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NUMERICAL COMPUTATION OF
THE INCOMPLETE LIPSCHITZ-
HANKEL INTEGRAL $J_0(a,z)$

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NUMERICAL COMPUTATION OF THE INCOMPLETE LIPSCHITZ-HANKEL INTEGRAL \( J_\nu(a,z) \)

Steven L. Dvorak and Edward F. Kuester

Abstract

Two factorial-Neumann series expansions for the incomplete Lipschitz-Hankel integral \( J_\nu(a,z) \), are derived. These expansions are used together with the Neumann series expansion, given by Agrest, in an algorithm which efficiently computes \( J_\nu(a,z) \) to a user defined number of significant digits. The source code for this algorithm is included in the Appendix.

Other expansions for \( J_\nu(a,z) \), which are found in the literature, are also discussed, but these expansions are found to offer no significant computational advantages when compared with the expansions used in the algorithm.

An asymptotic expansion for a related integral, \( Y_\nu(a,z) \), is also derived.
1. INTRODUCTION

The incomplete Lipschitz-Hankel integral is defined in [1] as

\[ Z_{\nu}(a,z) := \int_{0}^{\infty} e^{-at} t^{\nu} Z_{\nu}(t) \, dt; \quad a, z, \nu \in \mathbb{C}, \quad (1-1) \]

where \( Z_{\nu}(t) \) is one of the cylindrical functions \( J_{\nu}(t) \), \( Y_{\nu}(t) \), \( I_{\nu}(t) \), or \( K_{\nu}(t) \). For finite values of \( z \), the integral given in (1-1) will converge provided \( \text{Re}(2\nu + 1) > 0 \). The corresponding complete (ordinary) Lipschitz-Hankel integral is an improper integral of the form

\[ \int_{0}^{\infty} e^{-at} t^{\nu-1} Z_{\nu}(t) \, dt; \quad a, \nu, \mu \in \mathbb{C}, \quad (1-2) \]

where the conditions \( \text{Re}(\nu + \mu) > 0 \) and \( \text{Re}(a) > 0 \) guarantee the convergence of this integral.

The incomplete Lipschitz-Hankel integral is an important special function since it arises in a number of problems in mathematical physics. Agrest and Maksimov [1] describe a number of these, and provide references to a large body of literature on the subject (see also the report [2]). A typical example from acoustics, which involves the functions \( J_{\nu}(a,z) \) and \( Y_{\nu}(a,z) \), is the calculation of scattering from an absorbing strip [3].

Several papers have been written on the computation of these integrals. Agrest has developed various expansions (see [4]-[6]) which can be used to compute \( Z_{\nu}(a,z) \) in different regions of the variables \( a \) and \( z \). Earlier, Maximon [7] obtained a Neumann series expansion for a class of functions including \( J_{\nu}(a,z) \) and \( I_{\nu}(a,z) \), though no computational procedures were attempted. Also, Amos and Burgmeier [8] give a recurrence algorithm which can be used to compute \( J_{\nu}(a,z) \) and \( I_{\nu}(a,z) \), although no numerical tests were reported.
We encountered the incomplete Lipschitz-Hankel integral of zero order,

\[ J_{0}(a,z) = \int_{0}^{z} e^{-at} J_{0}(t) dt, \quad (1-3) \]

in an electromagnetic fields problem [9]. In this problem, we used spectral domain techniques to obtain the nearzone fields and the driving point impedance of a center-fed, printed strip dipole antenna. In the formulation, the two-dimensional inverse Fourier transforms are expressed in polar coordinates, and then the inner angular integral is rewritten in terms of one, or more incomplete Lipschitz-Hankel integrals of the form given in (1-3). This application motivated us to study \( J_{0}(a,z) \).

In this paper, we develop an algorithm which efficiently computes \( J_{0}(a,z) \) for \( z \in \mathbb{R} \) and \( a \in \mathbb{C} \). We only need to handle positive values of \( z \), since

\[ J_{0}(a,-z) = -J_{0}(-a,z). \quad (1-4) \]

In Section 2, we derive a convergent factorial-Neumann series expansion and an asymptotic factorial-Neumann series expansion for \( J_{0}(a,z) \). Then in Section 3, we outline the derivation for a Neumann series expansion which was found by Agrest in [5]. In Section 4, we compare the expansions given in Sections 2 and 3, with other expansions found in the literature.

In Section 5, we develop an algorithm using the series expansions given in Sections 2 and 3, which efficiently computes \( J_{0}(a,z) \) to a user defined number of significant digits. The Fortran source code for this algorithm is given in Appendix F.

In this paper, we restrict \( z \) to be real, but the analysis presented in this paper, and the computer program, can be modified for \( z \in \mathbb{C} \).
2. DEVELOPMENT OF FACTORIAL-NEUMANN SERIES EXPANSIONS FOR $J_{\nu}(a,z)$

In this section, we develop series expansions for the incomplete Lipschitz-Hankel integral $J_{\nu}(a,z)$, where $z > 0$ and $a \in \mathbb{C}$. In §2.1, a first-order nonhomogeneous recurrence relation is constructed for the functions $J_{n}(a,z)$. Then in §2.2, we use this recurrence relation for $n \geq 0$, to construct a factorial-Neumann series which converges rapidly for small to moderate values of $|z^{\sqrt{a^{2}+1}}|$. In §2.3, the recurrence relation given in §2.1 is used for $n \leq -1$ to obtain a second factorial-Neumann series. The resulting series expansion is used to asymptotically evaluate $J_{\nu}(a,z)$ for large values of $|z(a^{2}+1)|$.

In §2.4, an asymptotic series is developed for $Y_{\nu}(a,z)$. This section is included since it is a natural extension of the methods used in §2.1 and §2.3.

2.1 Derivation of a First-order Non-homogeneous Recurrence Relation for $J_{n}(a,z)$

The incomplete Lipschitz-Hankel integral for the Bessel function of the first kind is given by (see (1-1))

$$J_{\nu}(a,z) := \int_{0}^{Z} e^{-at} t^{\nu} J_{\nu}(t) dt.$$ (2-1)

It is convenient to define a related integral which has variable upper and lower limits:

$$J_{\nu}(a,\delta,z) := \int_{\delta}^{Z} e^{-at} t^{\nu} J_{\nu}(t) dt.$$ (2-2)

Integrating (2-2) by parts, with

$$u := e^{-at}J_{\nu}(t), \quad dv := t^{\nu}dt,$$
and applying the recurrence relation for the derivative of Bessel functions (C-4), gives

\[ J_{\nu}(a, \delta, z) = \frac{1}{(2\nu+1)} \left\{ J_{\nu+1}(a, \delta, z) + a \int_{\delta}^{z} e^{-at} t^{\nu+1} J_{\nu}(t) dt \right\} + e^{-at} t^{\nu+1} J_{\nu}(t) \bigg|_{\delta}^{z} \]  \hspace{1cm} (2-3)

Using a second integration by parts with

\[ u := e^{-at}, \quad dv := t^{\nu+1} J_{\nu}(t) dt, \]

and the indefinite integral (C-13), yields a first-order nonhomogeneous recurrence relation:

\[ J_{\nu}(a, \delta, z) = \left( \frac{a^2+1}{2\nu+1} \right) J_{\nu+1}(a, \delta, z) \]

\[ = e^{-at} t^{\nu+1} \left[ J_{\nu}(t) + aJ_{\nu+1}(t) \right] \bigg|_{\delta}^{z}; \quad \nu \in \mathbb{C}. \]  \hspace{1cm} (2-4)

In this paper, we will restrict \( \nu \) to the set of integer values.

Equation (2-4) can be rewritten in a compact form,

\[ J_{n}(a, \delta, z) + d_{n} J_{n+1}(a, \delta, z) = f_{n}(z) - f_{n}(\delta); \quad n = 0, \pm 1, \pm 2, \ldots \]  \hspace{1cm} (2-5a)

where

\[ d_{n} := - \left( \frac{a^2+1}{2n+1} \right), \quad \text{and} \]

\[ f_{n}(t) := \frac{e^{-at} t^{n+1}}{(2n+1)} \left[ J_{n}(t) + aJ_{n+1}(t) \right]. \]  \hspace{1cm} (2-5b)

If we choose \( \delta \) so that \( f_{n}(\delta) = 0 \), then we obtain the desired first-order nonhomogeneous recurrence relation:

\[ J_{n}(a, \delta, z) + d_{n} J_{n+1}(a, \delta, z) = f_{n}(z); \quad n = 0, \pm 1, \pm 2, \ldots \]  \hspace{1cm} (2-6)

This recurrence relation yields solutions that behave very differently for the two cases \( n \geq 0 \) and \( n \leq -1 \). These two cases are explored in §2.2 and §2.3 respectively.
2.2 Development of a Convergent Factorial–Neumann Series Expansion for $J_{\delta_0}(a,z)$

In this section, we will use the recurrence relation (2-6) for $n \geq 0$. First, we must choose $\delta$ so that $f_n(\delta) = 0$ for $n \geq 0$. This requirement is satisfied when $\delta = 0$, because of the behavior of Bessel functions of small argument (see (C-7)).

Reference to (2-1) and (2-2) shows that for the special case $\delta = 0$,

$$J_n(a,z) = J_n(a,0,z). \quad (2-7)$$

Therefore, the recurrence relation (2-6) can be rewritten as

$$J_n(a,z) + d_n J_{n+1}(a,z) = f_n(z); \quad n \geq 0, \quad (2-8a)$$

where

$$d_n = -\left(\frac{a^2+1}{2n+1}\right), \quad (2-8b)$$

and

$$f_n(z) = e^{-az} \left(\frac{z}{2n+1}\right)^{n+1} [J_n(z) + a J_{n+1}(z)]. \quad (2-8c)$$

This recurrence relation can be analyzed using the techniques summarized in Appendix A. The homogeneous solution is found using (A-4) and (C-20):

$$J_{\delta_0}^{(h)}(a) = \left(\frac{2}{a^2+1}\right)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}; \quad n \geq 0. \quad (2-9)$$

For large values of $n$, (2-9) can be approximated by using Stirling's formula (C-21), and (C-28), giving

$$J_{\delta_0}^{(h)}(a) \approx \sqrt{2} \left[\frac{2n}{a(a^2+1)}\right]^n; \quad n \to \infty. \quad (2-10)$$

An index of stability for the forward computation of $J_{\delta_0}(a,z)$ from $J_{\delta_k}(a,z)$ is given by (A-7):
\[ \alpha(k,n) = \left| \frac{J_e_k(a,z) J_e(h)_n(a)}{J_e_n(a,z) J_e(h)_k(a)} \right| = \frac{\rho_n}{\rho_k} , \quad (2-11a) \]

where
\[ \rho_n = \left| \frac{J_e_0(a,z) J_e(h)_n(a)}{J_e_n(a,z)} \right| . \quad (2-11b) \]

Using this index, we can determine whether the recurrence relation (2-8) can be used in the forward direction (see Appendix A for more details).

The behavior of \( \rho_n \), for large values of \( n \), is obtained by using (2-9) along with the asymptotic behavior of \( J_e_n(a,z) \) given in (3-4):
\[ \rho_n^{(1)} - |J_e_0(a,z)| - \frac{e^{\gamma} \frac{\Re(a)}{z}}{2 \sqrt{\pi} n} \left( \frac{4n}{e^{\gamma} |a^2 + 1|} \right)^{\frac{3}{2}} ; \]
\[ n \gg \kappa , \quad z > 0 , \quad a \in \mathbb{C} . \quad (2-12) \]

where \( \kappa = \max(z,|az|) \).

For the limiting case \( n \to \infty \), (2-12) can be simplified by using (C-21) again. This gives
\[ \rho_n^{(1)} - |J_e_0(a,z)| \cdot \frac{e^{\gamma} \Re(a)}{z} \cdot \frac{4^n}{2 \pi n} \left( \frac{2n}{e^{\gamma} |a^2 + 1|} \right)^{\frac{2n}{2}} ; \quad n \to \infty . \quad (2-13) \]

Since condition (A-9) is satisfied, forward recurrence using (2-8) is unstable.

The recurrence relation (2-8) can also be found in the paper by Agrest and Rikenglaz ([4], pg. 207). In that paper, the authors state that to analyze the incomplete Lipschitz-Hankel integrals with integer values of \( \nu = n \) it is sufficient to study them with \( \nu = 0 \). It may be possible to use (2-8) in the forward direction to calculate a few values of \( J_e_n(a,z) \), but as we have shown in (2-13), the recurrence will eventually become unstable. The point where the forward recurrence becomes unstable is dependent on the parameters \( a \) and \( z \).
We already know that forward recurrence is unstable when $n >> \text{Max}(z, |az|)$, since $\rho_n^{(1)}$ is monotone increasing (see (2-12)). Now, we need to determine the stability when $n < z$. We will assume that $\text{Re}(a) > 0$ for this analysis, but the case $\text{Re}(a) \leq 0$ can be handled using similar techniques.

We start the analysis by splitting the integral into two pieces,

$$J_{e_n}(a,z) = J_{e_n}(a,\infty) + J_{e_n}(a,\infty, z) ; \quad z > 0, \quad \text{Re}(a) > 0. \quad (2-14)$$

The first integral is known in closed form (C-17), and the second integral can be approximated by using (D-21). Therefore,

$$J_{e_n}(a, z) = \frac{2^n \Gamma(n + \frac{1}{2})}{\sqrt{\pi} (a^2 + 1)^{n + \frac{1}{2}}} - (-1)^n \sqrt{\frac{2}{\pi} \frac{e^{-az} z^n}{(a^2 + 1)}} \left[ a \cos \left( z + \frac{n \pi}{2} - \frac{\pi}{4} \right) - \sin \left( z + \frac{n \pi}{2} - \frac{\pi}{4} \right) \right] ; \quad z > 0, \quad \text{Re}(a) > 0, \quad \eta >> n \geq 0, \quad (2-15)$$

where $\eta = \min(z, |a \pm i|)$.

A stability index for using (2-8a) in the forward direction, when $0 < n < z$, is given by (2-11a), where

$$\rho_n^{(2)} = \frac{|J_{e_0}(a, z)|}{\left[ \frac{1}{(a^2 + 1)^{\frac{1}{2}}} - \sqrt{\frac{2}{z}} e^{-az} \left( \frac{-z(a^2 + 1)}{2} \right)^n \frac{\left[ a \cos \left( z + \frac{n \pi}{2} - \frac{\pi}{4} \right) - \sin \left( z + \frac{n \pi}{2} - \frac{\pi}{4} \right) \right]}{\Gamma(n + \frac{1}{2})(a^2 + 1)} \right] ; \quad z > 0, \quad \text{Re}(a) > 0, \quad \eta >> n \geq 0, \quad (2-16)$$

where (2-9) and (2-15) were substituted into (2-11b). The approximation for $\rho_n^{(2)}$, given in (2-16), indicates that forward recurrence will be relatively stable when $\eta >> n \geq 0$.

A comparison between $\rho_n^{(1)}$ and $\rho_n^{(2)}$, given in equations (2-12) and (2-16) respectively, shows that forward recurrence, using (2-8), becomes unstable somewhere in the vicinity of $n = \kappa$. 
Since $\rho_n(1)$ is monotone increasing as $n \to \infty$, it may be possible to calculate $J_{0}(a,z)$ by using backward recurrence, in the form of a Miller algorithm. In the idealized case where infinite precision arithmetic is used, a relative error bound can be obtained by applying (A-15). We find that

$$\left| \frac{J_{0}^{N}(a,z) - J_{0}(a,z)}{J_{0}(a,z)} \right| = \left| \frac{J_{N}(a,z)}{J_{N}^{(h)}(a) \cdot J_{0}(a,z)} \right| = \frac{1}{\rho_{N}^{(1)}} ; \quad N \gg \kappa ,$$  \hspace{1cm} (2-17)

where $\rho_{N}^{(1)}$ is given in (2-12). Reference to (2-12), or (2-13), shows that in the idealized case, Miller's algorithm can be used to compute $J_{0}(a,z)$ to any desired accuracy if $N$ is large enough.

The actual error can be determined by using (A-14). We find that the relative error is given by

$$\left| \frac{J_{0}^{N}(a,z) - J_{0}(a,z)}{J_{0}(a,z)} \right| \leq \left| \frac{J_{0}^{(h)}(a)}{J_{0}(a,z)} \right| \left\{ (1 + \varepsilon)^{N} \right\} \left[ \frac{\varepsilon + |J_{N}(a,z)|}{|J_{N}^{(h)}(a)|} \right]$$

$$+ \varepsilon \sum_{k=0}^{N-1} (1 + \varepsilon)^{k} \left| \frac{J_{k}(a,z)}{J_{k}^{(h)}(a)} \right|$$

$$- (1 + \varepsilon)^{N} \left[ \frac{\varepsilon}{|J_{0}(a,z)|} \right] \frac{\Gamma(\frac{1}{2})}{\Gamma(N+\frac{1}{2})} \frac{a^{2+1}}{2} \frac{N}{\rho_{N}^{(1)}}$$

$$+ \varepsilon \sum_{k=0}^{N-1} \frac{(1+\varepsilon)^{k}}{\rho_{k}^{(2)}} + \varepsilon \sum_{k=\kappa}^{N-1} \frac{(1+\varepsilon)^{k}}{\rho_{k}^{(1)}} ; \quad N \gg \kappa , \quad z > 0, \quad \text{Re}(a) > 0 ,$$  \hspace{1cm} (2-18)

where the previous results (2-9), (2-12) and (2-16) have been used. The $\varepsilon$ in (2-18) is a measure of the maximum error introduced due to finite precision arithmetic.

Equation (2-18) shows that errors due to finite precision arithmetic become significant, and must be included in the analysis, when either $\frac{\varepsilon}{\rho_{k}^{(2)}}$ or $\frac{\varepsilon}{\rho_{k}^{(1)}}$ become large for $0 \leq k \leq \kappa$ or $\kappa < k \leq N-1$, respectively. Reference to (2-16) shows that $\frac{\varepsilon}{\rho_{k}^{(2)}}$
may become large when \( z|a^2+1| \) is large. Therefore, one must be careful while using the Miller algorithm when this occurs. For small to moderate values of \( z|a^2+1| \), the error accumulation due to finite precision arithmetic can usually be ignored and the error bound given in (2-17) is adequate. Errors due to finite precision arithmetic will be discussed more thoroughly in Section 5.

When a sequence of solutions to a recurrence relation is desired, then a direct backward recurrence using (A-10) and (A-11) should be used, but when only one solution is desired, as in our case, the equivalent series representation (A-13) provides some computational advantages.

Substitution of (2-8c) and (2-9) into (A-13) yields

\[
J_{n}^{N}(a,z) = \frac{ze^{-az}}{2} r(n+\frac{1}{2}) \sum_{k=n}^{N-1} z^{k} \left( \frac{a^2+1}{2} \right)^{k-n} \frac{[J_{k}(z) + aJ_{k+1}(z)]}{r(k + \frac{3}{2})};
\]

\[n \geq 0, \quad z > 0, \quad a \in \mathbb{C}.\]

Reference to (2-13) and (2-17) shows that this series converges most rapidly for small to moderate values of \( z|\sqrt{a^2+1}| \).

Due to the presence of the \( z^{k} \) term in (2-19), this series is not a Neumann series. In the book [10], Nielsen classifies series, which can be written in the form

\[
\sum_{n=0}^{\infty} a_{n} z^{\alpha n} J_{n+\nu}(z),
\]

as \textit{Fakultätenreihe}. We will call series, which have this form, factorial-Neumann series. Therefore, since (2-19) is of the form (2-20), we will call (2-19) a convergent factorial-Neumann series.

In this paper, we are interested in the special case \( n = 0 \). The desired factorial-Neumann series expansion for \( J_{0}^{N}(a,z) \) is given by,
\[ J_{e_0}^N(a,z) = r^\left(\frac{3}{2}\right) e^{-az} \sum_{k=0}^{N-1} \left[ \frac{z(a^2+1)}{2} \right]^k \frac{[J_k(z) + aj_{k+1}(z)]}{r(k + \frac{3}{2})} \]

\[ z > 0, \quad a \in \mathbb{C} \]

(2-21)

2.3 Development of an Asymptotic Factorial-Neumann Series Expansion for \( J_{e_0}(a,z) \)

In the last section, we used the recurrence relation (2-6) for \( n \geq 0 \), to develop a factorial-Neumann series which rapidly converges for small to moderate values of \( z|\sqrt{a^2+1}| \) (see (2-21)). In this section, we find that using the recurrence relation (2-6) for \( n \leq -1 \), yields a second factorial-Neumann series which behaves differently than (2-21).

First, we make a change of variables, \( m = -(n+1) \), in (2-6):

\[ J_{e_{-m}}(a, \delta, z) + \frac{1}{d_{-m-1}} J_{e_{-m-1}}(a, \delta, z) = \frac{f_{-m-1}(z)}{d_{-m-1}} \] \[ m \geq 0. \]  

(2-22)

This equation can be rewritten in terms of a new integral,

\[ \hat{J}_{e_{-m}}(a, \delta, z) := \int_{\delta}^{z} e^{-at} t^{-m} J_m(t) dt, \]

(2-23)

by using the relationship between Bessel functions of integer order (C-2):

\[ e_{m}(a, \delta, z) + \hat{d}_{m} \hat{e}_{m+1}(a, \delta, z) = \hat{f}_{m}(z); \quad m \geq 0, \]

(2-24a)

where \( \hat{d}_{m} = \frac{1}{d_{m}} = -\left( \frac{2m+1}{a^2+1} \right) \), and

\[ \hat{f}_{m}(z) = \frac{e^{-az}}{(a^2+1)z^m} \left[ J_{m+1}(z) - a J_{m}(z) \right]. \]

(2-24b)

(2-24c)

The homogeneous solution for (2-24) is obtained by using (A-4),

\[ \hat{J}_{e_{m}}^{(h)}(a) = \prod_{k=0}^{m-1} [-\hat{d}_{k}]^{-1} = [J_{e_{m}}^{(h)}(a)]^{-1} \]

\[ = (\frac{a^2+1}{2})^m \frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{3}{2})} \]

(2-25a)
\[ - \frac{1}{\sqrt{2}} \left[ \frac{e(a^2+1)}{2m} \right]^m \rightarrow m \rightarrow \infty, \quad (2-25b) \]

where the results given in (2-9) and (2-10) were used.

Using (C-8a), it is easy to show that choosing

\[
\delta = \begin{cases} 
\infty & \text{Re}(a) \geq 0 \\
-\infty & \text{Re}(a) < 0 
\end{cases} \quad (2-26a) \]

\[
(2-26b) \]

satisfies the requirement that \( \hat{f}_m(\delta) = 0 \) for \( m \geq 0 \).

The stability index for forward recurrence defined in (A-7) is given by

\[
\hat{a}(k,m) = \frac{\hat{J}_e(a,\delta,\pi) \hat{J}_e(h)(a)}{\hat{J}_e(a,\delta,\pi) \hat{J}_e(h)(a)} = \frac{\hat{\rho}_m}{\hat{\rho}_k}, \quad (2-27a) \]

\[
(2-27b) \]

where

\[
\hat{\rho}_m = \frac{\hat{J}_e(0,\delta,\pi) \hat{J}_e(h)(a)}{\hat{J}_e(0,\delta,\pi)} \quad (2-27b) \]

for the recurrence relation (2-24). The asymptotic behavior of \( \hat{\rho}_m \) for large \( m \), is obtained by substituting (2-25b) and (D-14) into (2-27b):

\[
\hat{\rho}_m(1) = \begin{cases} 
& e^{2\text{Re}(a)} m \hat{J}_e(a,\delta,\pi) \quad m \rightarrow \infty, \ z > 0, \ a \neq 0 \\
& \hat{J}_e(0,\delta,\pi) \quad m \rightarrow \infty, \ z > 0, \ a = 0 \end{cases} \quad (2-28a) \]

\[
(2-28b) \]

Condition (A-9) is satisfied when \( a \in D_a \), where

\[
D_a := \{ a : |a^2+1| \geq 1 \cap a \neq 0 \}. \quad (2-29) \]

Therefore, forward recurrence using (2-24) is unstable if \( a \in D_a \).

It is also interesting to look at the behavior of \( \hat{\rho}_m \) for values of \( m < z \). This behavior is obtained for the case \( \text{Re}(a) \geq 0 \) by substituting (2-25a) and (D-19) into (2-27b):
\[ \hat{\rho}_m(2) = \left| \frac{e^{az(a^2+1)}[-z(a^2+1)]^m}{\Gamma(m+\frac{1}{2})[a \cos(z - \frac{m\pi}{2} - \frac{\pi}{4}) - \sin(z - \frac{m\pi}{2} - \frac{\pi}{4})]} \right| ; \]

\( z > 0, \quad \text{Re}(a) \geq 0, \quad \eta \gg m \gg 0, \)

where \( \eta = \min(|z|, |z|a \pm i|) \).

This equation shows that forward recurrence is unstable for large values of \( z|a^2+1| \) in the region where (2-30) holds.

Since \( \hat{\rho}_m(1) \) is monotone increasing as \( m \to \infty \) for \( a \in D_a \), it may be possible to use a Miller algorithm to calculate \( \hat{J}_e_0(a, \delta, z) \) in this region of the \( a \)-plane. In the idealized case where infinite precision arithmetic is assumed, the relative error bound given in (A-15) yields

\[ \left| \frac{\hat{J}_e_0^M(a, \delta, z) - \hat{J}_e_0(a, \delta, z)}{\hat{J}_e_0(a, \delta, z)} \right| = \left| \frac{\hat{J}_e_0^M(a, \delta, z)}{\hat{J}_e_0(a, \delta, z)} \right| = \left| \frac{1}{\hat{\rho}_M} \right| ; \quad M \to \infty, \]  

where \( \hat{\rho}_M(1) \) is given in (2-28). Therefore, in this idealized case, \( \hat{J}_e_0(a, \delta, z) \) can be computed to any number of significant digits by using Miller's algorithm when \( a \in D_a \).

When \( a \in D_a \), a series representation for \( \hat{J}_e_0(a, \delta, z) \) is obtained by substituting the homogeneous solution (2-25a), and the forcing function (2-24c), into (A-13):

\[ \hat{J}_e_0^M(a, \delta, z) = \frac{e^{-az}}{(a^2+1)\Gamma(m+\frac{1}{2})} \sum_{k=m}^{M-1} \frac{[\frac{2}{a^2+1}]^{k-m}}{z^k} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} [J_{k+1}(z) - aJ_k(z)] ; \]

\( z > 0, \quad a \in D_a. \)

The integral of interest, \( J_e_0(a, z) \), is related to \( \hat{J}_e_0(a, \delta, z) \) by the identity
\[ J_0(a, z) = -\hat{J}_0(a, \delta, 0) + \hat{J}_0(a, \delta, z), \]  
\[ (2-33) \]

where \( \delta \) is defined in (2-26). The first integral, \( \hat{J}_0(a, \delta, 0) \), is a special case of the integral in (2-14). If \( \text{Re}(a) \geq 0 \) and \( a \neq \pm i \), then

\[
\hat{J}_0(a, \infty, 0) = \int_0^\infty e^{-at} J_0(t) dt = -\frac{1}{\sqrt{a^2+1}}.
\]  
\[ (2-34a) \]

When \( \text{Re}(a) < 0 \), the change of variables \( \tau = -t \) gives

\[
\hat{J}_0(a, -\infty, 0) = \int_{-\infty}^0 e^{-at} J_0(t) dt = \int_0^\infty e^{a\tau} J_0(\tau) d\tau = \frac{1}{\sqrt{a^2+1}}.
\]  
\[ (2-34b) \]

The square roots in (2-34) are defined as

\[ \text{Re}(\sqrt{a^2+1}) \geq 0. \]  
\[ (2-35) \]

A convergent factorial-Neumann series is obtained by combining the results in (2-32), (2-33), and (2-34), giving

\[
J_0^M(a, z) = \frac{1}{[a^2+1]^\frac{1}{2}} + \frac{e^{-az}}{\Gamma(\frac{1}{2})(a^2+1)} \sum_{k=0}^{M-1} \left[ \frac{2}{z(a^2+1)} \right]^k \Gamma(k+\frac{1}{2}) [J_{k+1}(z) - aJ_k(z)];
\]  
\[ (2-36) \]

where the proper branch cut for the square root is given by

\[ \text{Re}(\sqrt{a^2+1}) \geq 0 \text{ if } \text{Re}(a) \geq 0, \text{ or } \]
\[ \text{Re}(\sqrt{a^2+1}) < 0 \text{ if } \text{Re}(a) < 0. \]
\[ (2-37a) \]

This branch cut is shown pictorially in Fig. 2-1.

When \( |a^2+1| \geq 1 \), but not too large, the series (2-36) won't yield an accurate approximation for \( J_0(a, z) \) until \( M \gg z \). Therefore, for large values of \( z \), a large number of terms may be required.

Some insight into the behavior of this series is obtained by looking at the special case \( |a^2+1| = 1 \), where \( a \neq 0 \). If \( z \) is large, then the magnitude
Fig. 2-1: Branch cut for the convergent series expansions.

Fig. 2-2: Branch cut for the asymptotic factorial-Neumann series expansion.
of the first few terms in the series will decrease rapidly due to the inverse powers in \( z \). At some point, the behavior of the gamma function in the numerator will become dominant, and the terms will start to increase in magnitude. When \( k \) becomes much larger than \( z \), the behavior of the Bessel functions dominates, and once again the terms decrease in magnitude. Now, if \( z|a^2+1| \) is large, what if we truncate the series when the magnitude of the terms reaches the first local minimum? This question can be answered by looking at the relative error bound (2-31a) for \( M \ll \eta = \text{Min} \left( z, z|a+i| \right) \). We find that

\[
\left| \frac{\hat{J}_0^M(a,\infty,z) - \hat{J}_0(a,\infty,z)}{\hat{J}_0(a,\infty,z)} \right| = \frac{1}{\rho_\text{m}^{(2)}} ; \quad \eta \gg M, \quad z > 0, \quad \text{Re}(a) \geq 0, \quad (2-38)
\]

where \( \rho_\text{m}^{(2)} \) is given in (2-30). Reference to (2-30) and (2-38) shows that for large values of \( z|a^2+1| \), it may be possible to obtain an accurate approximation for \( \hat{J}_0(a,\infty,z) \) by using the first few terms in the series (2-32).

Therefore, the factorial-Neumann series (2-36) can be used as an asymptotic series for large values of \( z|a^2+1| \), provided that \( z > 0 \) and \( \text{Re}(a) \geq 0 \).

For the case \( \text{Re}(a) \geq 0 \), we found that \( \hat{J}_m(a,\infty,z) \) can be approximated by (D-19) for large \( \eta \). We obtained this approximation by replacing the Bessel function in \( \hat{J}_m(a,\infty,z) \) with its asymptotic expansion for large argument, and then we integrated the result.

When \( z \) is a large positive number and \( \text{Re}(a) < 0 \), the Bessel function in \( \hat{J}_m(a,-\infty,z) \) can't be replaced by its asymptotic expansion, since the integration variable now ranges between \( z \) and \(-\infty\). If we rewrite \( \hat{J}_m(a,-\infty,z) \) as

\[
\hat{J}_m(a,-\infty,z) = \hat{J}_m(a,-\infty,z_0) + \hat{J}_m(a,z_0,z) \quad \quad (2-39)
\]

where \( z < z_0 \), then we can use (D-19) to show that
\[
\hat{\text{J}}_e_m(a, -\infty, z) - C_m(a) - \sqrt{z} \frac{\text{e}^{-az}}{\pi(a^2 + 1)z^{m+\frac{1}{2}}} \left[a \cos(z - \frac{m\pi}{2} - \frac{\pi}{4}) \right. \\
\left. - \sin(z - \frac{m\pi}{2} - \frac{\pi}{4})\right]; \quad z > 0, \quad a \leq 0, \quad \eta \gg m > 0,
\]

(2-40)

where \(C_m(a)\) is a function that is independent of \(z\).

Using (2-25a), (2-27b), and (2-40), we find that

\[
\rho_m(2) \frac{\hat{\text{J}}_e_0(a, -\infty, z)}{\Gamma(m+\frac{1}{2}) C_m(a) \left[\frac{2}{a^2+1}\right]^m} - \sqrt{\frac{2}{z}} \frac{\text{e}^{-az}}{\pi(a^2+1)} \left[\frac{2}{z(a^2+1)}\right]^m \left[a \cos(z - \frac{m\pi}{2} - \frac{\pi}{4}) - \sin(z - \frac{m\pi}{2} - \frac{\pi}{4})\right]; \quad z > 0, \quad \text{Re}(a) \leq 0, \quad \eta \gg m \geq 0.
\]

(2-41)

If we try to use (2-32) as an asymptotic expansion for \(\hat{\text{J}}_e_0(a, -\infty, z)\), then the relative error (see (2-38)) will be given by \(\frac{1}{\rho_m(2)}\), where \(\rho_m(2)\) is given in (2-41). Reference to (2-41) shows, that because of the function \(C_m(a)\), large \(z|a^2+1|\) may not make the relative error small. Therefore, (2-32) may not be used as an asymptotic for expansion for \(\hat{\text{J}}_e_0(a, -\infty, z)\) when \(\text{Re}(a) \leq 0\) and \(\eta\) is large.

The series given in (2-32) is an asymptotic expansion for some function, therefore if we let (see (2-32))

\[
\text{J}_e^m_0(a, z) - G(a) \sim \frac{\text{e}^{-az}}{\Gamma(\frac{1}{2})(a^2+1)} \sum_{k=0}^{m-1} \left[\frac{2}{z(a^2+1)}\right]^k \Gamma(k+\frac{1}{2})[\text{J}_k(z) - a\text{J}_k(z)],
\]

(2-42)

then all we need to do is find the function \(G(a)\).

The integral \(\text{J}_e_0(a, z)\) is continuous across the boundary \(\text{Re}(a) = 0\), therefore the asymptotic expansion for \(\text{J}_e_0(a, z)\) must also be continuous at this boundary. We previously found that (2-36) can be used as an asymptotic expansion for large \(z|a^2+1|\) when \(z > 0\) and \(\text{Re}(a) \geq 0\). Therefore, the function \(G(a)\) can be determined by equating (2-36) and (2-42) at the boundary \(\text{Re}(a) = 0\). We find that

\[
G(a) = \frac{1}{[a^2+1]^{\frac{1}{2}}}
\]

(2-43)
where the branch cut for the square root, must be defined as in Fig. 2-2.

The branch cut is defined analytically by

\[
\text{Re}(z) < 0; \quad \text{Re}(a) < 0 \cap |\text{Im}(a)| > 1
\]

\[
|z(a^2+1)z| > 0; \quad \text{otherwise.}
\]

(2-44a)

The desired asymptotic factorial-Neumann series expansion for $J_{\nu}(a,z)$ is given by

\[
J_{\nu}^{M}(a,z) = \frac{1}{[a^2+1]^{\frac{1}{8}}} + \frac{e^{-az}}{\Gamma(\frac{1}{2})(a^2+1)} \sum_{k=0}^{M-1} \frac{2^k}{z(a^2+1)^{\frac{k}{2}}} \Gamma(k+\frac{1}{2})[J_{k+1}(z) - aJ_k(z)];
\]

\[
z|a^2+1| \to \infty, \quad z > 0, \quad a \in \mathbb{C},
\]

(2-45)

where the proper branch of the square root is defined in (2-44).

It is interesting to compare the asymptotic expansion given in (2-45) with the convergent series given in (2-36). We find that the first term in (2-45) is not the same as in (2-36) when $\text{Re}(a) < 0$ and $|\text{Im}(a)| < 1$. In fact, they differ by a minus sign.

2.4 Development of an Asymptotic Series Expansion for $Y_{\nu}(a,z)$

In this section, we will develop an asymptotic expansion for the incomplete Lipschitz-Hankel integral $Y_{\nu}(a,z)$. This function, which is a special case of (1-1), is defined as

\[
Y_{\nu}(a,z) := \int_{0}^{Z} e^{-at} t^{\nu} Y_{\nu}(t) dt,
\]

(2-46)

where $Y_{\nu}(t)$ is the Bessel function of the second kind.

The procedure that we use to find the asymptotic expansion for $Y_{\nu}(a,z)$ is analogous to the procedure used to obtain the asymptotic expansion for $J_{\nu}(a,z)$. Therefore, we can use some of the results found in §2.1 and §2.3.

First, we define an integral related to (2-46),
\[ \mathcal{Y}_e(a, \delta, z) := \int_{\delta}^{z} e^{-at} t^\nu \mathcal{Y}_\nu(t) dt. \] (2-47)

Using the same techniques as in §2.1, we find that \( \mathcal{Y}_e(a, \xi, z) \) satisfies a first-order nonhomogeneous recurrence relation (see (2-6)),

\[ \mathcal{Y}_e(a, \delta, z) + d_n \mathcal{Y}_e(a, \delta, z) = f_n(z); \quad n = 0, \pm 1, \pm 2, \ldots, \] (2-48a)

where

\[ d_n := \frac{a^2+1}{2n+1}, \] (2-48b)

\[ f_n(z) := \frac{e^{-az}}{2n+1} \left[ \mathcal{Y}_n(z) + a \mathcal{Y}_{n+1}(z) \right], \] (2-48c)

and where we must choose \( \delta \) so that \( f_n(\delta) = 0 \).

If we define a new integral,

\[ \mathcal{Y}_e(a, \delta, z) := \int_{\delta}^{z} e^{-at} t^{-m} \mathcal{Y}_m(t) dt, \] (2-49)

then we find that (see (2-24)),

\[ \hat{\mathcal{Y}}_e(a, \delta, z) + \hat{d}_m \hat{\mathcal{Y}}_e(a, \delta, z) = \hat{f}_m(z); \quad m \geq 0, \] (2-50a)

where

\[ \hat{d}_m = -\frac{2m+1}{a^2+1}, \] (2-50b)

and

\[ \hat{f}_m(z) = \frac{e^{-az}}{(a^2+1)z^m} \left[ \mathcal{Y}_{m+1}(z) - a \mathcal{Y}_m(z) \right]. \] (2-50c)

The homogeneous solution of (2-50a) is the same as the homogeneous solution of (2-24a), and is given in (2-25).

Reference to (C-8) shows that the Bessel functions of the first and second kind have similar asymptotic behavior for large argument. Therefore, \( \delta \) can be chosen as in (2-26). Using the same techniques as in §2.3, we can show that
\[
\hat{\gamma}_m(a, z) - \frac{e^{-az}}{(a^2 + 1)\Gamma(m + \frac{1}{2})} \sum_{k=m}^{M-1} \left[ \frac{2}{a^2 + 1} \right] \frac{\Gamma(k + \frac{1}{2})}{z^k} [Y_{k+1}(z) - aY_k(z)];
\]

(2-51)

\[z|a^2 + 1| \to \infty, \ z > 0, \ \text{Re}(a) \geq 0.\]

Unlike the series expansion for \(\hat{\gamma}_m(a, \sigma, z)\) given in (2-32), the expansion for \(\hat{\gamma}_m(a, \delta, z)\) doesn't converge for any values of the variables \(a\) and \(z\).

The incomplete Lipschitz-Hankel integral \(\gamma_0(a, z)\) is related to \(\hat{\gamma}_m(a, \omega, z)\) by the identity

\[\gamma_0(a, z) = -\hat{\gamma}_0(a, \omega, 0) + \hat{\gamma}_0(a, \omega, z); \ z > 0, \ \text{Re}(a) \geq 0. \quad (2-52)\]

The first integral on the right-hand side of (2-52) is known in closed form (C-15). Therefore, we obtain

\[
\gamma_0(a, z) = -\frac{\ln(a + \sqrt{a^2 + 1})}{\sqrt{a^2 + 1}} + \frac{e^{-az}}{\Gamma(\frac{1}{2})(a^2 + 1)} \sum_{k=0}^{M-1} \left[ \frac{2}{z(a^2 + 1)} \right] k \Gamma(k + \frac{1}{2}) [Y_{k+1}(z) - aY_k(z)];
\]

(2-53)

\[z|a^2 + 1| \to \infty, \ z > 0, \ \text{Re}(a) \geq 0,\]

where \(\text{Re}(\sqrt{a^2 + 1}) \geq 0\) and the principal value for the logarithm is used.

We can obtain the asymptotic expansion for the region \(\text{Re}(a) < 0\) by using the same techniques as in §2.3. We find that eqn. (2-53) still holds if the proper branch for the square root is defined as in (2-44), and as shown pictorially in Fig. 2-2.
3. DEVELOPMENT OF A NEUMANN SERIES EXPANSION FOR \( J_{\nu}(a, z) \)

A very useful Neumann series expansion for \( J_{\nu}(a, z) \) is given by Agrest [5]. This series converges for \( z, a \in \mathbb{C} \), where \( a \neq \pm i \). In this section, we give a derivation for this Neumann series expansion.

The incomplete Lipschitz-Hankel integral \( J_{\nu}(a, z) \) is defined in (1-3). For \( \text{Re}(a) > 0 \), \( J_{\nu}(a, z) \) can be written as

\[
J_{\nu}(a, z) = \int_{0}^{\infty} e^{-at} J_{\nu}(t) dt - e^{-az} \int_{0}^{\infty} e^{-a(t-z)} J_{\nu}(t) dt
\]

\[
= \int_{0}^{\infty} e^{-at} J_{\nu}(t) dt - e^{-az} \int_{0}^{\infty} e^{-a\tau} J_{\nu}(\tau + z) d\tau ; \quad (3-1)
\]

where the change of variables \( \tau = t - z \) has been applied. The addition theorem (C-12b) can be applied to the Bessel function \( J_{\nu}(\tau + z) \), and then the resulting integrals can be evaluated using (C-14b). This yields

\[
J_{\nu}(a, z) = \int_{0}^{\infty} e^{-at} J_{\nu}(t) dt - e^{-az} \sum_{k=0}^{\infty} (-1)^{k} \epsilon_{k} J_{\nu}(z) \int_{0}^{\infty} e^{-a\tau} J_{\nu}(\tau) d\tau
\]

\[
= \frac{1}{\sqrt{a^2 + 1}} \left\{ 1 + e^{-az} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \epsilon_{k} J_{\nu}(z)}{[\sqrt{a^2 + 1} + a]^{k}} \right\} ; \quad z > 0, \text{ Re}(a) > 0, \quad (3-2)
\]

where \( \text{Re}(\sqrt{a^2 + 1}) > 0 \) and \( \epsilon_{k} \) is defined as

\[
\epsilon_{k} = \begin{cases} 
1; & k = 0 \\
2; & k = 1, 2, 3, \ldots 
\end{cases} \quad (3-3)
\]

The first term in this expansion is the same as the first term in the convergent factorial-Neumann series given in (2-36). We previously determined that the proper branch cut, for the square root in this first term, is given by (2-37), and is shown pictorially in Fig. 2-1. Since we know the proper branch cut for the squareroot, we can analytically continue the series given
in (3-2) into the rest of the a-plane. Re(a) ≤ 0, a ≠ ±i.

Therefore, the desired Neumann series expansion for \( J_0(a, z) \) is given by

\[
J_0(a, z) = \frac{1}{[a^2 + 1]^{1/2}} \left\{ 1 + e^{-az} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} e_k J_k(z)}{[a^2 + 1]^{k/2} + a^k} \right\}; \quad z > 0, \quad a \in \mathbb{C}, \quad a \neq ±i, \tag{3-4}
\]

where the branch cut for the square root is defined in (2-37).

A Neumann series expansion for \( J_0(a, z) \) can also be found in an earlier paper by Maximon ([7], eqn. (31')). By using the generating series (C-11b), and some algebra, it can be shown that Maximon's expansion is equivalent to (3-4).
4. OTHER EXPANSIONS FOR $J_0(a,z)$

In this section, we look at some of the other expansions for $J_0(a,z)$ that are given in the literature. We found that none of these expansions provide any significant computational advantages over the expansions developed in Sections 2 and 3, so they were not included in our algorithm.

First, we will look at a convergent series expansion for $J_0(a,z)$ that is given in the paper by Agrest and Rikenglaz ([4], eqn.(6)). This expansion is obtained by replacing the Bessel function in the integrand of (1-3), by the series expansion (C-5), and then integrating term-by-term. This procedure yields an expansion in terms of incomplete gamma functions $\gamma(n,x)$, which can be expressed in the form:

$$J_0(a,z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \frac{\gamma(2k+1, az)}{a^{2k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \frac{g_{2k}(a,z)}{2^n2^n}, \quad (4-1a)$$

where

$$g_n(a,z):=\frac{\gamma(n+1, az)}{a^{n+1}}; \quad n \geq 0. \quad (4-1b)$$

Incomplete gamma functions satisfy the recurrence relation (C-26), therefore $g_n(a,z)$ will satisfy

$$g_n(a,z) - \frac{a}{n+1} g_{n+1}(a,z) = \frac{z^{n+1} e^{-az}}{(n+1)!}; \quad n \geq 0. \quad (4-2)$$

The behavior of this recurrence relation can be obtained by applying the techniques in Appendix A. The homogeneous solution for (4-2) is found using (A-4):

$$g_n^{(h)}(a) = \prod_{k=0}^{n-1} \frac{k+1}{a} = \frac{\Gamma(n+1)}{a^n}; \quad n \geq 1. \quad (4-3)$$
The index of stability for forward recurrence is given by (see (A-7))

$$\alpha(k,n) = \left| \frac{g_k(a,z)g_n^{(h)}(a)}{g_n(a,z)g_k^{(h)}(a)} \right| = \frac{\rho_n}{\rho_k},$$

(4-4a)

where

$$\rho_n = \left| \frac{g_0(a,z)g_n^{(h)}(a)}{g_n(a,z)} \right|,$$

(4-4b)

and

$$g_0(a,z) = \frac{1 - e^{-az}}{a}.$$

(4-5)

Using equations (C-24), (C-25), and (4-1b), we find that

$$g_n(a,z) = e^{-az} z^{n+1} \Gamma(n+1) \sum_{k=0}^{\infty} \frac{(az)^k}{\Gamma(n+2+k)}.$$

(4-6)

Therefore, by using (4-3), (4-4b), (4-5), and (4-6), we find that

$$\rho_n = \left| \frac{e^{az} - 1}{(za)^{n+1}} \sum_{k=0}^{\infty} \frac{(az)^k}{\Gamma(n+2+k)} \right| \to \infty; \quad n \to \infty.$$

(4-7)

Since the condition in (A-9) is satisfied, forward recurrence using (4-2) is unstable.

In ([4], p. 210), a recurrence relation, which is equivalent to (4-2), is used in the forward direction to compute the functions in the expansion. When using (4-2) in the forward direction, one must be careful since the recurrence will eventually become unstable. It can be shown that the sequence of functions $g_n(a,z)$ can be computed using a backward recurrence algorithm.
It is interesting to compare the rate of convergence for the incomplete gamma function expansion (4-1) with the convergent factorial-Neumann series expansion (2-21). Using (4-1a), (4-6), and (C-21), we find that when 
\[ k \gg |az|, \] the \( k \)th term in the incomplete gamma function expansion is of the order
\[ \tau_k^{(1)} := \frac{(-1)^k}{(k!)^2 2^{2k}} g_{2k}(a,z) \]  
\[ = O \left( k^{-1} \left[ \frac{ze}{2k} \right]^{2k+1} \right); \quad k \gg z|a|. \]  
(4-8a)  
(4-8b)
Likewise, using (2-21), (C-10a), and (C-21), we find that the \( k \)th term in the factorial-Neumann series is given by
\[ \tau_k^{(2)} := \frac{\pi}{2} \left[ \frac{ze}{2k} \right]^{2k} \frac{\gamma_k(z+a\gamma_k(z))}{\Gamma(k+3/2)} \]  
\[ = O \left( k^{-3/2} \left[ \frac{zea^{2+1}}{2k} \right]^{2k} \right); \quad k \gg z. \]  
(4-9a)  
(4-9b)
Therefore, if \(|a^{2+1}| < 1\), then the convergent factorial-Neumann series (2-21) will probably converge faster than the incomplete gamma function expansion (4-1).

When \( z|a^{2+1}| \) is large, we found in §2.2 that rounding errors due to finite precision arithmetic couldn't be ignored when using (2-21). We can look at this problem in a second way. When \( \eta \) is large, we can approximate \( \Gamma_0(a,z) \) by (2-15):
\[ \Gamma_0(a,z) = \frac{1}{\sqrt{a^{2+1}}} - e^{-az} \frac{a(z - \frac{\pi}{4}) - \sin(z - \frac{\pi}{4})}{\pi z (a^{2+1})}; \eta \gg 0, \]  
(4-10)
where \( \eta := \min(z, z|a^{2+1}|) \). When \( z \gg k \), the \( k \)th term in the factorial-Neumann series, (4-9a), can be approximated by using (C-8a):
\[ \left| \tau_k^{(2)} \right| \leq \frac{\sqrt{e}}{2} e^{\text{Re}(a) \left[ \frac{z|a^{2+1}|}{2} \right] k} \frac{\max(1, |a|)}{\Gamma(k+3/2)}; \quad z \gg k. \]  
(4-11)
Comparing (4-11) to (4-10), we find that for large values of \( z|a^{2+1}| \), individual terms in the factorial-Neumann series expansion (2-21) may be much
larger than $J_{0}(a,z)$. Therefore, accuracy is lost when using (2-21), since the large terms are added together to yield a smaller result.

The same type of analysis can be applied to the incomplete gamma function expansion. When $z|a|$ is large, $g_{n}(a,z)$ can be approximated by using (C-23) and (C-27):

$$g_{n}(a,z) = \frac{\Gamma(n+1)}{a^{n+1}} - \frac{\Gamma(n+1, az)}{a^{n+1}}$$

$$- \frac{\Gamma(n+1)}{a^{n+1}} - \frac{z^{n}e^{-az}}{a}; \quad z|a| >> n.$$  \hspace{1cm} (4-12a)

Therefore,

$$|\tau^{(1)}_{k}| - \left| \frac{2\Gamma(2k+1)}{(k!)^{2}} \frac{e^{-az}}{2^{2k+1}} \frac{\left[ \frac{z}{2} \right]^{2k}}{a^{(k!)^{2}}} \right|; \quad z|a| >> k.$$  \hspace{1cm} (4-13)

This shows that when $z/2$ is large, accuracy may be lost when using (4-1) since large terms are added together yielding a smaller result. Comparison of (4-13) with (4-11) shows that when $z/2$ is large, and $|a^{2}+1| < 1$, accumulation of rounding errors is a more severe problem in (4-1) than in (2-21).

A similar type of analysis can be used to show that when $z$ is large, and $|a^{2}+1| > 1$, the Neumann series expansion given in (3-4) is better suited for computational purposes than (4-1).

In conclusion, we find that the incomplete gamma function expansion (4-1) doesn't offer any significant computational advantages when compared with the expansions given in (2-21), (2-45), and (3-4).

The paper by Agrest and Rikenglaz (see [4], (4) and (5)) also contains two asymptotic expansions for $J_{0}(a,z)$, for large $z|a^{2}+1|$. Since $J_{0}(a,z)$ must have a unique asymptotic expansion as $z|a^{2}+1| \to \infty$, the asymptotic expansions given in [4] must be equivalent to (2-45). We found that the asymptotic factorial-Neumann series (2-45) is better suited for computational purposes than either of the asymptotic expansions obtained by Agrest and Rikenglaz.
In the paper (see [8], eqn. 3.2), Amos and Burgmeier give a second order nonhomogeneous recurrence relation which can be used to obtain $J_{\nu}(a,z)$. For the special case that we are interested in, this recurrence relation is equivalent to the recurrence relation given by Agrest (see [5], eqn. 1.3). Agrest shows that using this second order recurrence relation is equivalent to summing the Neumann series (3-4).
5. AN ALGORITHM FOR COMPUTING \(J_{\alpha}(a,z)\)

In this section, we outline an algorithm which efficiently computes \(J_{\alpha}(a,z)\) to a user defined number of significant digits (SD). The Fortran source code for this algorithm is given in Appendix F.

We use the three series expansions given in (2-21), (2-45), and (3-4) to compute the incomplete Lipschitz-Hankel integral \(J_{\alpha}(a,z)\) for \(z > 0\) and \(a \in \mathbb{C}\). When \(z < 0\), we apply the identity (1-4).

The three expansions are rewritten here for convenience.

I. Convergent Factorial-Neumann Series Expansion

\[
J_{\alpha}(a,z) = \Gamma\left(\frac{3}{2}\right) z e^{-az} \sum_{k=0}^{N-1} \left[\frac{z(a^2+1)}{2}\right]^k \frac{[J_k(z) + aJ_{k+1}(z)]}{\Gamma(k + \frac{3}{2})} ; \quad z > 0, \quad a \in \mathbb{C} 
\]

II. Asymptotic Factorial-Neumann Series Expansion

\[
J_{\alpha}(a,z) = \frac{1}{[a^2+1]^\frac{1}{2}} + \frac{e^{-az}}{\Gamma(\frac{1}{2})(a^2+1)} \sum_{k=0}^{N-1} \left[\frac{2}{z(a^2+1)}\right]^k \Gamma(k + \frac{1}{2})[J_{k+1}(z) - aJ_k(z)]; \\
where \(z|a^2+1| \to \infty, \quad z > 0, \quad a \in \mathbb{C},\)
\]

III. Neumann Series Expansion

\[
J_{\alpha}(a,z) = \frac{1}{[a^2+1]^\frac{1}{2}} \left\{ 1 + e^{-az} \sum_{k=0}^{N-1} \frac{(-1)^k \varepsilon_k J_k(z)}{[[a^2+1]^\frac{1}{2} + a]^k} \right\} ; \quad z > 0, \quad a \in \mathbb{C}, \quad a \neq \pm i,
\]

where the branch cut for the square root is defined in (2-37) and \(\varepsilon_k\) is defined in (3-3).

The three expansions have very different properties, as we have previously shown. Now, we must determine which expansion to use for a given set of inputs \(a, z,\) and \(\text{SD}\).

In all three of these expansions, we need to compute a sequence of Bessel functions \(\{J_k(z)\}\). We use one of two different methods to compute them depending on the values of the inputs \(a, z,\) and \(\text{SD}\).
In the first method, we use Hankel's asymptotic expansion (C-9) to compute the starting functions \( J_0(z) \) and \( J_1(z) \), and then we use the recurrence relation (C-3) in the forward direction to obtain the higher order Bessel functions. Since an asymptotic expansion is used to obtain the starting functions, this method can only be used when \( z \) is large. We determined, using numerical tests, that Hankel's asymptotic expansion approximates the starting functions to SD significant digits if

\[
z > ZJASY = SD + 4. \quad (5-4)
\]

This method can only be used to compute the sequence \( \{J_k(z)\} \) for values of \( k \) that are less than \( z \), since forward recurrence using (C-3) becomes unstable for \( k > z \). (It may be possible to compute a few Bessel functions of order greater than \( k \), but the accuracy will start falling off rapidly).

If forward recurrence can't be used, then we generate the sequence of Bessel functions using backward recurrence. This method is outlined in Appendix B.

For large values of \( z \) and \( z|a^2+1| \), we would like to use the asymptotic factorial-Neumann series expansion (5-2) to approximate \( J_0(a,z) \). First, we must determine whether this expansion will yield an accurate result, for a given set of inputs.

Since \( z \) is large, we will use forward recurrence to compute the sequence of Bessel functions. Therefore, we must truncate the series at some \( k \), where \( k \leq z \). This means that the Bessel functions in (5-2) will have sinusoidal behavior. Reference to (5-2) shows that we will obtain the best approximation for \( J_0(a,z) \) when the factor,

\[
\left( \frac{2}{z(a^2+1)} \right)^k r(k + \frac{1}{2}), \quad (5-5)
\]

reaches the first minimum. Using the results given in (E-4), we find that (5-5) reaches a minimum when
\[ k = k_{\text{max}} = \frac{z|a^2 + 1| - 1}{2} \]  \hspace{1cm} (5-6) 

If \( k_{\text{max}} > z \), then the accuracy is limited by the number of computed Bessel functions. For this case, we will set

\[ k_{\text{max}} = z. \]  \hspace{1cm} (5-7) 

We can use the series (5-2) to obtain an approximation for \( J_0(a,z) \) that is accurate to \( SD \) significant digits if the \( k^{th} \) term, where \( k = k_{\text{max}} \), is small enough. Therefore, the following inequality must hold:

\[ \frac{1}{2} \times 10^{-SD} |J_0(a,z)e^{az}| > \left| \frac{\Gamma(k_{\text{max}} + \frac{1}{2})}{\Gamma(\frac{1}{2})(a^2 + 1)} \left[ \frac{2}{z(a^2 + 1)} \right]^{k_{\text{max}}} [J_{k_{\text{max}}+1}(z) - aJ_{k_{\text{max}}}(z)] \right|. \]  \hspace{1cm} (5-8) 

For large values of \( \eta \), we can approximate (5-8) by using (C-8a) and (C-21), yielding

\[ \frac{1}{2} \times 10^{-SD} |J_0(a,z)e^{az}| > \left| \frac{2}{\sqrt{\pi z}(a^2 + 1)} \left[ \frac{2k_{\text{max}} + 1}{ze(a^2 + 1)} \right]^{k_{\text{max}}} \text{Max}(1,|a|) \right|, \]  \hspace{1cm} (5-9) 

where \( \eta = \text{Min}(z, z|a \pm i|) \).

When \( z > k_{\text{max}} \), (5-9) can be further simplified by substituting (5-6) into (5-9):

\[ \frac{1}{2} \times 10^{-SD} |J_0(a,z)e^{az}| > \left| \frac{2^{\frac{z|a^2 + 1|}{\sqrt{\pi z}(a^2 + 1)}}}{\text{Max}(1,|a|)} \right|. \]  \hspace{1cm} (5-10) 

An approximation for \( |J_0(a,z)| \) can be obtained by using the previous result (2-15):

\[ |J_0(a,z)e^{az}| - \left| \frac{e^{az}}{[a^2+1]^\frac{1}{2}} - \sqrt{\frac{2}{\pi z}} \frac{1}{(a^2+1)} [a \cos(z - \frac{\pi}{4}) - \sin(z - \frac{\pi}{4})] \right|; \]  \hspace{1cm} (5-11) 

\( \eta \gg 0. \)
Reference to equations (5-9), (5-10), and (5-11) shows that the asymptotic factorial-Neumann series expansion (5-2) is most useful when \( z|a^2+1| \) is large.

We use this expansion when (5-9) holds, and \( z > ZJASY. \)

When \( z > ZJASY, \) but (5-9) isn't satisfied, we still prefer to use forward recurrence to compute the sequence of Bessel functions, but now we would like to use the convergent factorial-Neumann series expansion (5-1). Once again, we must first determine whether this expansion will provide the desired accuracy.

Since we are using forward recurrence to obtain \( \{J_k(z)\} \), the series (5-1) must be truncated at some \( k \leq k_{\text{int}} = \text{Int}(z) \). Therefore, we can use the series expansion (5-1) if

\[
\frac{1}{2} \times 10^{-SD} |J_{e_0}(a_1) e^{az}| > \left| \frac{z(a^2+1)}{\Gamma\left(\frac{3}{2}\right)} \right|^k_{\text{int}} \frac{\left[ J_k(z) + a J_{k+1}(z) \right]}{\Gamma(k_{\text{int}} + \frac{3}{2})}.
\]

Once again, we approximate (5-12) by applying (C-Ba) and (C-21):

\[
\frac{1}{2} \times 10^{-SD} |J_{e_0}(a_1) e^{az}| > \left| \sqrt{\frac{z}{\pi}} \frac{e^{3/2}}{2k_{\text{int}}+3} \left[ \frac{e^{(a^2+1)}}{2k_{\text{int}}+3} \right] \right|^k_{\text{int}} \text{Max}(1, |a|).
\]

When \( z|a^2+1| \geq 2 \), we can use the approximation for \( |J_{e_0}(a_1)| \) given in (5-11). On the other hand, when \( z|a^2+1| < 2 \), we can obtain an adequate approximation for \( |J_{e_0}(a_1)| \) by keeping the first term in (5-1), yielding

\[
|J_{e_0}(a_1) e^{az}| = |z[J_0(z) + a J_{1}(z)]| = \left| \sqrt{\frac{2z}{\pi}} \left\{ \cos(z - \frac{\pi}{4}) + a \cos(z - \frac{3\pi}{4}) \right\} \right| ; \ z|a^2+1| < 2, \quad (5-14)
\]

where the asymptotic expansion (C-Ba) was applied.

We use forward recurrence to compute the Bessel functions and then use the convergent factorial-Neumann series expansion (5-1) when \( z > ZJASY \) and (5-13) holds.

If this expansion will not work, then we must compute the sequence of Bessel functions by using backward recurrence, and then use one of the expansions given in (5-1) or (5-3).
When \( z|a^2+1| < 2 \), we use the convergent factorial-Neumann series expansion (5-1), since it converges faster than the Neumann series expansion (5-3).

When \( z|a^2 + 1| > 2 \) and \( |a^2 + 1| \leq 1 \), the convergent factorial-Neumann series expansion will still converge faster than the Neumann series expansion, but now, as was shown in §2.2, we need to worry about round-off errors.

In §2.2, we found that the \( \frac{1}{\rho_k^{(2)}} \) term, in (2-18), may become large and cause round-off error problems when \( z|a^2 + 1| \) is large. Since we want to obtain an approximation to \( J_0(a,z) \), which is accurate to SD significant digits, reference to (2-18) and (2-16) shows that

\[
\frac{1}{2} \times 10^{-SD} |J_0(a,z)| > \varepsilon \left| \frac{J_0(a,z)}{\rho_k^{(2)}} \right| \quad (5-15)
\]

\[
= \varepsilon \left| \frac{1}{[a^2+1]^{1/2}} - \sqrt{\frac{z}{2}} e^{-az} \left( a^2+1 \right) \right| k_{\text{max}} \left[ a \cos(z + \frac{\pi}{2k_{\text{max}}} - \frac{\pi}{4}) - \sin(z + \frac{\pi}{2k_{\text{max}}} - \frac{\pi}{4}) \right] \frac{a^2 + 1}{\Gamma(k_{\text{max}} + 1) (a^2 + 1)}
\]

where \( k_{\text{max}} \) is chosen to maximize the second term on the right hand side of (5-15). \( k_{\text{max}} \) is obtained by applying (E-4):

\[
k_{\text{max}} = \frac{z|a^2+1|-1}{2} \quad (5-16)
\]

Round-off error won't be a problem if the second term in (5-15) is smaller than the first term. If this is the case, then (5-1) can be used to compute \( J_0(a,z) \).

On the other hand, when the second term is larger than the first term, we must account for the round-off errors. Application of (C-21) and (5-16) to (5-15) yields

\[
\varepsilon < \frac{1}{2} \times 10^{-SD} \left| J_0(a,z) (a^2+1) \right| \frac{\sqrt{\pi z} e^{-\left( |a^2+1| \right)}}{\text{Max}(1,|a|)} \quad , \quad (5-17)
\]
where we have dropped the first term on the right hand side of (5-15).

This inequality gives us an estimate for the maximum tolerable error, \( \varepsilon \), that can exist if (5-1) is to be used when \( z|a^2+1| \) is large and \(|a^2+1| \leq 1\).

We can define

\[
SDN := SD - \log_{10}\left[ \frac{z[\Re(a) - \frac{|a^2+1|}{2}]}{\max(1, |a|)} \right], \tag{5-18}
\]

where SDN is the number of significant digits required in all operations.

If the accuracy of the computer is less than SDN, then (5-1) can't be used.

If (5-1) can be used, then in order to obtain SD significant digits in \( J_0(a,z) \), we must use SDN instead of SD when we calculate the sequence of Bessel functions (see Appendix B).

Now, if neither factorial-Neumann series expansion has been used to compute \( J_0(a,z) \), then we use the Neumann series expansion (5-3). An algorithm, which is structured as outlined in this section, will use the most efficient expansion to compute \( J_0(a,z) \).
6. CONCLUSION

Two factorial-Neumann series expansions for $J_0(a,z)$ have been derived. These two expansions are used in conjunction with the Neumann series expansion, given by Agrest, in an algorithm which efficiently computes $J_0(a,z)$ for $z \in \mathbb{R}$ and $a \in \mathbb{C}$.

For comparison purposes, we used an 8-panel, Newton-Cotes, adaptive quadrature routine (QUANC8) [17] to compute $J_0(a,z)$. We found that the algorithm given in this paper, provides a much faster method for computing $J_0(a,z)$ than QUANC8.

The amount of CPU time saved, when this algorithm is used instead of QUANC8, depends on the parameters $a,z$ and $SD$. For example, when $z|a^2+1|$ is large and $a$ is pure imaginary, the integrand of $J_0(a,z)$ is highly oscillatory, so a large number of function evaluations are required when using QUANC8. On the other hand, only a few terms of the factorial-Neumann series expansion are required to obtain an approximation for $J_0(a,z)$. Also, the amount of time saved by using this algorithm increases as the desired number of significant digits increases.

The expansions developed in this paper provide a very efficient method for computing the incomplete Lipschitz-Hankel integral $J_0(a,z)$. This is further demonstrated in [9].
APPENDIX A

First-order Nonhomogeneous Recurrence Relations

In this Appendix, the computational use of first-order nonhomogeneous recurrence relations is discussed. The primary source for this Appendix is the book by Wimp [11].

First-order nonhomogeneous recurrence relations of the general form

$$Y_n + d_n Y_{n+1} = f_n; \quad n \geq 0,$$

(A-1)

where $d_n \neq 0$, are considered in this Appendix. The solution of this nonhomogeneous equation is given by a linear combination of the homogeneous and the particular solutions

$$Y_n = C Y_n^{(h)} + y_n^{(p)},$$

(A-2)

where $C$ is a constant.

The homogeneous solution, which must satisfy the homogeneous equation

$$Y_n^{(h)} + d_n Y_{n+1}^{(h)} = 0; \quad n \geq 0,$$

(A-3)

is obtained by choosing the initial condition $Y_0^{(h)} = 1$, and then recursing in the forward direction using (A-3). The resulting expression is

$$y_n^{(h)} = \prod_{k=0}^{n-1} [-d_k]^{-1},$$

(A-4)

where empty products are to be interpreted as 1.

The particular solution takes on different forms when the recurrence relation (A-1) is used in the forward and backward directions. When forward recurrence is used,

$$y_n^{(p)} = -y_n^{(h)} \sum_{k=0}^{n-1} \frac{f_k}{y_k^{(h)}},$$

(A-5)

and for backward recurrence

$$y_n^{(p)} = y_n^{(h)} \sum_{k=n}^{\infty} \frac{f_k}{y_k^{(h)}},$$

(A-6)
Now, let $w_n$ be the desired solution of (A-1). Before we can use (A-1) to calculate the sequence of solutions \{w_n\} in either the forward or backward direction, we must determine whether the recurrence is stable in that direction.

First, we look at forward recurrence. Assume that an initial value, $w_k$, is known. An index of stability for the forward computation of $w_n$ from $w_k$ is given by ([11], eqn. (2.17))

$$\alpha(k,n) = \frac{w_k y_n}{w_n y_k} = \frac{\rho_n}{\rho_k}, \quad (A-7)$$

where

$$\rho_n = \frac{w_o y_n}{w_n}. \quad (A-8)$$

When recursing from $k$ to $n$, the error is increased if $\rho_n \geq \rho_k$ and decreased if $\rho_n < \rho_k$. So, if $\rho_n$ is monotone decreasing, then the recurrence relation (A-1) may be used safely in the forward direction to compute $w_n$. On the other hand, if

$$\lim_{n \to \infty} \rho_n = \infty, \quad (A-9)$$

then forward recurrence is unstable for computing $w_n$.

When condition (A-9) is satisfied and $\rho_n$ is monotone increasing, there exists the possibility that the recurrence may be used safely in the backward direction. The Miller algorithm can usually be used when the above conditions are satisfied.

In the Miller algorithm, the initial condition,

$$\gamma^N = 0, \quad (A-10)$$

is chosen because of the asymptotic behavior given in (A-9). Then, $\gamma^N_n$ is obtained using

$$\gamma^N_n = f_n - d_n \gamma^N_{n+1} \quad (A-11)$$
for \( n = N-1, N-2, \ldots, 0 \). The constant in (A-2) is determined by enforcing the initial condition (A-10):

\[
C = \frac{-\gamma^{(p)}_N}{\gamma^{(h)}_N} = -\sum_{k=N}^{\infty} \frac{f_k}{\gamma^{(h)}_k}.
\]  

(A-12)

Therefore, the solution for the Miller algorithm, represented as a series, is

\[
\gamma^{N}_n = \gamma^{(h)}_n \sum_{k=n}^{N-1} \frac{f_k}{\gamma^{(h)}_k}.
\]  

(A-13)

A relative error bound for the Miller algorithm is given by ([11], eqn. (3.15))

\[
\left| \frac{\gamma^{N}_n - w_n}{w_n} \right| \leq \left| \frac{\gamma^{(h)}_n}{w_n} \right| \left\{ (1 + \varepsilon)^{N-n} \left[ \frac{\varepsilon + |w_N|}{|\gamma^{(h)}_N|} \right] \right. \\
+ \varepsilon \sum_{j=n}^{N-1} (1 + \varepsilon)^{j-n} \left| \frac{w_j}{\gamma^{(h)}_j} \right| \left\} ; \quad 0 \leq n \leq N,
\]

(A-14)

where it is assumed that \( w_n \neq 0 \), and \( \varepsilon \) is the maximum error due to finite precision arithmetic, introduced during the recurrence.

In the idealized case where infinite precision arithmetic is used, the absolute error is given by

\[
\left| \gamma^{N}_n - w_n \right| = \left| \gamma^{(h)}_n \right| \left| \frac{w_N}{\gamma^{(h)}_N} \right| = O \left( \left| \frac{w_N}{\gamma^{(h)}_N} \right| \right), \quad N \to \infty.
\]

(A-15)

This equation shows that for the idealized case, \( w_n \) can be calculated to any desired accuracy using the Miller algorithm if the condition (A-9) is satisfied. If the Miller algorithm converges, then the series representation (A-13) will also converge to the desired solution.
In cases where the errors due to finite precision arithmetic can be ignored, the absolute error given in (A-15), or

\[ |y_n^N - w_n| = |y_n^{(h)}| \sum_{k=N}^{\infty} \frac{f_k}{y_k^{(h)}} \]

can be used.
APPENDIX B

Generation of Bessel Functions \( J_n(x) \) using Backward Recurrence

In this Appendix, we discuss a backward recurrence algorithm which efficiently computes a sequence of Bessel functions of integer order and real argument. Many authors have written about this subject; the paper by Gautschi [12], and the book by Wimp [11] are two examples.

We will use an algorithm which is based upon a combination of the algorithms due to J.C.P. Miller and F.W.J. Olver. This combination algorithm is presented in [13]. Background information on the Olver algorithm can be found in [14]. We chose the combination algorithm because it provides automatic control of the truncation error, and it is very well suited for the calculation of \( J_n(x) \).

In this paper, we have derived three series expansions that involve Bessel functions of integer order (see (5-1), (5-2), and (5-3)). In order to compute \( J_0(a,z) \) to SD significant digits using (5-1) or (5-3), we need accurate approximations for all the Bessel functions \( J_n(x) \) that are larger in magnitude than the test value \( T \), where

\[
T := |J_0(x)| \cdot \frac{1}{2} \times 10^{-SD}.
\]

The test value, \( T \), can be approximated by

\[
T = \begin{cases} 
\frac{1}{2} \times 10^{-SD} & ; 0 \leq x \leq 1 \\
\frac{1 \times 10^{-SD}}{\sqrt{2\pi x}} & ; x > 1
\end{cases}
\]

The combination algorithm, given in [13], provides a very efficient method for accomplishing this.

First, a sequence \( \{ p_M \} \), \( \{ p_{M+1} \} \), ..., is computed by using the recurrence relation for Bessel functions (C-3) in the forward direction starting with \( p_M = 0 \) and...
\( p_{M+1} = 1 \), where \( M = \lfloor x \rfloor \) (\( M \) is the largest integer less than or equal to \( x \)).

The sequence of \( p_m \)'s will be nondecreasing in absolute value since the dominant solution to the recurrence, \( Y_n(x) \), is nondecreasing for \( n > \lfloor x \rfloor \).

Forward recurrence continues until the error condition,

\[
\left| \frac{1}{p_{N+1}} \right| < T, \quad (B-3)
\]

is satisfied (see [14], eqn. (4.12)), where \( T \) is given in (B-2).

Once the error condition in (B-3) is satisfied, backward recurrence on (C-3), beginning with \( Z_N = 0 \) and \( Z_{N-1} = \frac{1}{p_N} \), gives the sequence \( Z_N, Z_{N-1}, \ldots, Z_0 \).

Now, the desired solutions \( J_n(x) \) are determined by applying the normalization condition (C-11a):

\[
J_n(x) = \frac{Z_n}{Z_0 + 2Z_2 + 2^2Z_4 + \ldots + 2^{N-1}Z_{N-1}}. \quad (B-4)
\]
APPENDIX C

Some Useful Special Functions

In this Appendix, we have listed the properties of special functions that are used in this paper. The properties can be found in either the book by Abramowitz and Stegun [15], or Gradshteyn and Ryzhik [16].

C.1 Bessel Functions

Let \( Z_n(z) = J_n(z) \) or \( Y_n(z) \) \hspace{1cm} (C-1)

Bessel Functions of Negative Integer Order

\( Z_{-n}(z) = (-1)^n Z_n(z) \) \hspace{1cm} (C-2)

Recurrence Relations

\( Z_{\nu-1}(z) + Z_{\nu+1}(z) + \frac{2\nu}{z} Z_{\nu}(z) \) \hspace{1cm} (C-3)

\( Z'_{\nu}(z) = -Z_{\nu+1}(z) + \frac{\nu}{z} Z_{\nu}(z) \) \hspace{1cm} (C-4)

Ascending Series

\( J_{\nu}(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{z^2}{4})^k}{k!\Gamma(\nu+k+1)} \hspace{1cm} (C-5) \)

Analytic Continuation

\( J_{\nu}(ze^{\pm i}) = e^{\nu\pi i} J_{\nu}(z) \); \hspace{0.5cm} m = integer \hspace{1cm} (C-6)

Limiting Form for Small Argument

\( J_{\nu}(z) \sim \left( \frac{z}{2} \right)^\nu \frac{1}{\Gamma(\nu+1)} \); \hspace{0.5cm} z \to 0 \hspace{0.5cm} \nu \neq -1, -2, -3, ... \hspace{1cm} (C-7) \)
Principle Asymptotic Forms for Large Argument

\[ J_v(z) = \sqrt{\frac{2}{\pi z}} \{ \cos(z - \frac{v\pi}{2} - \frac{\pi}{4}) + e^{i\text{Im}(z)}O(|z|^{-1}) \}; \quad |z| \to \infty, \quad |\arg(z)| < \pi \]  
\[ (C-8a) \]

\[ Y_v(z) = \sqrt{\frac{2}{\pi z}} \{ \sin(z - \frac{v\pi}{2} - \frac{\pi}{4}) + e^{i\text{Im}(z)}O(|z|^{-1}) \}; \quad |z| \to \infty, \quad |\arg(z)| < \pi \]  
\[ (C-8b) \]

Hankel's Asymptotic Expansion

\[ J_v(z) = \sqrt{\frac{2}{\pi z}} \{ P(v,z) \cos x - Q(v,z)\sin x \}; \quad |z| \to \infty, \quad |\arg(z)| < \pi, \]  
\[ (C-9) \]

where

\[ x = z - (\frac{v}{2} + \frac{1}{4})\pi, \]

\[ P(v,z) = \sum_{k=0}^{\infty} (-1)^k \frac{(v,2k)}{(2z)^{2k}} \]

\[ Q(v,z) = \sum_{k=0}^{\infty} (-1)^k \frac{(v,2k+1)}{(2z)^{2k+1}} \]

and Hankel's symbol is defined as

\[ (n,k) = \frac{\Gamma(\frac{1}{2}+n+k)}{k!\Gamma(\frac{1}{2}+n-k)} \]

Principle Asymptotic Forms for Large Order

\[ J_v(z) \sim \frac{1}{\sqrt{2\pi v}} \left( \frac{ez}{2v} \right)^v; \quad v \to \infty \]  
\[ (C-10a) \]

\[ Y_v(z) \sim -\sqrt{\frac{2}{\pi v}} \left( \frac{ez}{2v} \right)^{-v}; \quad v \to \infty \]  
\[ (C-10b) \]

Generating Functions

\[ 1 = J_0(z) + 2J_2(z) + 2J_4(z) + 2J_6(z) + \ldots \]  
\[ (C-11a) \]

\[ e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{k=-\infty}^{\infty} t^k J_k(z) \quad (t \neq 0) \]  
\[ (C-11b) \]
Addition Theorem

\[ J_n(u \pm v) = \sum_{k=-\infty}^{\infty} J_{n+k}(u) J_k(v) \]  \hspace{1cm} (C-12a)

\[ J_0(u \pm v) = \sum_{k=0}^{\infty} \varepsilon_k(\frac{v}{2})^k J_k(u) J_k(v) \]  \hspace{1cm} (C-12b)

where \( \varepsilon_k = \begin{cases} 1, & k = 0 \\ 2, & k = 1, 2, \ldots \end{cases} \)

C.2 Integrals of Bessel Functions

\[ \int x^{p+1} J_p(x) \, dx = x^{p+1} J_{p+1}(x), \]  \hspace{1cm} (C-13)

where \( J_p(x) \) is defined in (C-1).

\[ \int_0^\infty e^{-ax} J_\nu(\beta x) \, dx = \frac{e^{-\sqrt{a^2+\beta^2} - a} \nu}{\sqrt{a^2 + \beta^2}} \]  \hspace{1cm} (C-14a)

\[ = \frac{\beta^\nu}{\sqrt{a^2 + \beta^2} [\sqrt{a^2 + \beta^2} + a]^\nu} ; \]  \hspace{1cm} (C-14b)

\( \text{Re}(\nu) > -1, \text{ Re}(a \pm i\beta) > 0 \)

\[ \int_0^\infty e^{-ax} Y_\nu(\beta x) \, dx = \frac{-2}{\pi \sqrt{a^2 + \beta^2}} \ln \frac{a + \sqrt{a^2 + \beta^2}}{\beta} \]  \hspace{1cm} (C-15)

\( \text{Re}(a) > |\text{Im} \beta| \)

\[ \int_0^\infty x^\mu J_\nu(ax) \, dx = 2^\mu a^{-\mu-1} \frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{2} + \frac{\mu}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{\nu}{2} - \frac{\mu}{2}\right)} ; \]  \hspace{1cm} (C-16)

\( -\text{Re}(\nu) - 1 < \text{Re}(\mu) < \frac{1}{2}, \ a > 0 \)

\[ \int_0^\infty e^{-ax} J_\nu(\beta x)x^\nu \, dx = \frac{(2\beta)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\alpha^2 + \beta^2)^{\nu + \frac{1}{2}}} ; \]  \hspace{1cm} \( \text{Re}(\nu) > -\frac{1}{2}, \text{ Re}(a) > |\text{Im}(\beta)| \)  \hspace{1cm} (C-17)
\[
\int_0^\infty e^{-ax}j_\nu(\beta x)x^{\nu-1}dx = \frac{(\beta)^\nu}{2\alpha^\nu \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right);
\]

\[\text{Re}(\nu+\mu) > 0, \quad \text{Re}(\alpha \pm i\beta) > 0\] (C-18)

### C.3 Gamma Functions

\[\Gamma(z+1) = z\Gamma(z) = z!\] (C-19)

\[\Gamma(n + \frac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^n} \Gamma\left(\frac{1}{2}\right)\] (C-20)

Stirling's Formula

\[\Gamma(z) = e^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3}\right.\]

\[\left. - \frac{571}{2488320z^4} + \ldots\right]; \quad z \to \infty, \quad |\text{arg}(z)| < \pi\] (C-21)

### C.4 Incomplete Gamma Functions

\[\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt \quad \text{Re}(a) > 0\] (C-22)

\[\Gamma(a,x) = \Gamma(a) - \gamma(a,x) = \int_0^\infty e^{-t} t^{a-1} dt\] (C-23)

\[\gamma^*(a,x) = \frac{x^{-a}}{\Gamma(a)} \gamma(a,x)\] (C-24)

Series Expansion

\[\gamma^*(a,z) = e^{-z} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(a+n+1)}\] (C-25)

Recurrence Formula

\[\gamma(a+1,x) = a\gamma(a,x) - x^a e^{-x}\] (C-26)
Asymptotic Expansions

\[ r(a, z) - z^{a-1} e^{-z} \left[ 1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \ldots \right]; \]

\[ z \to \infty, \quad |\text{arg}(z)| < \frac{3\pi}{2} \]  \hspace{1cm} (C-27)

C.5 Miscellaneous Functions

Exponential Function

\[ \lim_{m \to \infty} \left( 1 + \frac{z}{m} \right)^m = e^z \]  \hspace{1cm} (C-28)

Hypergeometric Functions

\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \]  \hspace{1cm} (C-29a)

\[ = \frac{r(c)}{r(a)r(b)} \sum_{n=0}^{\infty} \frac{r(a+n)r(b+n)}{r(c+n)} \frac{z^n}{n!} \]  \hspace{1cm} (C-29b)
APPENDIX D

Asymptotic Approximations for Some Integrals

In this Appendix, we derive asymptotic approximations for the integrals we encounter while performing the error analysis in Section 2.

D.1 Asymptotic Behavior of $J_\nu(a,z)$ for Large $n$

The function of interest is given by (2-1), where $\nu = n$:

$$J_{\nu}(a,z) = \int_0^z e^{-at} t^{\nu} J_n(t) dt; \quad n \geq 0, \; z > 0, \; a \in \mathbb{C}. \quad (D-1)$$

If $n \gg z$, then an asymptotic expansion for this integral can be obtained by replacing the Bessel function by its asymptotic expansion (C-10a):

$$J_{\nu}(a,z) \sim \frac{1}{\sqrt{2\pi n}} \left( \frac{e}{2\pi} \right)^n \int_0^z e^{-at} t^{2n} dt; \quad n \gg z. \quad (D-2)$$

The integral on the right hand side of (D-2) is an incomplete gamma function (see (C-22) and (C-24)) which has a series expansion given in (C-25).

$$J_{\nu}(a,z) \sim \frac{z}{\sqrt{2\pi n}} \left( \frac{z e^{2\pi}}{2\pi} \right)^n \Gamma(2n+1) \gamma^*(2n+1,az) \quad (D-3)$$

$$= \frac{ze^{-az}}{\sqrt{2\pi n}} \left( \frac{ze}{2\pi} \right)^n \Gamma(2n+1) \sum_{k=0}^{\infty} \frac{(az)^k}{\Gamma(k+2n+2)}; \quad n \gg z.$$  

The desired asymptotic behavior is obtained by only keeping the first term in the series:

$$J_{\nu}(a,z) \sim \frac{ze^{-az}}{\sqrt{2\pi n}} \left( \frac{ze}{2\pi} \right)^n \frac{1}{2n+1}; \quad n \gg \kappa, \; z > 0, \; a \in \mathbb{C}. \quad (D-4a)$$

where

$$\kappa := \text{Integer (Max}(z, |az|)). \quad (D-4b)$$
D.2 Asymptotic Behavior of $\hat{J}_{e m}(a, \delta, z)$ for Large $m$

We are interested in finding the asymptotic behavior of

$$\hat{J}_{e m}(a, \delta, z) = \int_{\delta}^{z} e^{-at} t^{-m} J_{m}(t) dt; \quad m \geq 0, \quad z > 0, \quad a \in \mathbb{C}, \quad (2-23)$$

where $\delta$ is defined in (2-26). The integral in (2-23) is divided into two pieces,

$$\hat{J}_{e m}(a, \delta, z) = e_{m}(a, \delta, 0) + \hat{J}_{e m}(a, 0, z), \quad (D-5)$$

and the asymptotic behavior of each piece is obtained separately. The results are then added together, to obtain the asymptotic behavior of the original integral (2-23).

The asymptotic behavior of $\hat{J}_{e m}(a, 0, z)$ is obtained by replacing the Bessel function with its asymptotic expansion (C-10a):

$$\hat{J}_{e m}(a, 0, z) = \frac{1}{\sqrt{2\pi m}} \left( \frac{e}{2m} \right)^{m} \int_{0}^{z} e^{-at} dt$$

$$= \begin{cases} 
\frac{1}{\sqrt{2\pi m}} \left( \frac{e}{2m} \right)^{m} \left( 1 - e^{-az} \right) \frac{a}{a} ; & m \gg z > 0, \quad a \neq 0 \\
\frac{z}{\sqrt{2\pi m}} \left( \frac{e}{2m} \right)^{m} ; & m \gg z > 0, \quad a = 0. 
\end{cases} \quad (D-6a) \quad (D-6b)$$

The other integral, $\hat{J}_{e m}(a, \delta, 0)$, will exhibit different asymptotic behavior for the two cases of $\delta$ given in (2-26). When $\text{Re}(a) \geq 0$ and $a \neq 0$, $\hat{J}_{e m}(a, \infty, 0)$ can be expressed as a hypergeometric function using (C-18):

$$\hat{J}_{e m}(a, \infty, 0) = -\frac{1}{a 2^{m} \Gamma(m+1)} F(\frac{1}{2}, 1; m+1; -\frac{1}{a^2}). \quad (D-7)$$
The asymptotic behavior is obtained by only keeping the first term in the series expansion for the hypergeometric function (C-29a):

\[ \hat{J}_m(a, \omega, 0) = \frac{-1}{a^2 \Gamma(m+1)} ; \quad m \gg \left| \frac{1}{a^2} \right|, \quad \text{Re}(a) \geq 0, \quad a \neq 0. \] (D-8)

For the special case \( a = 0 \), the integral is known in closed form (see (C-16)):

\[ \hat{J}_m(0, \omega, 0) = \frac{-\Gamma(\frac{1}{2})}{2^m \Gamma(m+\frac{1}{2})}. \] (D-9)

The asymptotic behavior of \( \hat{J}_m(a, \omega, 0) \) can be expressed in a form similar to (D-6) by using Stirling's formula (C-21):

\[ \hat{J}_m(a, \omega, 0) \approx \begin{cases} 
- \frac{e}{a \sqrt{2 \pi m}} \left( \frac{e}{2m(1 + \frac{1}{m})} \right)^m ; & m \to \infty, \quad a \neq 0, \quad \text{Re}(a) \geq 0 \quad (D-10a) \\
- \sqrt{\frac{e}{2}} \left( \frac{e}{2m(1 + \frac{1}{2m})} \right)^m ; & m \to \infty, \quad a = 0. \quad (D-10b)
\end{cases} \]

Application of the identity (C-28), yields

\[ \hat{J}_m(a, \omega, 0) \approx \begin{cases} 
- \frac{1}{a \sqrt{2 \pi m}} \left( \frac{e}{2m} \right)^m ; & m \to \infty, \quad a \neq 0, \quad \text{Re}(a) \geq 0 \quad (D-11a) \\
- \frac{1}{\sqrt{2}} \left( \frac{e}{2m} \right)^m ; & m \to \infty, \quad a = 0. \quad (D-11b)
\end{cases} \]

The asymptotic behavior of \( \hat{J}_m(a, -\omega, 0) \) for the case \( \text{Re}(a) < 0 \), is obtained by making the changes of variables \( \tau = -t \) and \( b = -a \) in (2-23). This yields

\[ \hat{J}_m(a, -\omega, 0) = (-1)^{m+1} \int_{-\infty}^{0} e^{-b \tau} \tau^{-m} J_m(-\tau) d\tau \]

\[ = - \hat{J}_m(b, \omega, 0), \] (D-12)
where the analytic continuation equation (C-6) was applied to the Bessel function. Now, the previous result (D-11a) can be applied since \( \text{Re}(b) > 0 \):

\[
\hat{J}_m(a, -\infty, 0) - \frac{1}{b\sqrt{a^2m}} \left( \frac{e}{2m} \right)^m = -\frac{1}{a\sqrt{a^2m}} \left( \frac{e}{2m} \right)^m; \quad m \to \infty, \quad \text{Re}(a) < 0.
\]

(D-13)

Finally, the asymptotic behavior of \( \hat{J}_m(a, \delta, z) \) is obtained by combining the results given in (D-5), (D-6), (D-11), and (D-13):

\[
\begin{align*}
\hat{J}_m(a, \delta, z) &\left\{ \begin{array}{l}
- \frac{e^{-az}}{a\sqrt{a^2m}} \left( \frac{e}{2m} \right)^m; \quad m \to \infty, \quad z > 0, \quad a \neq 0 \\
- \frac{1}{\sqrt{2}} \left( \frac{e}{2m} \right)^m; \quad m \to \infty, \quad z > 0, \quad a = 0.
\end{array} \right.
\end{align*}
\]

(D-14a)  \hspace{1cm} (D-14b)

\section*{D.3 Asymptotic Behavior of \( \hat{J}_m(a, \infty, z) \) for Large \( z \)}

The integral of interest is a special case of (2-23):

\[
\hat{J}_m(a, \infty, z) = \int_{\infty}^{z} e^{-at} t^{-m} J_m(t) dt; \quad m \geq 0, \quad z > 0, \quad \text{Re}(a) \geq 0.
\]

(D-15)

When \( z \gg m \), the Bessel function can be replaced by its principal asymptotic form (C-8a):

\[
\hat{J}_m(a, \infty, z) - \sqrt{\frac{2}{\pi}} \int_{\infty}^{z} e^{-at} t^{-m-\frac{1}{2}} \cos(t - \frac{m}{2} - \frac{\pi}{4}) dt, \quad z \gg m \geq 0, \quad \text{Re}(a) \geq 0.
\]

(D-16)

The analysis is simplified by initially assuming that \( a \) is a real variable, and then later extending the results to complex values of \( a \) by using analytic continuation. This assumption enables (D-16) to be written as

\[
\hat{J}_m(a, \infty, z) - \sqrt{\frac{2}{\pi}} \text{Re} \left\{ e^{\frac{\pi i}{2} (m+\frac{1}{2})} \int_{\infty}^{z} e^{-t(a+i)} t^{-m-\frac{1}{2}} dt \right\},
\]

\[
a \geq 0, \quad z \gg m \geq 0.
\]

(D-17)
Making the change of variable, \( \tau = t(a \pm i) \) enables us to put the integral into the form of an incomplete gamma function (see (C-23)):

\[
\hat{J}_e_m(a, \omega, z) = \sqrt{\frac{2}{\pi}} \text{Re} \left\{ e^{\frac{\pi}{2} (m+\frac{1}{2})} (a \pm i)^{m-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\tau} \tau^{-m-\frac{1}{2}} d\tau \right\} \\
= -\sqrt{\frac{2}{\pi}} \text{Re} \left\{ e^{\frac{\pi}{2} (m+\frac{1}{2})} (a \pm i)^{m-\frac{1}{2}} \Gamma(\frac{1}{2} - m, z(a \pm i)) \right\};
\]

\( a \geq 0, \ z \gg m \geq 0. \quad (D-18) \)

The desired asymptotic expansion is obtained by replacing the incomplete gamma function by the first term of (C-27):

\[
\hat{J}_e_m(a, \omega, z) = -\sqrt{\frac{2}{\pi}} e^{-az} \text{Re} \left\{ e^{\frac{\pi}{2} \left[ z - \frac{\pi}{2} (m+\frac{1}{2}) \right]} \right\} \\
= -\sqrt{\frac{2}{\pi}} e^{-az} \frac{(a \pm i)^{m+\frac{1}{2}}}{(a^2 + 1)z^{m+\frac{1}{2}}} [a \cos(z - \frac{\pi}{2} - \frac{\pi}{4}) - \sin(z - \frac{\pi}{2} - \frac{\pi}{4})];
\]

\( z > 0, \ a \geq 0, \ \eta \gg m \geq 0, \quad (D-19a) \)

where

\[ \eta := \text{Min}(z, z|a \pm i|). \quad (D-19b) \]

Now, by using analytic continuation, (D-19) can be extended to hold for \( z > 0, \ \text{Re}(a) \geq 0, \ a \neq \pm i \).

With the added restriction, \( R(a) \neq 0, \ \hat{J}_e_m(a, \omega, z) \) will be defined for all positive and negative values of \( m \). Using equations (2-2), (D-15), and (C-2) we find that

\[
\hat{J}_e_n(a, \omega, z) = (-1)^n \hat{J}_e_{-n}(a, \omega, z); \quad z > 0, \ \text{Re}(a) > 0. \quad (D-20)
\]
Therefore, the asymptotic expansion for $J_n(a, \infty, z)$ is found using the results in (D-19):

$$J_n(a, \infty, z) - (-1)^n \sqrt{\frac{2}{\pi}} \frac{e^{-az}z^{n-\frac{1}{2}}}{(a^2+1)^{\frac{n}{2}}} [a \cos\left(z + \frac{n\pi}{2} - \frac{\pi}{4}\right) - \sin\left(z + \frac{n\pi}{2} - \frac{\pi}{4}\right)];$$

$$z > 0, \quad \text{Re}(a) > 0, \quad n >> n \geq 0. \quad \text{(D-21)}$$
APPENDIX E

Finding the Local Minimum of $P_\xi$

In this Appendix, we determine what value of $\xi$ minimizes $P_\xi$, where

$$P_\xi := \left(\frac{2}{\pi|a^2+1|}\right)^{\xi} \Gamma(\xi); \quad \xi > 0. \quad (E-1)$$

The gamma function, in (E-1), can be approximated by the first term in Stirling's formula for moderate to large values of $\xi$.

$$P_\xi = \frac{\sqrt{2\pi}}{\xi} \left(\frac{2\xi}{e|a^2+1|}\right)^{\xi}; \quad \xi \gg 0. \quad (E-2)$$

Now, the value of $\xi$ which minimizes $P_\xi$ can be found by differentiating (E-2), and setting the result to zero:

$$0 = \frac{\partial P_\xi}{\partial \xi} = P_\xi \left[1 + \ln \left(\frac{2\xi}{e|a^2+1|}\right) - \frac{1}{2\xi}\right]; \quad \xi \gg 0. \quad (E-3)$$

Solving for $\xi$ yields

$$\xi = \frac{z|a^2+1|e^{2\xi}}{2} = \frac{z|a^2+1|}{2}; \quad \xi \gg 0. \quad (E-4)$$
APPENDIX F

Fortran Source Code

This Appendix contains the Fortran source code for an algorithm which computes $J_0(a,z)$. The structure for this algorithm is given in Section 5.
PROGRAM TJE0

C (Test JEO)

------------------------------------------------------------------------------

PURPOSE:

The purpose of this routine is to test the function JEO.

INPUTS: (From the keyboard)

A = Factor in the exponential of the incomplete Lipschitz-Hankel integral. (Complex)

Z = Upper limit of integration. (Real)

OUTPUTS: (To the screen)

RESULT = Computed value for JEO(A,Z).

SD = Number of significant digits.

TIME1 = Amount of CPU time required to compute JEO(A,Z).

WHICH = Tells which subroutine was used to compute JEO(A,Z).

SUBRoutines and Functions Called:

JEO = A function which computes JEO(A,Z).

INITIAL = A subroutine which initializes the variables used in the program.

ETIME = A function which provides a reading of the clock. (This is a non-standard fortran-77, machine dependent intrinsic function)

VARIABLE DECLARATIONS:

COMPLEX*16 A,JEO,RESULT
DOUBLE PRECISION Z
REAL TIME1,TIME2,ETIME,TARRAY(Z),ERROR,ZJASY
INTEGER SD,I
CHARACTER*6 WHICH
LOGICAL ZNEG

EXTERNAL JEO
C
C Initialize the constants
C
CALL INITIAL
C
1 WRITE(*,*) ' INPUT A, Z : To stop, input Z = 0.'
READ(*,*) A,Z
WRITE(*,10) A,Z
10 FORMAT(' A=',2D12.4,' Z=',D12.4)
C
IF Z=0. THEN JEO(A,Z)=0.
IF (Z .EQ. 0.) THEN
RESULT=DCMPLX(0.0D0,0.0D0)
WRITE(*,*) 'JEO = ',RESULT
STOP
ELSE IF (Z .LT. 0.) THEN
C Set a flag and make the proper change of variables to insure
C that Z is positive, i.e. JEO(A,-Z)=JEO(-A,Z)
ZNEG=.TRUE.
Z=-Z
A=-A
ELSE
   ZNEG=.FALSE.
END IF

C Calculate J_0(A,Z) for different values of SD
DO 100,SD=5,15,5
   ZJASY=REAL(SD+.4)
C Compute the desired relative error bound.
   ERROR=0.5*10.**(-SD)
   TIME1=ETIME(TARRAY)
C Compute J_0 30 times in order to obtain a more
C accurate time reading.
DO 200,I=1,30
   RESULT=J_0(A,Z,ERROR,ZJASY,WHICH)
200   CONTINUE
   TIME2=ETIME(TARRAY)
C Calculate the computation time.
   TIME1=(TIME2-TIME1)/30.
C If Z < 0., then J_0(A,-Z)=-J_0(-A,Z).
   IF (ZNEG) RESULT=-RESULT
C Output the results to the screen.
   WRITE(*,*) WHICH,' J_0= ',RESULT
   WRITE(*,*) 'SD= ',SD,' TIME = ',TIME1
100   CONTINUE
C Start the test over.
   GOTO 1
END
SUBROUTINE INITIAL

(Initialization Routine)

--------------------------------------------------------------------

PURPOSE:

The constants that are used in this program are initialized
in this subroutine. The constants are passed in common blocks.

--------------------------------------------------------------------

VARIABLE DECLARATIONS:

DOUBLE PRECISION TEMP
INTEGER K, I, K2, T1, T2, T3, T4

--------------------------------------------------------------------

COMMON BLOCKS

DOUBLE PRECISION FACTOR(0:200), PQARRY(4, 25)
COMMON/FACT/FACOR, PQARRY

COMPLEX*16 IM
PI, DSQRT(PI)
DOUBLE PRECISION PI, SQPI
COMMON/CONST/IM, PI, SQPI

***********************************************************************

PI=DCOS(-1.D0)
SQPI=DSQRT(PI)
IM=(0.D0, 1.D0)

Set up the factorial constants which are used in the subroutines
CFNEUM, FREC, and AFNEUM.
FACTOR(K)=SQPI/(2**(K+1)*GAMMA(K+1.5))
FACTOR(0)=1.D0
DO 100, K=1, 200
FACTOR(K)=FACTOR(K-1)/DBLE(2*K+1)
100 CONTINUE

Set up the constants that are used for Hankel’s asymptotic
expansion (see (C-9)) in the subroutine JHAEO1.
T2=-1
T4=3
I=1
DO 200, K=1, 25
K2=K*2
I=I+2
T1=T2
T2=-I*I
TEMP=DBLE(K2*(K2-1))*64.D0
PQARRY(1, K) = (0.2*K)*(-1/4)**K, where Hankel’s symbol is
defined as (N, K) = GAMMA(0.5+N+K)/(GAMMA(K+1)*GAMMA(.5-N-K)
PQARRY(1, K) = DBLE(-1*T1*T2)/TEMP
T3=T4
T4=4-I*I
PQARRY(2, K) = (1,2*K)*(-1/4)**K
PQARRY(2, K) = DBLE(-1*T3*T4)/TEMP
I=I+2
T1=T2
T2=-I*I
TEMP=DBLE(K2*(K2+1))*64.D0
PQARRY(3, K) = 0.5*(0.2*K+1)*(-1/4)**K
PQARRY(3, K) = DBLE(-1*T1*T2)/TEMP
T3=T4
T4=4-I*I

C PQARRY(4,K) = 0.5*(1,2*K+1)*(-1/4)**K
PQARRY(4,K)=DBLE(-1*T3*T4)/TEMP

200 CONTINUE
RETURN
END
COMPLEX*16 FUNCTION JE0(A,Z,ERROR,ZJASY,WHICH)

PURPOSE:

This routine evaluates the incomplete Lipschitz-Hankel integral,

\[ \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(-\alpha^2) \cos(\alpha z) d\alpha \]

where \( z \) is a positive real number and \( \alpha \) is a complex number.

The integral is evaluated to SD significant digits. The relative error is given by \( \text{ERROR} = 0.5 \times 10.0^{(-SD)} \).

In this routine, we determine which routine will be the most efficient for the evaluation of \( JE0(A,Z) \).

INPUTS:

A = Factor in the exponential of the incomplete Lipschitz-Hankel integral. (Complex)

Z = Upper limit of integration. (Positive Real)

ERROR = Desired relative error bound for \( JE0(A,Z) \).

ZJASY = SD + 4 : Test to see if Hankel's asymptotic expansion can be used.

OUTPUTS:

JE0 = Computed value for the incomplete Lipschitz-Hankel integral.

WHICH = Tells which series was used to compute \( JE0(A,Z) \).

SUBROUTINES AND FUNCTIONS CALLED:

CFNEUM = Convergent factorial-Neumann series. Backward recurrence used for the Bessel functions.

FREC = Convergent factorial-Neumann series. Forward recurrence used for the Bessel functions.

AFNEUM = Asymptotic factorial-Neumann series. Forward recurrence used for the Bessel functions.

BRNEUM = Neumann series. Backward recurrence used for the Bessel functions.

VARIABLE DECLARATIONS:

COMPLEX*16 A,CFNEUM,FREC,AFNEUM,BRNEUM
DOUBLE PRECISION Z,ERP
REAL ZJASY,ERP,MAGA,COMPAR,APKJE0,ERROR
INTEGER KMAX,INTZ
CHARACTER*6 WHICH
E = EXP(1.)
REAL E
PARAMETER (E=2.71828)
COMMON BLOCKS

DOUBLE PRECISION COSX, SINX
COMMON /SINE/COSX, SINX

IM=CDSQRT(-1), ASP1=A*A+1, EXPMAZ=CDEXP(-A*Z), SQASPI=CDSQRT(ASP1)
COMPLEX*16 IM, ASP1, EXPMAZ, SQASPI
PI, SQPI=DSQRT(PI)
DOUBLE PRECISION PI, SQPI
ABASPI=CDABS(ASP1)
REAL ABASPI
COMMON/CONST/IM, PI, SQPI, ASP1, ABASPI, EXPMAZ, SQASPI

******************************************************************************
EXTERNAL CPNEUM, FREC, AFNEUM, BRNEUM
ZSP=REAL(2)
ASP1=A*A+1.

Check to see if only the first term of the Neumann series is required
if \( \exp(-A^2) \cdot \text{ERROR} \), then only use first term.
First, check to see if the first term will give at least 5 SD accuracy.
IF (ZSP*REAL(A) .GT. 12.2) THEN
Now, make sure the first term will give the desired
number of significant digits accuracy.
IF (ZSP*REAL(A) .GT. -ALOG(ERROR)) THEN
Use the first term in (3-4)
Note: Re(A) > 0, so this is the correct branch cut
J0=1.20/SQRT(ASP1)
WHICH='FIRST'
RETURN
END IF
END IF

Set up the constants.
MAGA=ABS(CMPLX(A))
EXPMAZ=EXP(-Z*A)
INT2=IDINT(Z)
ABASPI=ABS(CMPLX(ASP1))

Test to see if Hankel's asymptotic expansion (C-9) can be used to
find J0 and J1 (see (5-4)).
IF (ZSP .GT. ZJASY) THEN
Forward recurrence can be used for the Bessel functions.
Find CSQRT(A*A+1). The branch cut is given in (2-37).
SQASPI=SQRT(ASP1)
IF ((REAL(A) .LT. 0.) .OR. (REAL(A) .EQ. 0.) .AND.
& AIMAG(A) .LT. -1.)) SQASPI=-SQASPI
COSX=Z-P/4.D0
SINX=DSIN(COSX)
COSX=DCOS(COSX)
IF (ZSP*ABASPI .LE. 2.) THEN
Approximate ABS(COS(A)*EXP(A)*Z)*ERROR. See (5-14).
AFXJED=SQR(ZSP)*7.978845E-1)*CABS(CMPLX(COSX*A+SINX))
& *ERROR
Construct the right hand side of (5-13).
TMP=DBLE(2*INTZ+3)
& TMP=(ZSP*ABASPI*TMP)**INTZ*2.52852*MAX(1.,MAGA)
& SQR(TMP)/TMP
Test the inequality in (5-13).
IF (TMP .LT. AFXJED) THEN
Use the convergent factorial-Neumann series (2-21).
JED=FREC(A,Z,ERROR)
WHICH='FREC'
RETURN
END IF
ELSE
Approximate \( \text{CABS}(J_{e0}(A,Z) \cdot \text{CEXP}(A*Z)): \) See (5-11). 
\( \text{APXJE0} = \text{ABS}(1.00/(\text{SQASPI} \cdot \text{EXPMAZ}) - \text{DSQRT}(2.00/(\text{PI*Z})) \cdot \\
(A \cdot \text{COSX-SINX})/\text{ASP1}) \cdot \text{ERROR} \)

Find \( K_{\text{MAX}} \) (see (5-6)). 
\( K_{\text{MAX}} = \text{AINT}((\text{ZSP} \cdot \text{ABASP1} - 1.)/2.) \)

Construct the right hand side of (5-9). 
\( \text{TMP} = \text{DBLE}(2.00/(\text{INTZ} \cdot \text{ABASP1})) ** \text{DBLE}(\text{KMAX}) \)

Test the inequality in (5-9). 
\( \text{IF} (\text{TMP} . \text{LT.} \cdot \text{APXJE0}) \text{ THEN} \)

Use the asymptotic factorial-Neumann series (2-45). 
\( \text{JE0} = \text{AFNEUM}(A,Z,\text{ERROR},K_{\text{MAX}}) \)

RETURN 

END IF

IF (ABASP1 .LT. .75) THEN

Construct the right hand side of (5-13). 
\( \text{TMP} = \text{DBLE}(2.00/\text{INTZ} + 3) \)

Test the inequality in (5-13). 
\( \text{IF} (\text{TMP} . \text{LT.} \cdot \text{APXJE0}) \text{ THEN} \)

Use the convergent factorial-Neumann series (2-21). 
\( \text{JE0} = \text{FREC}(A,Z,\text{ERROR}) \)

RETURN 

END IF

IF (ABASP1 .GT. 1.0) THEN

Use the Neumann series (3-4). 
\( \text{JE0} = \text{BRNEUM}(A,Z,\text{ERROR}) \)

RETURN 

END IF

BACKWARD RECURRENCE MUST BE USED FOR THE BESSEL FUNCTIONS.

IF (ABASP1.ZSP .LE. 2.0) THEN

Use the convergent factorial-Neumann series (2-21). 
\( \text{JE0} = \text{CFNEUM}(A,Z,\text{ERROR},\text{ERROR}) \)

RETURN 

ELSE IF (ABASP1 .LE. 1.0) THEN

Look at round-off error problems (see (5-17)).

\( \text{TMP} = \text{ERROR} \cdot \text{ABASP1} \cdot 1.77245 \cdot \text{SQRT}(2.00) \cdot \text{DBLE}(E) ** \text{DBLE}(ZSP \cdot \\
(\text{REAL}(A) - \text{ABASP1}/2.0)) / \text{MAX}(1.0,\text{MAGA}) \)

See if the computer has enough accuracy.
\( \text{IF} (\text{TMP} . \text{GT.} \text{.50D-16}) \text{ THEN} \)

Use the convergent factorial-Neumann series (2-21). 
\( \text{JE0} = \text{CFNEUM}(A,Z,\text{ERROR},\text{REAL}(\text{TMP})) \)

RETURN 

END IF

END IF

Find \( \text{CSQRT}(A*Z+1). \) The branch cut is given in (2-37). 
\( \text{SQASPI} = \text{SQRT}(\text{ASP1}) \)

Use the Neumann series (3-4). 
\( \text{JE0} = \text{BRNEUM}(A,Z,\text{ERROR}) \)

RETURN 

END
COMPLEX*16 FUNCTION CFNEUM(A,Z,ERROR,ERRJ)
(Convergent Factorial-Neumann Series - Bessel Functions. Obtained
Using Backward Recurrence)

PURPOSE:
This routine computes the incomplete Lipschitz-Hankel integral,
\[ J_0(A,Z) \], by using the convergent factorial-Neumann series expansion,
\[ J_0(A,Z) = Z^{*-1} \sum_{k=0}^{K-1} \frac{\text{COEFF}(k) \times (J(z) + A*J'(z))}{k! \times (z+1)!} \]
where \( \text{COEFF}(k) = (z^*(A^*A+1.)/2.)^k \times \Gamma(k+1.5)/\Gamma(k+1.5) \),
and \( \Gamma(k) \) is the Gamma function. The Bessel functions are
calculated using backward recurrence.

INPUTS:
A = Factor in the exponential of the incomplete Lipschitz-Hankel
integral. (Complex)
Z = Upper limit of integration. (Positive Real)
ERROR = Desired relative error bound for \( J_0(A,Z) \).
ERRJ = Required relative error for the Bessel functions.

OUTPUTS:
CFNEUM = Computed value for the incomplete Lipschitz-Hankel integral.

SUBROUTINES AND FUNCTIONS CALLED:
JBREC = A subroutine which computes the sequence of Bessel
functions using a backward recurrence algorithm.

Inputs:
Z = Argument of the Bessel functions.
ERRJ = Desired relative error bound.

Outputs:
JBESS = An array of unnormalized Bessel functions passed
in a common block.
NORM = The normalization constant.
NOTERM = Number of Bessel functions in the array.

VARIABLE DECLARATIONS:
COMPLEX*16 A,COEFF,TERM,ZASP1
DOUBLE PRECISION Z,NORM
REAL ERROR,ERRJ
INTEGER X,NOTERM,I,COUNT
COMMON BLOCKS

DOUBLE PRECISION JBESS(0:200)
COMMON /WORK/ JBESS

FACTOR(K)=SQRT(2.*K+1)*GAMMA(K+1.5))
DOUBLE PRECISION FACTOR(0:200)
COMMON/FACT/FACTOR

IM=CDSQRT(-1), ASP1=A*A+1, EXPMAZ=CDEXP(-A*Z)
COMPLEX*16 IM,ASP1,EXPMAZ

PI, DSQRT(PI)
DOUBLE PRECISION PI,SQPI

ABASPI=CDABS(ASP1)
REAL ABASPI
COMMON/CONST/IM,PI,SQPI,ASP1,ABASPI,EXPMAZ

C******************************************************************************
C Compute the Bessel functions using backward recurrence
CALL JBREC(Z,ERRJ,NORM,NOTERM)
C******************************************************************************
C Initialize the constants
ZASP1=Z*ASP1
COEFF=Z/NORM
C Initialize the series
CFNEUM=(JBESS(0)+A*JBESS(1))*COEFF
C Sum the series
C Don't need to test the convergence until K > Z
DO 100,K=1,INT(REAL(Z))
   COEFF=COEFF*ZASP1
   CFNEUM=CFNEUM+COEFF*((JBESS(K)+A*JBESS(K+1))*FACTOR(K))
100   CONTINUE
DO 110,I=K,NOTERM-1
   COEFF=COEFF*ZASP1
   TERM=COEFF*((JBESS(I)+A*JBESS(I+1))*FACTOR(I))
   CFNEUM=CFNEUM+TERM
C Test for convergence. Two consecutive terms must satisfy
C the error criterion.
   IF (CABS(COMPLEX(TERM))/(COMPLEX(CFNEUM)) .LT. ERROR) THEN
      IF (I.EQ. COUNT+1) GOTO 120
      COUNT=I
   END IF
110   CONTINUE
120   CFNEUM=CFNEUM*EXPMAZ
RETURN
END
COMPLEX*16 FUNCTION FREC(A,Z,ERROR)

(Convergent Factorial-Neumann Series - Bessel Functions Obtained
Using Forward Recurrence)

PURPOSE:
This routine computes the incomplete Lipschitz-Hankel integral,
Je (A,Z), by using the convergent factorial-Neumann series expansion,

\[ \sum_{k=0}^{K} \frac{Z^k}{\text{COEFF}(k)^* \left( J\left( \frac{Z}{2} \right) + A*J\left( \frac{Z}{2} \right) \right)^k} \]

where \( \text{COEFF}(k) := (Z^k(A^k)!) / (2^k) \) \( A^k \) \( k \cdot k+1 \),
and \( \text{GAMMA}(k) \) is the Gamma function.

The Bessel functions, \( J\left( \frac{Z}{2} \right) \), are computed using forward recurrence
with the following recurrence relation:

\[ J_{k+1}\left( \frac{Z}{2} \right) = 2k*J_k\left( \frac{Z}{2} \right) - J_{k-1}\left( \frac{Z}{2} \right) \]

The starting functions, \( J_0\left( \frac{Z}{2} \right) \), and \( J_1\left( \frac{Z}{2} \right) \) are calculated using

Hankel's asymptotic expansion (C-9).

Note: Recurrence in the forward direction is stable as long as
\( k <= Z \).

INPUTS:
A = Factor in the exponential of the Incomplete Lipschitz-Hankel integral. (Complex)
Z = Upper limit of integration. (Positive Real)
ERROR = Desired relative error bound for \( J\left( \frac{Z}{2} \right) \).

OUTPUTS:
FREC = Computed value for the Incomplete Lipschitz-Hankel integral.

SUBROUTINES AND FUNCTIONS CALLED:
JHAEO1 = A subroutine which returns asymptotic approximations for
the values for \( J_0\left( \frac{Z}{2} \right) \), and \( J_1\left( \frac{Z}{2} \right) \).
Inputs:
Z = Argument of the Bessel functions.
ERROR = Desired relative error bound.
Outputs:
JBESS(0) = \( J_0\left( \frac{Z}{2} \right) \)
JBESS(1) = \( J_1\left( \frac{Z}{2} \right) \)
JBESS(1) = J(Z)

VARIABLE DECLARATIONS:
COMPLEX*16 A, COEFF, TERM, TEMP
DOUBLE PRECISION Z, ZD2
REAL ERROR
INTEGER COUNT, K

COMMON BLOCKS

FACTOR(K) = SQRT(/(2.**(K+1) + GAMMA(K+1.5)))
DOUBLE PRECISION FACTOR(0:200)
COMMON /FACT/FACTOR

WORK ARRAY FOR COMPUTING THE BESSEL FUNCTIONS
DOUBLE PRECISION JBESS(0:200)
COMMON /WORK/ JBESS

IM = CDPSQRT(-1), ASP1 = A**A1, EXPMAZ = CDEXP(-A*Z)
COMPLEX*16 IM, ASP1, EXPMAZ
PI, DSQRT(PI)
DOUBLE PRECISION PI, SQPI
ABASP1 = CDABS(ASP1)
REAL ABASP1
COMMON/CDABS/IM, PI, SQPI, ASP1, ABASP1, EXPMAZ

C******************************************************************************

C Obtain the starting values J(Z), and J'(Z)
CALL JHAEO1(Z, JBESS(0), JBESS(1), ERROR)

C Initialize the constants
COEFF = Z
ZD2 = Z/2.0
TERM = Z*ASP1
COUNT = -1

C Initialize the series
FREC = JBESS(0) + A*JBESS(1)*Z

C Recurse forward on the J's and sum the series
C Note: Forward recurrence becomes unstable for K > Z, but it
C is possible to recurse a little past this point.
DO 100, K = 1, INT (REAL(Z)) + 5
C
C Calculate the next Bessel function, J(Z)
JBE=(K+1) = DBLE(K + JBESS(K)) / ZD2 - JBESS(K-1)
C C C C C C
C Find the next term in the series
TERM = COEFF*JBESS(K) + A*JBESS(K+1)*FACTOR(K)
FREC = FREC + TERM
C
C Determine whether the series has converged to within
C the desired error.
IF (CMPLX(TERM)/CMPLX(FREC)) .LT. ERROR) THEN
C
IF (K .EQ. COUNT+1) GOTO 120
COUNT = K
END IF

100 CONTINUE
120 FREC = EXPMAZ * FREC
RETURN
END
COMPLEX*16 FUNCTION AFNEUM(A,Z,ERROR,KMAX)

(Asymptotic Factorial-Neumann Series - Bessel Functions Obtained
Using Forward Recurrence)

PURPOSE:
This routine computes the incomplete Lipschitz-Hankel integral,
J_e (A,Z), by using the asymptotic factorial-Neumann series expansion,
\[ J_e (A,Z) = \frac{1}{CSQRT(A*A+1)} + \]
\[ \sum_{k=0}^{K} \left( CEXP(-A*Z)/(A*A+1) \right) * \left( COEFF(k) * (J_{k+1}(Z)-A*J_k(Z)) \right), \]

where \( COEFF(k) := (Z/(Z*(A*A+1.)))^{k} \cdot GAMMA(K+0.5)/GAMMA(0.5), \)
and \( GAMMA(K) \) is the Gamma function. The proper branch cut for
the square root is defined by:
\[ Re(A*A+1.) < 0. \quad \text{if} \quad Re(A) < 0., \text{ and Abs(Im(A)) > 1.} \]
\[ Re(A*A+1.) \geq 0. \quad \text{otherwise}. \]
The Bessel functions, \( J_k(Z) \), are computed using forward recurrence
\( k \)
with the following recurrence relation:
\[ J_{k+1}(Z) = (2k+1)(Z)J_k(Z) - (k-1)(Z)J_k(Z), \]
The starting functions, \( J_0(Z) \), and \( J_1(Z) \) are calculated using Hankel's
asymptotic expansion (C-9).

Note: Recurrence in the forward direction is stable as long as
\( k <= Z. \)

INPUTS:
A = Factor in the exponential of the Incomplete Lipschitz-Hankel
integral. (Complex)
Z = Upper limit of integration. (Positive Real)
ERROR = Desired relative error bound for \( J_e (A,Z) \).
KMAX = The value of \( K \) where the magnitude of the terms is at a
local minimum. The desired asymptotic approximation
should be obtained before \( K = KMAX. \)

OUTPUTS:
AFNEUM = Computed value for the Incomplete Lipschitz-Hankel integral.
SUBROUTINES AND FUNCTIONS CALLED:

JHAEO1 - A subroutine which returns asymptotic approximations for the values of J (Z), and J (Z).

Inputs:
- Z = Argument of the Bessel functions.
- ERROR = Desired relative error bound.

Outputs:
- JBESS(0) = J (Z)
- JBESS(1) = J (Z)

VARIABLE DECLARATIONS:

COMPLEX*16 A, COEFF, TERM, ZASP1
DOUBLE PRECISION Z, ZD2
REAL ERROR
INTEGER NMAX, COUNT, K

COMMON BLOCKS

FACTOR(K)=SQPI/(2.**(K+1)*GAMMA(K+1.5))
DOUBLE PRECISION FACTOR(0:200)
COMMON/FACT/FACTOR

WORK ARRAY FOR COMPUTING THE BESSEL FUNCTIONS
DOUBLE PRECISION JBESS(0:200)
COMMON /WORK/ JBESS

IM=CDSQRT(-1), ASP1=A*A+1, EXPMAZ=CDEXP(-A*Z), SQASP1=CDSQRT(ASPI)
COMPLEX*16 IM, ASP1, EXPMAZ, SQASP1
PI, SQPI=DSQRT(PI)
DOUBLE PRECISION PI, SQPI
AABASP1=CDABS(ASPI)
REAL AABASP1
COMMON/CONST/IM, PI, SQPI, ASP1, AABASP1, EXPMAZ, SQASP1

Obtain the starting values J (Z), and J (Z)
CALL JHAEO1(Z, JBESS(0), JBESS(1), ERROR)

Take the proper branch cut for SQASP1
IF ( (REAL(A) .LT. 0 .AND. ABS(AIMAG(A)) .LE. 1.) ) SQASP1=-SQASP1

Initialize variables
ZD2=Z/2.DO
ZASP1=Z*ASP1
COEFF=EXPMAZ/ASP1
COUNT=-1

Initialize the series
AFNEUM=1.DO/SQASP1+COEFF*(JBESS(1)-A*JBESS(0))

Recursively forward on Jk and sum the series
DO 100, K=1, NMAX
  find the next Jk
  JBESS(K+1)=DBLE(K)*JBESS(K)/ZD2-JBESS(K-1)
  COEFF=COEFF/ZASP1
  Calculate the next term in the series
  TERM=COEFF*(JBESS(K+1)-A*JBESS(K))/FACTOR(K-1)
  AFNEUM=AFNEUM+TERM

Determine whether the desired error has been obtained
IF (CABS(CMPLX(TERM)/CMPLX(AFNEUM)) .LT. ERROR) THEN
  IF (K .EQ. COUNT+1) GOTO 120
  COUNT = K
END IF
100 CONTINUE
120 RETURN
END
COMPLEX*16 FUNCTION BRNEUM(A,Z,ERROR)

(Neumann Series Expansion - Bessel Functions Obtained Using Backward Recurrence)

----------------------------------------------------------------------------------
PURPOSE:
This routine computes the incomplete Lipschitz-Hankel integral,

\[ J_{e}(A,Z) = \frac{1 - \exp(-A^2)}{COEFF(k)*J(k)} / CSQRT(A^2+1.0) \]

\[ \text{for } k=0 \]

where \( COEFF(k) := E^{-(1/(CSQRT(A^2+1.0)+A))^k} \) and \( k \)

\[ \{ 1 \ ; \ k = 0 \]
\[ \{ 2 \ ; \ k = 1,2,3,... \]

The proper branch cut for the square root is defined by:

\[ \text{Re}(A^2+1.0) \geq 0 \ ; \ \text{Re}(A) \geq 0 \]
\[ \text{Re}(A^2+1.0) < 0 \ ; \ \text{Re}(A) < 0 \]

The Bessel functions are calculated using backward recurrence.

----------------------------------------------------------------------------------
INPUTS:
A = Factor in the exponential of the Incomplete Lipschitz-Hankel integral. (Complex)
Z = Upper limit of integration. (Positive Real)
ERROR = Desired relative error bound for \( J_{e}(A,Z) \).
ERRJ = Required relative error for the Bessel functions.

----------------------------------------------------------------------------------
OUTPUTS:
BRNEUM = Computed value for the Incomplete Lipschitz-Hankel integral.

----------------------------------------------------------------------------------
SUBROUTINES AND FUNCTIONS CALLED:
JBREC = A subroutine which computes the sequence of Bessel functions using a backward recurrence algorithm.

Inputs:
Z = Argument of the Bessel functions.
ERROR = Desired relative error bound.

Outputs:
JBESS = An array of unnormalized Bessel functions passed in a common block.
NORM = The normalization constant.
NORM = Number of Bessel functions in the array.
VARIABLE DECLARATIONS:

COMMON*16 A, COEFF, TEMP, TERM
DOUBLE PRECISION Z, NORM
REAL ERROR
INTEGER COUNT, NOTERM, I, K

COMMON BLOCKS

DOUBLE PRECISION JBESS(0:200)
COMMON /WORK/ JBESS

IM=CDSQRT(-1), ASPL=A*A+1, EXPMAZ=CDEXP(-A*2), SQASP1=CDSQRT(ASPL)
COMPLEX*16 IM, ASPL, EXPMAZ, SQASP1

PI, SQPI=DSQRT(PI)
DOUBLE PRECISION PI, SQPI

ABASP1=CDABS(ASPL)
REAL ABASP1
COMMON/CONST/IM, PI, SQPI, ASPL, ABASP1, EXPMAZ, SQASP1

C******************************************************************************
C Compute the unnormalized Bessel functions using backward recurrence
CALL JBEZC(Z, ERROR, NORM, NOTERM)
C******************************************************************************

Initialize the constants
COEFF=0.5D0/NORM
TEMP=1.0D0/(SQASP1+A)
BRNEUM=0.5D0/EXPMAZ-COEFF*JBESS(0)
COEFF=2.0D0*COEFF*TEMP
BRNEUM=BRNEUM+COEFF*JBESS(1)
COUNT=-1

Sum the series

Note: The series will probably not converge until I > Z.
DO 100, I=2, INT(REAL(Z))
   COEFF=COEFF*TEMP
   BRNEUM=BRNEUM+COEFF*JBESS(I)
100 CONTINUE
DO 200, K=1, NOTERM
   COEFF=COEFF*TEMP

Calculate the next term in the series
TERM=COEFF*JBESS(K)
BRNEUM=BRNEUM+TERM

Test for convergence. Two consecutive terms must satisfy
the error criterion.
IF (CABS(CMPLX(TERM))/CMPLX(BRNEUM)) .LT. ERROR) THEN
   IF (K .EQ. COUNT+1) GOTO 300
   COUNT=K
END IF
200 CONTINUE
300 BRNEUM=BRNEUM*EXPMAZ*2.0D0/SQASP1
RETURN
END
SUBROUTINE JBREC(Z,ERROR,NORM,NOTERM)

PURPOSE:
The Bessel functions, J(z), are computed using a backward recurrence
algorithm by Olver (see Appendix B)

INPUTS:
Z = Argument of the Bessel function. (Positive Real)
ERROR = Desired relative error for the Bessel functions.

OUTPUTS:
JBESS = An array of unnormalized Bessel functions passed in
a common block.
NORM = The normalization constant.
NOTERM = Number of computed Bessel functions.

VARIABLE DECLARATIONS:
NOTE: JBESS(0:200) WILL TEMPORARILY HOLD PR
DOUBLE PRECISION Z,JBESS(0:200),NORM,ZD2
REAL ERROR,INVER
INTEGER NOTERM,I,K
COMMON /WORK/ JBESS

ZD2=Z/2.0D0
IF (REAL(Z) .GT. 1.) THEN
   INVER=1.7725*SQR(R(REAL(ZD2)))/ERROR
ELSE
   INVER=1./ERROR
END IF

THE STARTING VALUE MUST BE EVEN
I=2*INT(ZD2)
JBESS(I)=0.0D0
JBESS(I+1)=1.0D0

Use the recurrence relation (C-3) in the forward direction
DO 11,K=I+1,198,2
   JBESS(K+1)=DBLE(K)*JBESS(K)/ZD2-JBESS(K-1)
   JBESS(K+2)=DBLE(K+1)*JBESS(K+1)/ZD2-JBESS(K)
11 CONTINUE
WRITE(*,*) 'NOT ENOUGH TERMS IN ARRAY JBESS'
RETURN

Initialize for the backward recurrence
20 JBESS(K+3)=0.0D0
   NOTERM=K+2
   JBESS(NOTERM)=1.0D0
   NORM=0.0D0
   DO 30,K=NOTERM-3,-2
      JBESS(K-1)=DBLE(K)*JBESS(K)/ZD2-JBESS(K+1)
      JBESS(K-2)=DBLE(K-1)*JBESS(K-1)/ZD2-JBESS(K)
30 CONTINUE
   Compute the normalization constant (see (C-11a))
   NORM=NORM+JBESS(K-1)
RETURN
30 CONTINUE
JBESS(0) = JBESS(1) / ZD2 - JBESS(2)
NORM = NORM * 2. DO + JBESS(0)
RETURN
END
SUBROUTINE JHAEOI(Z,J0,J1,ERROR)

PURPOSE:

This subroutine uses Hankel’s asymptotic expansion (C-9) to calculate
the Bessel function of the first kind, of orders 0 and 1.

INPUTS:

Z = Argument of the Bessel function. (Positive Real)
ERROR = Desired relative error for the Bessel functions.

OUTPUTS:

J0 = Bessel function of the first kind, order zero.
J1 = Bessel function of the first kind, order one.

VARIABLE DECLARATIONS:

DOUBLE PRECISION Z,P0,Q0,P1,Q1,ZSQ,SQMP,JO,JI,TERMO,TERM1
REAL ERROR
INTEGER K

COMMON BLOCKS

FACTOR(K)=SQRT/(2.*(K+1)*GAMMA(K+1.5))
DOUBLE PRECISION FACTOR(0:200),POARRY(4,25)
COMMON/FACT/FACTOR,POARRY

DOUBLE PRECISION COSX,SINX
COMMON /SINE/COSX,SINX

IM=CDSQRT(-1)
COMPLEX*16 IM
PI, DSQRT(PI)
DOUBLE PRECISION PI,SQPI
COMMON/CONST/IM,PI,SQPI

---

Initialize the constants for J0
ZSQ=Z*Z
P0=1.0D0
Q0=1.0D0/(8.0D0*Z)
J0=COSX-Q0*SINX

Initialize the constants for J1
P1=1.0D0
Q1=-3.0D0*Q0
J1=SINX+Q1*COSX

Calculate J0 and J1 using (C-9)

Note: POARRY was computed in INITIAL and passed in a common block

DO 100 K=1,25
  POARRY(1,K) = (0.2*K)*(-1/4)**K, where Hankel’s symbol is
defined as (N,K) = GAMMA(.5+N+K)/(GAMMA(K+1)*GAMMA(.5+N-K))
  P0=P0*POARRY(1,K)/ZSQ
  POARRY(2,K) = (1.2*K)*(-1/4)**K
  P1=P1*POARRY(2,K)/ZSQ
  POARRY(3,K) = 0.5*(0.2*K+1)*(-1/4)**K
100 CONTINUE
QK0 = QK0 * PQARRY(3, K) / ZSQ
PQARRY(4, K) = 0.5 * (1.2 * K + 1) * (-1/4)**K
QK1 = QK1 * PQARRY(4, K) / ZSQ
TERM0 = PK0 * COSX - QK0 * SINX
J0 = J0 + TERM0
TERM1 = PK1 * SINX + QK1 * COSX
J1 = J1 + TERM1
IF (MAX(ABS(REAL(TERM0)/REAL(J0)), ABS(REAL(TERM1)/REAL(J1)))
   .LT. ERROR) GOTO 200
100 CONTINUE
   WRITE(*, *) 'NOT ENOUGH TERMS IN JHAE01'
200 SQTMP = DSQRT(2.0D0/FI*2))
   J0 = SQTMP * J0
   J1 = SQTMP * J1
RETURN
END
REFERENCES


