0}{f(\Delta z)} e^{-ikz} \left[ e^{-(x-x_0)/2f^2(\Delta z)} J_3 \left( \frac{ia(x-x_0)}{f^2(\Delta z)} \right) \left( \frac{2ia^2}{\pi f^2(\Delta z)} \right) - e^{-(x+x_0)^2/2f^2(\Delta z)} J_3 \left( \frac{ia(x+x_0)}{f^2(\Delta z)} \right) \left( \frac{2ia^2}{\pi f^2(\Delta z)} \right) \right]. \tag{4.8}
\]

where we define "a complex waist parameter" \(f(\Delta z)\) as

\[
f^2(\Delta z) = \frac{w_0^2}{2} \left[ 1 - i \frac{4a^2}{\pi w_0^2} \frac{\Delta z}{z_{11}} \right] = \frac{2}{\Delta z} - i \frac{\Delta z}{k} \tag{4.9}
\]

using (A.12), (4.8) can be reduced to

\[
E_y(x, z) = \frac{w_0}{f(\Delta z)} e^{-ikz} \sum_{m=-\infty}^{\infty} \left[ e^{-\frac{(x-x_0+2ma^2)^2}{2f^2(\Delta z)}} - e^{-\frac{(x+x_0+2ma^2)^2}{2f^2(\Delta z)}} \right]. \tag{4.10}
\]

Now,

\[
\frac{1}{2f^2(\Delta z)} = \frac{1}{2w^2(\Delta z)} + \frac{i\Delta z/\Delta w}{2 \Delta w (\Delta z)} \tag{4.11}
\]
where \( w(\Delta z) \) is given by

\[
    w^2(\Delta z) = \frac{2}{w_0} + \left(\frac{\Delta z}{kw_0}\right)^2 \quad .
\]

(4.12)

From (4.10), we can see that the field at any point inside the guide can be represented by an infinite series of Gaussian beams, each broadened from its focal plane \( z = qz_{11} \) as it is propagating. As \( \Delta z \) increases, more and more "image" beams start to contribute significantly to the field within \( 0 < x < a \). For example, at \( \Delta z = z_{11}/2 \), the waist size becomes,

\[
    w^2(z_{11}/2) = \frac{2}{w_0} + \left(\frac{2a^2}{\pi w_0^2}\right)^2 \approx \frac{4a^4}{\pi w_0^4} \gg a^2 \quad .
\]

Thus we may require quite a few terms of the image series (4.10) in order to compute the field at certain \( z \).

In order to improve the efficiency of computation, another scheme was used by transforming (4.7) into somewhat different form. Applying (A.9), we have

\[
    E_y(x,z) = \frac{w_0}{a} \frac{e^{-ikz}}{\sqrt{2}} \sum_{r=0}^{p-1} e^{-\pi r^2 22z/z_{11} - \pi^2 r^2 w_0^2/2a^2} \times
\]

\[
    \left\{ e^{+i\pi r(x-x_0)/z} \right\}, \left\{ e^{-i\pi r(x-x_0)/a} \right\} \left( p\left[\frac{\pi(x-x_0)}{2a} + \frac{2\pi r z}{z_{11}} + i r \frac{2w_0^2}{2a^2} \right] \right) \left( p\left[\frac{\pi(x+x_0)}{2a} + \frac{2\pi r z}{z_{11}} + i r \frac{2w_0^2}{2a^2} \right] \right) \right) \]}

(4.13)

for any specified \( p \). Now, let \( z = z_{pq} + \Delta z_p \), where now \( -z_{11}/2p < \)
\[ \Delta z_p < z_{11}/2p. \] The periodicity properties of (A.3) and (A.4) allow us to replace \( z \) by \( \Delta z_p \) in (4.12). Finally, applying (A.8) results in

\[
E_y(x, z) = \frac{w_0}{p^f(\Delta z_p)} e^{-ikz} \sum_{r=0}^{p-1} e^{2\pi i r^2 q/p} \left\{ e^{-(x-x_0)^2/2f^2(\Delta z_p)} \right. \\
\left. - e^{-(x+x_0)^2/2f^2(\Delta z_p)} \right. \\
\left. - \frac{ia(x-x_0)}{pf^2(\Delta z_p)} - \frac{\pi}{p} \frac{2ia^2}{\pi p^2 f^2(\Delta z_p)} \right. \\
\left. - e^{-(x-x_0+2ma/p)^2/2f^2(\Delta z_p)} - e^{-(x+x_0+2ma/p)^2/2f^2(\Delta z_p)} \right. \\
\left. \right\} 
\]

(4.14)

where \( f \) is defined in (4.9). For \( p = 1 \), this reduces to (4.10) as expected.

Finally, applying (A.12), we obtain

\[
E_y(x, z) = \frac{w_0}{f(\Delta z_p)} e^{-ikz} \sum_{r=0}^{p-1} e^{2\pi i r^2 q/p} \left\{ \sum_{m=-\infty}^{\infty} e^{-2\pi i m r/p} \\
\left[ e^{-(x-x_0+2ma/p)^2/2f^2(\Delta z_p)} - e^{-(x+x_0+2ma/p)^2/2f^2(\Delta z_p)} \right] \right. \\
\left. \right\} . 
\]

(4.15)

Clearly, equation (4.15) represents a string of Fresnel images at \( z_{pq} \), broadened by their additional propagation distance \( \Delta z_p \) as evidenced by the factor \( f(\Delta z_p) \). A slight rearrangement of (4.15) yields a single summation (letting \( m \to -n \):
\[
E_y(x,z) = \frac{w_0}{f(\Delta z_p)} e^{-ikz} \sum_{n=-\infty}^{\infty} C_n(p,q) \left\{ \begin{array}{c}
\frac{e^{-(x-x_0-2na/p)^2/2f^2(\Delta z_p)}}{-(x+x_0-2na/p)^2/2f^2(\Delta z_p)} \\
-e \end{array} \right\}
\]

(4.16)

where \(C_n\) is defined in (2.13).

Equation (4.16) is a very pleasant result; physically, it says that when the exciting field is a Gaussian beam, the field at any \(z\) can be represented by a string of Gaussian beams at \(z_{pq}\), broadened by propagation to an additional distance \(\Delta z_p\).

Numerical results were obtained for (4.16). Here we use a symmetrical beam excitation \((x_0=a/2)\), also we choose \(ka = 157\), \(n_1 = 1.02\), \(n_2 = 1.00\). Notice that when \(ka = 157\), this means that we have \(V/\pi = 10\) admissible modes to propagate.

\(p\) in (4.16) was set equal to the nearest integer to \(a/w_0\).

(This means we allow about \(a/w_0\) Gaussian beam images to be accomodated inside the guide without severe overlap. \(N\) was chosen to be equal to \(3pw(\Delta z_p)/a+4\) (this value was chosen to assure good accuracy when \(\Delta z_p\) varies from 0 to \(z_{11}/2p\)).

The graphical results for the power \(|E_y|^2\) distributed inside the guide from \(z = 0\) to \(z_{11}/8\) were displayed in Appendix B. Notice that at \(z_{16,1}\), we obtained two identical images inside the guide. However, at \(z = z_{81}\), we obtained one fully reconstructed image. These results agree with what we had predicted in last chapter [10].

Suppose now we have two identical parallel slab waveguides put adjacent to each other, and one guide is excited by a Gaussian beam
at this input plane. According to the discussion of the last chapter, we can treat this problem as two separate parts: a symmetrical mode system combined with an antisymmetrical mode system. Each system can be put into the form (4.16). So, the electric fields in guide 1 and guide 2 now can be expressed as,

\[
E_y(x_1,z) = \frac{w_0}{2} e^{-ikz} \sum_{n=-\infty}^{\infty} C_n(p,q_s,a) \times \\
\left\{ \frac{1}{f(\Delta z_p)} \right\} \left[ e^{-\left(\frac{x_1-x_0}{p} - \frac{2na}{p} \right)} \frac{2}{f^2(\Delta z_p^s)} - e^{-\left(\frac{x_1+x_0}{p} - \frac{2na}{p} \right)} \frac{2}{f^2(\Delta z_p^s)} \right] \\
+ \frac{1}{f(\Delta z_p^a)} \left[ e^{-\left(\frac{x_1-x_0}{p} - \frac{2na}{p} \right)} \frac{2}{f^2(\Delta z_p^a)} - e^{-\left(\frac{x_1+x_0}{p} - \frac{2na}{p} \right)} \frac{2}{f^2(\Delta z_p^a)} \right]
\]

(4.17)
in guide 1

\[
E_y(x_2,z) = \frac{w_0}{2} e^{-ikz} \sum_{n=-\infty}^{\infty} C_n(p,q_s,a) \times \\
\left\{ \frac{1}{f(\Delta z_p)} \right\} \left[ e^{-\left(\frac{x_2-x_0}{p} - \frac{2na}{p} \right)} \frac{2}{f^2(\Delta z_p^s)} - e^{-\left(\frac{x_2+x_0}{p} - \frac{2na}{p} \right)} \frac{2}{f^2(\Delta z_p^s)} \right] \\
- \frac{1}{f(\Delta z_p^a)} \left[ e^{-\left(\frac{x_2-x_0}{p} - \frac{2na}{p} \right)} \frac{2}{f^2(\Delta z_p^a)} - e^{-\left(\frac{x_2+x_0}{p} - \frac{2na}{p} \right)} \frac{2}{f^2(\Delta z_p^a)} \right]
\]

(4.18)
in guide 2
where

\[ \Delta z^a_p = z^a - z^a_{pq} \]
\[ \Delta z^s_p = z^s - z^s_{pq} \]  \hspace{1cm} (4.19)

and

\[ z^a_{pq} = \frac{q_a}{p} z^a_{11} \]
\[ z^s_{pq} = \frac{q_s}{p} z^s_{11} \]  \hspace{1cm} (4.20)

\( z^a_{11} \) and \( z^s_{11} \) are defined in (3.32), (3.33). Also,

\[ f^2(\Delta z^a_p) = w_0^2 [1 - \frac{i}{2} \frac{4a^2}{\pi} \frac{\Delta z^a_p}{z^a_{11}}] = w_0^2 - i \frac{\Delta z^a_p}{k} z^a_{11} \]  \hspace{1cm} (4.21)

\[ f^2(\Delta z^s_p) = w_0^2 [1 - \frac{i}{2} \frac{4a^2}{\pi} \frac{\Delta z^s_p}{z^s_{11}}] = w_0^2 - i \frac{\Delta z^s_p}{k} z^s_{11} \]  \hspace{1cm} (4.22)

We have calculated the power distribution \( |E_y|^2 \) in both guides. The parameters used are the same as those in the single guide case (Appendix B), only the distance between these two guides was chosen to equal to 0.278175 \( \mu m \), so that the total power coupling distance \( z_c = 5 \times z_{11} = 0.0115327 \) m. Notice that the value \( d = 0.278175 \) \( \mu m \) is much smaller than the guide width \( a = 10 \) \( \mu m \). And the smaller the distance \( d \) is, the shorter the \( z_c \) will be. The graphical results are shown in Appendix C. Here we can see that the power is coupled back and forth between guide 1 and guide 2 as \( z \) progresses. When \( z \) reaches \( z_c \), we have a fully
reconstructed image appear in guide 2 and nothing in guide 1. After \( z_c \), the sequence reverses, the power starts to coupling back into guide 1. At \( z = 2z_c \), we recover the original input image at guide 1. These results follow the general conclusion we had obtained in (3.28), (3.19). In (3.28) and (3.29), we had seen that the field in guide 1 varied consinusoidally and field in guide 2 varied sinusoidally. When \( z = z_c \), because the factor \( \cos \frac{m \pi}{2} = 0 \), \( m = 1, 3, 5... \) in (3.28) the field in guide 1 vanishes; conversely, because the factor \( \sin \frac{m \pi}{2} = 1 \), \( m = 1, 3, 5... \) in (3.29), the field in guide 2 is identical to the input field. The same kind of argument can apply to \( z = 2z_c \).

It is also interesting to investigate the case of off-centered excitation. Here we choose \( x_0 \), the center of the input Gaussian beam equal to \( 3a/4 = 7.5 \) \( \mu \)m. The graphical results are shown in Appendix D. Notice that \( z = z_c \), we have two identical images in both guide 1 and guide 2. This is because of the reason that when \( x_0 = 3a/4 \), the input field can be treated as the combination of an even term and an odd term, (see Fig. 10). From the discussion in Chapter II, we recall that \( z_c \) is the coupling distance for \( m = \) odd modes, so at \( z = z_c \), all the odd order modes coupled into guide 2 and all the even order modes remain in guide 1. Because here we are looking at \( |E_y|^2 \), the power patterns in guide 1 and guide 2 are the same.
Figure 10. Decomposition of the off-centered excited Gaussian beam.
CHAPTER V

POWER CALCULATION OF TWO COUPLED SLAB WAVEGUIDES

In the last chapter, we explored the imaging properties associated with the coupling between two guides. We saw that at the distance $z_c$, the power could be totally coupled to the other guide. It will be interesting to ask here if we can formulate a mathematical theory that can predict the amount of the power transfer at any arbitrary distance $z$, again without the necessity of computing a large number of modes. If we can achieve this, it might serve as a guideline for the design of optical waveguide couplers later on.

Neglecting the constants, the powers in guide 1 and guide 2 are:

$$p_1(z) = \int_0^a |E_y(x_1,z)|^2 \, dx_1$$  \hspace{1cm} (5.1)

$$p_2(z) = \int_0^a |E_y(x_2,z)|^2 \, dx_2$$  \hspace{1cm} (5.2)

$a$ in this chapter denotes the effective guide width. $E_y(x_1,z)$ and $E_y(x_2,z)$ can be put into the integrated form,

$$E_y(x_1,z) = \int_0^a E_{y_0}(x',0) G_1(x_1,x';z) dx'$$ \hspace{1cm} (5.3)

$$E_y(x_2,z) = \int_0^a E_{y_0}(x',0) G_2(x_2,x';z) dx'$$ \hspace{1cm} (5.4)

The Green's functions $G_1, G_2$ are known from (3.28), (3.29)
\[ G_1(x_1, x'; z) = \frac{2}{a} e^{-ikz} \sum_{m=1}^{\infty} \sin \frac{m\pi x_1}{a} \sin \frac{m\pi x'}{a} e^{\frac{2im^2z}{z_c}} \cos \frac{m\pi z}{2z_c} \]  

(5.5)

\[ G_2(x_2, x'; z) = \frac{2}{a} e^{-ikz} \sum_{m=1}^{\infty} \sin \frac{m\pi x_2}{a} \sin \frac{m\pi x'}{a} e^{\frac{2im^2z}{z_c}} \sin \frac{m\pi z}{2z_c} \]  

(5.6)

Putting (5.3), (5.4), (5.5), (5.6) into (5.1), we have

\[ p_1(z) = \int_{0}^{a} |E_y(x_1, z)|^2 \, dx_1 \]

\[ = \int_{0}^{a} \int_{0}^{a} E_{yo}(x_1, 0) E_{yo}^*(x', 0) \widetilde{G}_1(x_1, x'; z) \, dx_1 \, dx' \]  

(5.7)

where

\[ \widetilde{G}_1(x_1, x'; z) = \int_{0}^{a} G_1^*(x, x'; z) G_1(x, x_1; z) \, dx \]

\[ = \left( \frac{2}{a} \right)^2 \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} \sin \frac{m\pi x_1}{a} \sin \frac{m\pi x'}{a} \int_{0}^{a} \sin \frac{m\pi x}{a} \sin \frac{m'\pi x}{a} \, dx \sin \frac{2\pi m^2 z}{2z_c} \sin \frac{2\pi m' z}{2z_c} \]

\[ \times \cos \frac{m\pi z}{2z_c} \cos \frac{m'\pi z}{2z_c} e^{2\pi i(m^2 - m'^2)z/2z_c} \]

\[ = \frac{2}{a} \sum_{m=1}^{\infty} \sin \frac{m\pi x_1}{a} \sin \frac{m\pi x'}{a} \cos \frac{m\pi z}{2z_c} \]

\[ = \frac{2}{a} \sum_{m=1}^{\infty} \sin \frac{m\pi x_1}{a} \sin \frac{m\pi x'}{a} \left[ \frac{1}{4} e^{im\pi z/2z_c} + \frac{1}{4} e^{-im\pi z/2z_c} + \frac{1}{2} \right] \]

\[ = \frac{1}{8a} \left \{ \vartheta_3 \left( -\frac{1}{2a} \left| \frac{z}{z_c} \right| \right) - \vartheta_3 \left( -\frac{1}{2a} \left| \frac{z}{z_c} \right| \right) + \vartheta_3 \left( \frac{1}{2a} \left| \frac{z}{z_c} \right| + \frac{1}{2} \right) \right \} \]

\[ - \vartheta_3 \left( \frac{1}{2a} \left| \frac{z}{z_c} \right| - \frac{1}{z_c} \right) + 2\vartheta_3 \left( \frac{1}{2a} \left| \frac{z}{z_c} \right| - 0 \right) - 2\vartheta_3 \left( \frac{1}{2a} \left| \frac{z}{z_c} \right| - 0 \right) \}

(5.8)
If we now have a Gaussian beam input, $E_{y_0}(x,0) = e^{-(x-x_0)^2/2w_o^2}$, and this input beam is well confined inside the guide ($w_o \ll a$), we can extend the integration interval from $-\infty$ to $+\infty$. Now

$$
\int_0^a E_y(x,0)\tilde{G}_1(x_1,x';z)dx_1
= \int_{-\infty}^{\infty} E_y(x,0)\tilde{G}_1(x_1,x';z)dx_1
$$

consider the following integration

$$
\int_{-\infty}^{\infty} e^{-(x-x_0)^2/2w_o^2} J_3\left(\frac{\pi(x+b)}{2a}|\tau\right)dx
$$

by changing the variable, $u = \pi(x+b)/2a$, we have

$$
\frac{2a}{\pi} \int_{-\infty}^{\infty} e^{-(\frac{2a}{\pi}u-b-x_0)^2/2w_o^2} J_3(u|\tau)du
\quad - \frac{2a}{\pi} \frac{u^2}{2w_o^2} + \frac{2(b+x_0)a}{\pi w_o^2} - \frac{(b+x_0)^2}{2w_o^2} J_3(u|\tau)du
$$

$$
= \frac{2a}{\pi} \int_{-\infty}^{\infty} e^{-(\frac{2a}{\pi}u-b-x_0)^2/2w_o^2} J_3(u|\tau)du
$$

$$
= \sqrt{2\pi} w_o J_3\left(\frac{\pi(b+x_0)}{2a}, |\tau + \frac{i\pi w_o^2}{2a^2}\right)
$$

The last integration was obtained by applying (A.11) and some cancellations.
Using (5.8) and (5.10), (5.9) thus can be expressed as,

\[
\int_0^a E_{y_0}(x_1, 0) \tilde{G}_1(x_1, x'; z) dx_1
\]

\[
= \frac{\omega_0}{4a} \sqrt{\frac{\pi}{2}} \left\{ J_3 \left( \frac{\pi(x_0 - x')}{2a} \right) \frac{z}{z_c} + \frac{i\pi \omega^2}{a} \right\} - J_3 \left( \frac{\pi(x_0 + x')}{2a} \right) \frac{z}{z_c} + \frac{i\pi \omega^2}{a} \right\} 
\]

\[
+ J_3 \left( \frac{\pi(x_0 - x')}{2a} \right) \frac{i\pi \omega^2}{a} \right\} - J_3 \left( \frac{\pi(x_0 + x')}{2a} \right) \frac{i\pi \omega^2}{a} \right\} 
\]

\[
+ 2J_3 \left( \frac{\pi(x_0 - x')}{2a} \right) \frac{i\pi \omega^2}{a} \right\} - 2J_3 \left( \frac{\pi(x_0 + x')}{2a} \right) \frac{i\pi \omega^2}{a} \right\} 
\]

(5.11)

Similarly,

\[
\int_0^a \int_0^a E_{y_0}(x', 0) E_{y_0}(x_1, 0) \tilde{G}_1(x_1, x'; z) dx_1 dx'
\]

\[
= \frac{\omega_0}{4a} \sqrt{\frac{\pi}{2}} \sqrt{\frac{\pi}{2}} \omega_0 \left\{ J_3(0) \frac{z}{z_c} + \frac{i\pi \omega^2}{a} \right\} - J_3 \left( \frac{\pi x}{a} \right) \frac{z}{z_c} + \frac{i\pi \omega^2}{a} \right\} 
\]

\[
+ J_3(0) \frac{i\pi \omega^2}{a} \right\} - J_3 \left( \frac{\pi x}{a} \right) \frac{i\pi \omega^2}{a} \right\} 
\]

\[
+ 2J_3(0) \frac{i\pi \omega^2}{a} \right\} - 2J_3 \left( \frac{\pi x}{a} \right) \frac{i\pi \omega^2}{a} \right\} 
\]

(5.12)

Applying (A.9), we have the following relationship,
\[ \mathcal{V}_3 \left( \frac{\pi x}{a} \left( z - \frac{\pi w^2}{a^2} \right) \right) \]

\[ p^{-1} \sum_{r=0}^{p-1} \pi r^2 \left( \frac{z}{z_c} + \frac{i\pi w^2}{a^2} \right) + 2i \frac{\pi x}{a} \]

Now let \( z = 2sz_c/p + \Delta z \), where \( |\Delta z| < z_c/p \). From (A.3), (A.4) also from (A.8). We have
\[ \mathcal{V}_3 \left( \frac{\pi x_o}{a} \mid \frac{z}{z_c} + \frac{i \pi w_o^2}{a^2} \right) \]

\[ = \sum_{r=0}^{p-1} \pi r \frac{z}{z_c} - \pi r \frac{w_o^2}{a^2} + 2 \pi r x_o \frac{z}{z_c} \times \left( \frac{1}{p} \right)^{1/2} \]

\[ \times e^{-2 \left[ i \pi w_o^2 / a^2 + \pi r \frac{\Delta z}{z_c} + \pi x_o / a \right]^2 / \left[ p^2 \pi w_o^2 / a^2 - ip \right]} \]

\[ \times e^{\left( \frac{\pi x_o}{a} + \frac{\pi r \Delta z}{a} \right)^2 / \left( \frac{2}{a} \right)^2 - \left( \frac{1}{p} \right)^{1/2} \}

\[ \times \mathcal{V}_3 \left( \frac{\pi x_o}{a} + \frac{\pi r \Delta z}{a} \right) \frac{2}{a^2} \frac{\pi w_o^2}{p} \left( \frac{\Delta z}{z_c} + i \frac{w_o^2}{a} \right) \]

\[ = \frac{1}{p \sqrt{\pi}} \frac{a}{f_o(\Delta z)} e^{-\frac{x_o^2}{f_o^2}(\Delta z)} \pi r \frac{z}{z_c} + \frac{\Delta z}{z_c} \frac{\pi w_o^2}{a^2} \sum_{r=0}^{p-1} \frac{2 \pi r x_o}{f_o(\Delta z)} - \frac{\pi r}{p} \frac{a}{f_o^2}(\Delta z) \]

where

\[ f_o^2(\Delta z) = w_o^2 - i a^2 \frac{\Delta z}{z_c} \]

using (A.12)

\[ \mathcal{V}_3 \left( \frac{\pi x_o}{p f_o^2(\Delta z)} - \frac{\pi r}{p} \frac{a}{f_o^2(\Delta z)} \right) = \sum_{n=-\infty}^{\infty} e^{2i \pi r n / p} \frac{-(x_o - n a / p)^2}{f_o^2(\Delta z)} \]

(5.15)

Put (5.15) back into (5.14), we have

\[ \mathcal{V}_3 \left( \frac{\pi x_o}{a} \mid \frac{z}{z_c} + \frac{i \pi w_o^2}{a^2} \right) = \frac{a}{f(\Delta z)} \left( \frac{1}{\sqrt{\pi}} \right) \sum_{n=-\infty}^{\infty} c_n(p, s) e^{-\frac{(x_o - n a / p)^2}{f_o^2(\Delta z)}} \]

(5.16)

where \( c_n(p, s) \) are given in (2.13). We can apply the same technique
to all the other terms in (5.12). Since,

\[ P_1(0) \approx \int_{-\infty}^{\infty} |E_y(x,0)|^2 \, dx \]

\[ = \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{\omega_0^2}} \, dx \]

\[ = \int_{-\infty}^{\infty} e^{-\frac{x^2}{\omega_0^2}} = \omega_0 \sqrt{\pi} \]  

(5.17)

then,

\[ \frac{P_1(z)}{P_1(0)} \approx \sum_{n=-\infty}^{\infty} c_n(p,s) \frac{\omega_0}{4f_o(\Delta z)} \left[ e^{-\frac{(na/p)^2}{f_o^2(\Delta z)}} - e^{-\frac{(x_o-na/p)^2}{f_o^2(\Delta z)}} \right] 

+ \sum_{n=-\infty}^{\infty} c_n(p,-s) \frac{\omega_0}{4f_o(-\Delta z)} \left[ e^{-\frac{(na/p)^2}{f_o^2(-\Delta z)}} - e^{-\frac{(x_o-na/p)^2}{f_o^2(-\Delta z)}} \right] 

+ \sum_{n=-\infty}^{\infty} c_n(p,0)^{1/2} \left[ e^{-\frac{(na/p)^2}{\omega_0^2}} - e^{-\frac{(x_o-na/p)^2}{\omega_0^2}} \right]. \]  

(5.18)

Figures 11, 12, 13 are plots of ratio of total power in guide 1 as a function of axial distance. The parameters used are the same as last chapter, the only difference being in \( w_o \), the input Gaussian beam half beam width, which is chosen to be equal to 2 \( \mu \)m, 1 \( \mu \)m, 0.5 \( \mu \)m for three different cases. In Fig. 10 (\( w_o = 2 \mu m \)).

We see clearly the power transfer from guide 1 to guide 2 in a rather smooth manner. When it reaches \( z_c \), total power transfer has occurred, after that, the process reverses. We attribute this smoothness to the fact that the input Gaussian beam looks roughly like the \( m = 1 \) mode of the guide, so that the focusing-defocusing effect, which affects the degree of excitation of other modes, is
Figure 11. Ratio of total power in guide 1 as a function of the axial distance along the coupling structure. $x_o = a/2$, $\omega_o = 2.0 \, \mu m$. 
Figure 12. Ratio of total power in guide 1 as a function of the axial distance along the coupling structure. $x_o = a/2$, $w_o = 1.5 \mu m$. 
Figure 13. Ratio of total power in guide 1 as a function of the axial distance along the coupling structure. $x_0 = a/2$, $w_0 = 1.0 \mu m$. 
rather minor. Because of this, most coupling occurs for \( m = 1 \) mode, and the coupling distance of the \( m = 1 \) mode is \( z_c \) (the reason is clear from Chap. III), which explains Fig. 11.

When \( w_o \), the beam width, gets smaller and smaller, more and more higher order modes start to contribute significantly in the process of the power transfer. Eventually we have a rather complicated power transfer process, as shown in figures like Fig. 12 and Fig. 13. In Fig. 12 \((w_o = 1.5 \ \mu m)\), we can see that the effect of the \( m = 3 \) mode coupling is very prominent (the coupling length of the \( m = 3 \) mode is \( z_c/9 \)). In Fig. 13 \((w_o = 1.0 \ \mu m)\), we see that the effect of the \( m = 5 \) mode comes in (the coupling length of the \( m = 5 \) mode is \( z_c/25 \)).

Figure 14 shows the case of off-centered excitation \((x_o = 3a/4)\). We can see that at \( z = z_c \), there is only 50% power remaining in guide 1, the other 50% power coupled into guide 2. The reason for this 50%-50% power split was made clear in the last chapter. This kind of excitation is of special interest, because it fulfills the requirement of a 3-db coupler.
Figure 14. Ratio of total power in guide 1 as a function of the axial distance along the coupling structure. $x_0 = 3a/4$, $w_0 = 1.0 \mu m$. 

POWER IN GUIDE 1

0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

2.0
1.5
1.0
0.5
0.0

$Z_e$
CHAPTER VI

CONCLUDING REMARKS

The imaging and coupling properties of two identical parallel weakly coupled slab waveguides have been discussed. Knowing that the coupling occurs mainly between those degenerate modes, we, instead of dealing with a fairly complicated matrix, introduce a much simpler method which treat each degenerate coupling-modes pair separately. A special case of a Gaussian beam excitation has been studied in detail. It is shown that the oscillatory behaviors of the coupling depend very heavily on the input beam width. It is also shown that when the Gaussian beam is excited at the center of the guide, total power transfer is possible, when the Gaussian beam is excited at 3/4 of the guide a 3-db power coupler can be achieved. The qualitative results show agreement with numerical results of Yeh et al. [28] for the case of closely coupled fibers. It seems likely that the present theory can be extended to deal with single to multimode waveguide couplers [29]. The coupling properties of two strongly coupled dielectric waveguides has been studied by Suematsu and Kishino [30]. When the waveguides separation is sufficiently large, their analysis shows good agreement with ours.
BIBLIOGRAPHY


APPENDIX A

PROPERTIES OF $\mathcal{J}_3(z|\tau)$

In this appendix, we derive a number of useful properties of the theta-function $\mathcal{J}_3(z|\tau)$, defined by

$$\mathcal{J}_3(z|\tau) = \sum_{m=-\infty}^{\infty} e^{\frac{2}{\tau} \pi i m + 2miz} \quad (A.1)$$

where, for our purposes, $z$ is an arbitrary complex number, while $\tau$ lies in the upper half-plane $\text{Im}(\tau) > 0$. If $\tau$ lies on the real axis, we likewise require that $z$ be real, and in this case $\mathcal{J}_3(z|\tau)$ must be treated as a generalized function.

From Whittaker and Watson [27], we can obtain a number of periodicity and parity relations:

$$\mathcal{J}_3(z|\tau) = \mathcal{J}_3(-z|\tau) \quad (A.2)$$

$$\mathcal{J}_3(z|\tau + 2\pi n) = \mathcal{J}_3(z|\tau) \quad (A.3)$$

$$\mathcal{J}_3(z+n\pi|\tau) = \mathcal{J}_3(z|\tau) \quad \begin{cases} \text{for } n = 0, \pm 1, \pm 2, \ldots \end{cases} \quad (A.4)$$

$$\mathcal{J}_3(z+n\pi|\tau) = e^{-\pi \text{in}^2 \tau - 2inz} \mathcal{J}_3(z|\tau) \quad (A.5)$$

All four relations are easy consequences of the definition $(A.1)$. It is also interesting to note that $\mathcal{J}_3$ satisfies the parabolic equation
\[
\frac{\pi i}{4} \frac{\partial^2 \mathcal{J}_3(z|\tau)}{\partial z^2} + \frac{\partial \mathcal{J}_3}{\partial \tau} = 0
\] (A.6)

which is obtained from the Helmholtz equation in the paraxial approximation, and is similarly easily verified [cf. Eq. (8)].

For real \( z \), \( \mathcal{J}_3 \) at \( \tau = 0 \) (as a generalized function) can be evaluated as [31],[32]:

\[
\mathcal{J}_3(z|0) = \sum_{m=-\infty}^{\infty} e^{2miz} = \pi \sum_{n=-\infty}^{\infty} \delta(z-n\pi).
\] (A.7)

Another useful relation, which holds for general \( z \) and \( \tau \), is obtained from Jacobi's imaginary transformation [27]:

\[
\mathcal{J}_3(z|\tau) = \tau^{-1/2} e^{i\pi/4+z^2/2i\tau} \mathcal{J}_3\left(\frac{z}{\tau} \mid -\frac{1}{\tau}\right).
\] (A.8)

This relation can be verified using the Poisson summation formula [24].

The identities which form the basis for the "image-splitting" properties of the theta-function can be deduced from a more general expression given by Krazer [33]. These relations, which might be referred to as modular relations, are

\[
\mathcal{J}_3(z|\tau) = \sum_{r=0}^{p-1} e^{\pi ir^2\tau+2irz} \mathcal{J}_3(p[z+\pi r\tau]|p^2\tau)
\] (A.9)

\[
\mathcal{J}_3(z|\tau) = \frac{1}{p} \sum_{r=0}^{p-1} \mathcal{J}_3\left(\frac{z+\pi r}{p} \mid \frac{\tau}{p}\right).
\] (A.10)

Equation (A.9) is verified by writing \( m = p\tau+r \), \( \tau = 0,1,\ldots,(p-1) \) in (A.1); (A.10) follows by substituting (A.1) into the right-hand
side. Actually (A.9) and (A.10) can also be verified from each other using (A.8) as well.

An integral which could not be found in the literature is given below:

\[
\int_{-\infty}^{\infty} e^{-ax^2 + bx} \mathcal{V}_3(x|\tau) dx = \sqrt{\pi} \frac{b^{2/4a}}{2a} \mathcal{V}_3 \left( \frac{b}{2a} \left| \tau + \frac{i}{\pi a} \right. \right). \tag{A.11}
\]

The derivation is straightforward, proceeding by integrating (A.1) term-by-term.

Finally, we note that

\[
\mathcal{V}_3(i z | i \tau) = e^{z^2/\pi \tau} \sum_{m=-\infty}^{\infty} e^{-(z+m\pi \tau)^2/\pi \tau} \tag{A.12}
\]

i.e., \( \mathcal{V}_3 \) can be related to a string of displaced Gaussian functions.
APPENDIX B

EVALUATION OF POWER DISTRIBUTION OF GAUSSIAN BEAM OVER -1/8 CYCLE

Parameters:

\[ K_0 \] - Free space propagation constant \( - 15.7 \times 10^6 \text{ m}^{-1} \)

\[ A \] - Guide width \( - 10 \times 10^{-6} \text{ m} \)

\[ N_1 \] - Refractive index in guide 1 \( - 1.02 \)

\[ N_2 \] - Refractive index in guide 2 \( - 1.00 \)

\[ X_0 \] - Center of the Gaussian beam \( - 5.0 \times 10^{-6} \text{ m} \)

\[ W_0 \] - Gaussian beam waist width \( - 1.0 \times 10^{-6} \text{ m} \)
APPENDIX C

EVALUATION OF POWER DISTRIBUTION OF TWO COUPLED SLAB WAVEGUIDES

(CENTERED EXCITED GAUSSIAN BEAM)

Parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
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<td>Ko</td>
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</tr>
<tr>
<td>A</td>
<td>Guide width</td>
<td>$10 \times 10^{-6} , \text{m}$</td>
</tr>
<tr>
<td>N1</td>
<td>Refractive index in guide 1</td>
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<td>N2</td>
<td>Refractive index in guide 2</td>
<td>1.00</td>
</tr>
<tr>
<td>Xo</td>
<td>Center of the Gaussian beam</td>
<td>$5.0 \times 10^{-6} , \text{m}$</td>
</tr>
<tr>
<td>Wo</td>
<td>Gaussian beam waist width</td>
<td>$1.0 \times 10^{-6} , \text{m}$</td>
</tr>
<tr>
<td>D</td>
<td>Guides separation distance</td>
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</tbody>
</table>
APPENDIX D

EVALUATION OF POWER DISTRIBUTION OF TWO COUPLED SLAB WAVEGUIDES

(OFF-CENTERED EXCITED GAUSSIAN BEAM)

Parameters:

- $K_0$ - Free space propagation constant $= 15.7 \times 10^6 \text{ m}^{-1}$
- $A$ - Guide width $= 10 \times 10^{-6} \text{ m}$
- $N_1$ - Refractive index in guide 1 $= 1.02$
- $N_2$ - Refractive index in guide 2 $= 1.00$
- $X_0$ - Center of the Gaussian beam $= 7.5 \times 10^{-6} \text{ m}$
- $W_0$ - Gaussian beam waist width $= 1.0 \times 10^{-6} \text{ m}$
- $D$ - Guides separation distance $= 0.278175 \times 10^{-6} \text{ m}$
$Z = 10Z_{11}$