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Planar Model Characterization of Compensated Microstrip Bends

by

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ABSTRACT

This report discusses planar analysis of microstrip right-angle bend discontinuities. The bend discontinuity region is approximated using the planar waveguide model for microstrip line. The discontinuity configuration is broken up into multi-port regular segments. Each multi-port segment is characterized by its impedance matrix which is computed from the impedance Green’s function for that particular segment. The overall characterization of the discontinuity configuration is found by combining the impedance matrices for various segments using segmentation and/or desegmentation method.

A faster way of computing the impedance matrix elements for an isosceles triangular segment can be achieved by carrying out one of the two infinite summations in the Green’s function expression analytically using trigonometric Fourier series. The expressions for computing the impedance matrix elements for circular sector and annular sector segments are derived and implemented. These expressions are based on the doubly infinite series Green’s functions.

The computer program for planar analysis is developed. This program is used to analyze compensated microstrip right-angle bend discontinuities. The various compensated bend configurations analyzed are chamfered bends, bends rounded at the outside corner, bends rounded at both the outside and inside corners, and outer-recess bends. The results are compared with some published experimental results.
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CHAPTER 1

INTRODUCTION

An accurate and reliable characterization of microwave/millimeter-wave circuit components is one of the basic prerequisites of a successful Computer Aided Design (CAD) of microwave/millimeter-wave circuits. The degree of accuracy to which the performance of microwave/millimeter-wave integrated circuits can be simulated depends upon the accuracy of the characterization and modelling of components. Inadequacy of the modelling of junctions and discontinuities in transmission structures has been recognized as one of the key bottlenecks in microwave/millimeter-wave CAD.

Various methods available for characterization of microstrip components/discontinuities may be divided into three groups:

• 1. Quasi-static techniques for evaluation of capacitances and inductances associated with various discontinuity configurations.

• 2. Planar model analysis leading to frequency dependent scattering matrix evaluation of discontinuities like steps, bends, tee-junctions, cross-junctions, etc.

• 3. Full-wave analysis techniques using spectral domain and/or moment method or finite-element analysis or TLM approaches.

The second group, planar model analysis, is discussed in this report. Planar analysis of microstrip discontinuities is based on the parallel-plate waveguide planar model of the microstrip line fields. Planar analysis is more attractive for
interactive CAD environment due to the simplicity of the analysis compared to the rigorous full-wave analysis. Also, the results obtained from planar analysis are more accurate than those obtained from quasi-static analysis.

One type of microstrip discontinuities is a microstrip right-angle bend. Right-angle bend are used in microstrip circuits frequently. At the discontinuity section (i.e. at the bend) both the fields and the currents get disturbed. The current lines are concentrated at the inner corner, while at the outer corner there exists an excess discontinuity capacitance associated with the electric field near the corner. This excess capacitance introduces mismatch in the designed circuit. A typical value of $VSWR$ is 1.42 for a 50 ohms microstrip right-angle bend on a Rexolite substrate ($\varepsilon_r = 2.62$ and substrate height $h = 3.2$ mm). Hence, this unwanted reactance need to be eliminated, and can be eliminated or reduced by compensations. Basically there are two approaches for compensating discontinuity reactances. The first one is compensation by modifying the circuit, this is carried out during circuit optimization. The second one is compensation by modifying the discontinuity configuration. Many results on compensated microstrip bends have been reported [1], [2], [3], [4], [5]. Almost all of the results reported are based on experiments, except those reported in [1]. Compensation of microstrip bends by modifying the discontinuity configuration can be carried out in two different ways. A first way is by removing the excess discontinuity capacitances associated with the unmodified bend, i.e., by modifying the outer corner of the bend. This approach results in configurations such as chamfered bends, rounded bends, and outer recess bends [1], [2], [3], [4], [5]. A second way is by introducing inductance at the discontinuity region such that this inductance cancels out the excess capacitance associated with
the unmodified bend, i.e., by modifying the inner corner of the bend. This approach results in configurations such as inner recess bends [5].

This report discusses the planar analysis of compensated microstrip right-angle bend discontinuities using a segmentation/desegmentation method; these methods are described in [6], [7], [8], [9], [10], [11], [12]. Before the discussion of segmentation/desegmentation methods, the planar waveguide models of microstrip line and microstrip discontinuities are presented. Chapter 4 presents a faster formulation of the expressions for the impedance matrix elements for isosceles triangular segments. The expression for the impedance matrix elements for circular sector and annular sector segments are derived in Chapter 5. The computer code developed for planar analysis of microstrip discontinuities, the parameters that influence the convergence of the results, the computations of the zeros of Bessel’s functions, and the shift of reference planes are discussed in the chapter on numerical implementation. The numerical results presented are for chamfered bends, bends rounded at the outside corner, bends rounded at both the outside and inside corners, and outer recess bends. These results are compared with the unmodified bends and with some published experimental results. Some concluding remarks are given in the last chapter.
CHAPTER 2

PLANAR WAVEGUIDE MODELS OF MICROSTRIP LINES AND DISCONTINUITIES

In this chapter, the planar waveguide model of microstrip line is discussed (see [2], [10], [13], [14]). The frequency dependent quantities such as the effective width, characteristic impedance, effective dielectric constant, and losses are discussed first. The last section describes the modelling of microstrip discontinuities using the planar waveguide model of microstrip line.

2.1 The Effective Width of Planar Waveguide Model

The planar waveguide model of microstrip line is shown in Figure 2.1. The ideal planar waveguide consists of two parallel conductors bounded by magnetic walls (i.e. no normal component of the electric field, nor tangential component of the magnetic field) in the transverse directions. The electric and magnetic fields in the dielectric region are uniform because there are no fringing fields outside the magnetic walls (see Figure 2.1(b)).

The width of the model is larger than the physical width in order to account for the fringing/edge fields of microstrip. This width is called the effective width \( w_e(f) \). The effective width is determined from the characteristic impedance \( Z_0(f) \) and the effective dielectric constant \( \epsilon_r(f) \) according to the parallel plate waveguide formula:

\[
w_e(f) = \frac{\eta_0 h}{Z_0(f)\sqrt{\epsilon_r(f)}}
\]  

(2.1)

where \( \eta_0 \) is the wave impedance in air/vacuum (= 120\( \pi \)\( \Omega \)); \( h \) is the substrate
height; $Z_0(f)$ is the dynamic characteristic impedance; and $\epsilon_r(f)$ is the
dynamic effective dielectric constant.

2.2 Characteristic Impedance and Effective Dielectric Constant

An accurate closed-form expression for characteristic impedance and
effective dielectric constant has been reported by Hammerstad and Jensen [15].
The expressions for $Z_0\left(\frac{w}{h}, t = 0\right)$ in a homogeneous media ($\epsilon_r = 1$) and for
$\epsilon_r\left(\frac{w}{h}, \epsilon_r, t = 0\right)$ are derived from functional approximation of analytical equations for $t$ (the strip thickness) equal to zero. The expression for $Z_0\left(\frac{w}{h}, t = 0\right)$ for $\epsilon_r = 1$ based on [15] is given as follows:

$$Z_0\left(\frac{w}{h}, t = 0\right) = \frac{\eta_0}{2\pi} \ln\left(\frac{f\left(\frac{w}{h}\right)}{\frac{w}{h}} + \sqrt{1 + \left(\frac{2h}{w}\right)^2}\right)$$  \hspace{1cm} (2.2)

where

$$f\left(\frac{w}{h}\right) = 6 + (2\pi - 6)\exp\left(-\left(30.666\frac{h}{w}\right)^{0.7528}\right)$$ \hspace{1cm} (2.3)
This expression valid for $0 < \frac{w}{h} < \infty$. The maximum error of Equation (2.2) is 0.01% for $\frac{w}{h} < 1$ and 0.03% for $\frac{w}{h} < 1000$ compared to the exact solutions.

The expression for $\epsilon_{\tau}\left(\frac{w}{h}, \epsilon_{\tau}, t = 0\right)$ given by [15] is an extension of the Schneider’s equation [16]:

$$
\epsilon_{\tau}\left(\frac{w}{h}, \epsilon_{\tau}, t = 0\right) = \frac{\epsilon_{\tau} + 1}{2} + \frac{\epsilon_{\tau} - 1}{2} \left(1 + \frac{10h}{\epsilon_{\tau}}^{(\epsilon_{\tau})} - a\left(\frac{w}{h}\right)b\left(\epsilon_{\tau}\right)\right)
$$

(2.4)

where

$$
a\left(\frac{w}{h}\right) = 1 + \frac{1}{49} \ln\left(\left(\frac{w}{h}\right)^4 + \left(\frac{w}{h}\right)^2\right) + \frac{1}{18.7} \ln\left(1 + \left(\frac{w}{18.1h}\right)^3\right)
$$

(2.5)

$$
b\left(\epsilon_{\tau}\right) = 0.564\left(\frac{\epsilon_{\tau} - 0.9}{\epsilon_{\tau} + 3}\right)^{0.053}
$$

(2.6)

The accuracy of this expression is better than 0.2% for $0.01 < \frac{w}{h} < 100$ and $1 \leq \epsilon_{\tau} \leq 128$.

2.2.1 Expressions for $Z_0$ and $\epsilon_{\tau}$ taking into account the effect of finite conductor thickness of the strip. The expression for $Z_0\left(\frac{w}{h}, l, \epsilon_{\tau}\right)$ and $\epsilon_{\tau}\left(\frac{w}{h}, l, \epsilon_{\tau}\right)$ are derived by Hammerstad and Jensen [15] from a functional approximation of the numerical results for conductor thickness $t > 0$ and conductor width $w_e = w + \Delta w$ ($w_e$ not to be confused with $w_e(f)$ in Equation (2.1)). The resulting expressions are very accurate for narrow strips and for substrates with low dielectric constant. For homogeneous media ($\epsilon_{\tau} = 1$) the width correction is [15]:

$$
\Delta w_{t,0} = \frac{t}{\pi} \ln\left(1 + \frac{4\exp(1)}{(l/h) \coth^2(\sqrt{6.517}\frac{w}{h})}\right)
$$

(2.7)

where $\exp(1) = e = 2.71828$ (base of the natural logarithm); and $w_{t,0} = w + \Delta w_{t,0}$ is the corrected strip width for $\epsilon_{\tau} = 1$. For $\epsilon_{\tau} > 1$, the width correction is [15]:

$$
\Delta w = \frac{\Delta w_{t,0}}{2} \left(1 + \frac{1}{\cosh(\sqrt{\epsilon_{\tau} - 1})}\right)
$$

(2.8)
where \( w_c = w + \Delta w \) is the corrected width for \( \epsilon_r > 1 \). Thus, using Equations (2.7) and (2.8) the effect of strip thickness can be included in the computations of \( Z_0 \) and \( \epsilon_{\epsilon_r} \) [15] as follow:

\[
Z_0 \left( \frac{w}{h}, \frac{t}{h}, \epsilon_r \right) = \frac{Z_0 \left( \frac{w_e}{h}, t = 0, \epsilon_r \right)}{\sqrt{\epsilon_{\epsilon_r} \left( \frac{w_e}{h}, t = 0, \epsilon_r \right)}}
\]

(2.9)

and

\[
\epsilon_{\epsilon_r} \left( \frac{w}{h}, \frac{t}{h}, \epsilon_r \right) = \left( \frac{Z_0 \left( \frac{w_e}{h}, t = 0 \right) \epsilon_{\epsilon_r} \left( \frac{w_e}{h}, t = 0, \epsilon_r \right)}{Z_0 \left( \frac{w_e}{h}, t = 0, \epsilon_r \right)} \right)^2
\]

(2.10)

where \( Z_0 \left( \frac{w_e}{h}, t = 0 \right) \) and \( Z_0 \left( \frac{w_e}{h}, t = 0 \right) \) are given by Equation (2.2); and \( \epsilon_{\epsilon_r} \left( \frac{w_e}{h}, t = 0, \epsilon_r \right) \) is given by Equation (2.4).

2.2.2 Expressions for \( Z_0 \) and \( \epsilon_{\epsilon_r} \) taking into account the effect of dispersion in microstrip. Both characteristic impedance and effective dielectric constant vary with frequency because microstrip propagation is not purely TEM. Getsinger [17] used the Longitudinal-Section Electric (LSE) mode dispersion model to account for the dispersion in the effective dielectric constant. The model consists of a wave medium that allows LSE modes to propagate (see Figure 2.2). The model configuration (see Figure 2.2(b)) is defined by electric and magnetic walls. Since the field distribution in the model is different than in the original circuit hence, the dispersion model does not have exactly the same dispersion function \( \epsilon_{\epsilon_r}(f) \) as the original circuit. A good approximation can be achieved because the fields in microstrip are concentrated below the parallel plate capacitor section and in the conductor edges (fringing fields). Near the conductor edges the electric field is parallel and the magnetic
Figure 2.2. LSE dispersion model: (a) original circuit; (b) LSE dispersion model.

field is perpendicular to the dielectric boundary. Thus, the model is divided into three regions (see Figure 2.2(b)).

Region I is a planar waveguide with dielectric constant $\varepsilon_r$ (the same $\varepsilon_r$ as the substrate of the original circuit) and an arbitrarily chosen plate separation $h$ which generates the parallel plate capacitor field. The other regions, II and III, represent the fields in air of the original configuration. The planar waveguides of regions II and III have larger plate separation because the air-filled section is larger than the substrate. The E-field lines are parallel and the H-field lines are perpendicular to the dielectric boundary in the model. The dispersion of the model is adjusted to be the same as the dispersion of the original configuration by choosing the right values of $b/a$, $c/h$ and $b/h$. The right choice of $b/a$ and $c/h$ will match the values of $Z_0(\frac{w}{h}, \frac{c}{h}, \varepsilon_r)$ and $\varepsilon_r(\frac{w}{h}, \frac{c}{h}, \varepsilon_r)$ of the model to the original configuration. The dispersion function $\varepsilon_r(f)$ of the
model is determined by the transverse resonance method [18]. The value of \( b/h \) is determined such that the result matches the numerically determined dispersion function of the original configuration. The resulting dispersion function is [17]:

\[
\epsilon_r(e)(f) = \epsilon_r - \frac{\epsilon_r - \epsilon_r(w, \frac{t}{h}, \epsilon_r)}{1 + G \left( \frac{f}{f_p} \right)^2}
\]  

(2.11)

where \( f \) is the operating frequency and \( f_p = Z_0(\frac{w}{h}, \frac{t}{h}, \epsilon_r)/(2\mu_0 h) \) is an approximation to the first TE-mode cut-off frequency. \( G \) is a dimensionless factor which is empirically determined. The expression for \( G \) in [17] fits experimental data for alumina substrate only. Hammerstad and Jensen [15] gave an expression for \( G \) for any types of substrate now in use:

\[
G = \frac{\pi^2}{12} \frac{\epsilon_r - 1}{\epsilon_{re}(w, \frac{t}{h}, \epsilon_r)} \sqrt{\frac{2\pi Z_0(\frac{w}{h}, \frac{t}{h}, \epsilon_r)}{\eta_0}}
\]  

(2.12)

Based on the parallel-plate model, the frequency dependent characteristic impedance can be expressed as follows [15]:

\[
Z_0(f) = Z_0(\frac{w}{h}, \frac{t}{h}, \epsilon_r) \sqrt{\frac{\epsilon_{re}(w, \frac{t}{h}, \epsilon_r) \epsilon_r(e)(f) - 1}{\epsilon_{re}(w, \frac{t}{h}, \epsilon_r) - 1}}
\]  

(2.13)

where \( Z_0(\frac{w}{h}, \frac{t}{h}, \epsilon_r) \) is given by Equation (2.9) and \( \epsilon_{re}(w, \frac{t}{h}, \epsilon_r) \) is given by Equation (2.10). These last two equations for \( Z_0 \) and \( \epsilon_{re} \) (Equations (2.11) and (2.13)) are used in computing the effective width of the microstrip as given by Equation (2.1).

### 2.3 Losses in Microstrip Line

The conductor loss is computed [10] by assuming that a very small electric field (small enough not to disturb the field configuration) exists on the surface of the conductors. Thus, the Poynting vector has a small component normal to the conductors. The attenuation constant is derived from the power
attenuation rate, knowing the surface impedance of a metallic material when the conductivity $\sigma$ is much larger than $(\omega \varepsilon)$. Hence, the attenuation constant due to conductor loss can be expressed as [10]:

$$\alpha_c = \omega \sqrt{\mu \varepsilon} \frac{r}{2h} \quad (2.14)$$

where $h$ is the plate separation and $r = \sqrt{\frac{2}{\omega \mu \sigma}}$ is the skin depth of the conductor.

Similarly, the dielectric loss is derived from the power attenuation rate due to dielectric loss [10]. The expression for the attenuation constant due to the dielectric loss is given below:

$$\alpha_d = \omega \sqrt{\mu \varepsilon} \frac{\tan \delta}{2} \quad (2.15)$$

where $\tan \delta$ is the loss tangent delta of the dielectric.

Comparing Equations (2.14) and (2.15), one can define an equivalent loss tangent delta to account for the conductor loss as:

$$\tan \delta_e = \frac{1}{h \sqrt{\mu \pi f \sigma}} \quad (2.16)$$

Thus, the effective loss tangent of the dielectric (including the conductor loss) can be expressed as:

$$\tan \delta_e = \tan \delta + \tan \delta_c$$

$$= \tan \delta + \frac{1}{h \sqrt{\mu \pi f \sigma}} \quad (2.17)$$

where $\mu = \mu_0$ is the free space permeability; $f$ is the operating frequency; and $\sigma$ is the conductivity of the metallic material.

2.4 Planar Waveguide Model of Microstrip Discontinuities

Planar waveguide models of microstrip discontinuities can be derived from the planar waveguide model of microstrip line. The planar waveguide models of four microstrip discontinuities are illustrated in Figure 2.3.
Figure 2.3. Planar waveguide models of four microstrip discontinuities: (a) right-angle bend; (b) chamfered right-angle bend; (c) step-in-width; (d) tee-junction.
In all of these cases, the straight section of microstrip lines are modelled using the planar waveguide model of microstrip line with their effective widths, effective dielectric constants, and effective loss tangent computed using Equations (2.1), (2.11), and (2.17) respectively. The discontinuity region is modelled by extrapolating the effective dimensions. In Figure 2.3(a), the outer corner (between planes $T - T$) is obtained by extending the outer dimensions to meet at $oc$. In Figure 2.3(b), the points $x$ and $y$, which are obtained by replacing $w$ by $w_*(f)$, are joined together. Various other options available in this case are discussed later in Section 7.1. In Figure 2.3(c), the junction planes $T - T$ is shifted towards the narrower line by an amount $\Delta l$ (obtained from the open-end capacitance of microstrip or $(w_2 - w_1)$). In Figure 2.3(d), the planar model of the tee-junction is obtained by extending the effective boundary as shown.
CHAPTER 3

ANALYSIS OF PLANAR STRUCTURES

Three methods of analyzing the two-dimensional planar structures are presented in this chapter. The first one is the Green's function approach which is used if the structure is made up of a single regular segment only. The second method is the segmentation method which is used when the structure is a composite of several regular segments for which the Green's functions are available. The third method is complementary to the segmentation method and is called desegmentation method. These three methods are used in analyzing microstrip discontinuities; the details of this discontinuity analyses are discussed in Chapter 6.

3.1 Green's Function Approach

Consider an N-port arbitrarily shaped planar circuit shown in Figure 3.1. The ports are located along the periphery and their widths are $w_i, w_j,...$ etc. The rest of the periphery is an open boundary. Thus, the magnetic wall boundary condition $(\frac{\partial E_x}{\partial n} = 0)$ holds. The excitation is obtained by coupling an external circuit to one of the ports of the planar circuit. The dimensions in $x, y$ coordinates are comparable to the wavelength but, the substrate height is much smaller than the wavelength. Hence, there is no variation of fields in the $z$-direction. That is, $\frac{\partial}{\partial z} = 0$ and $H_z = E_z = E_y = 0$, and the field components satisfy the two-dimensional Helmholtz wave equation. Therefore, for homogeneous and isotropic substrate the two-dimensional Helmholtz
The voltage for this two-dimensional component is related to the electric field $E_z$ as:

$$v(x, y) = -h E_z(x, y)$$

(3.4)

where $h$ represents the spacing between the planar conductor and the ground conductor (i.e. the substrate height). This voltage is related to a $z$-directed source current $i(x_0, y_0)$ by an impedance Green's function defined as:

$$v(x, y) = \int \int G(x, y | x_0, y_0) i(x_0, y_0) dx_0 dy_0$$

(3.5)

The impedance Green's function $G(x, y | x_0, y_0)$ is the solution of:

$$(\nabla_T^2 + k^2)G(\vec{r} | \vec{r}_0) = -j\omega \mu \epsilon \delta(\vec{r} - \vec{r}_0)$$

(3.6)

where $\nabla_T^2$ is defined by Equation (3.2); $k$ is defined by Equation (3.3); and $\vec{r}$ represents the voltage point and $\vec{r}_0$ represents the source location.

If the planar circuit is fed by another planar circuit as shown in Figure 3.2, the current flow $J_z(x_0, y_0)$ is in the $x-y$ plane and not in the $z$-direction as assumed in the evaluation of the Green's function above. However, this feed current can be expressed in terms of an equivalent fictitious electric current density ($z$-directed) at the edge of the planar circuit being fed by considering the vertical interface between the two planar circuits (see Figure 3.2(b)).
Figure 3.1. Microstrip type planar circuit: (a) side view; (b) top view of the planar segment.
Figure 3.2. Equivalence between the port current and the z-directed current density at the junction between two planar circuits (the feed structure and the circuit being fed): (a) isometric view; (b) side view.
The feed current can be expressed as \( \bar{J}_s = \bar{n} \cdot \bar{J}_s \) and it is related to the magnetic field in the planar structure by \( \bar{J}_s = \bar{n} \times \bar{H} \) where \( \bar{n} \) underneath the strip is in \(-z\)-direction. The equivalent electric current density at the interface between the feed structure and the circuit being fed may now be expressed as:

\[
\bar{J}_s = \bar{n} \times \bar{H} = \bar{z} \mid \bar{H} \mid
\]

(3.7)

Since the feed current \( \bar{J}_s = \bar{y} \mid \bar{J}_s \mid = -\bar{z} \times \bar{H} \), using this in Equation (3.7) will give the following:

\[
\bar{J}_s = \bar{z} \mid \bar{H} \mid = \bar{z} \mid \bar{J}_s \mid
\]

(3.8)

Thus, \( \bar{J}_s \) for a planar structure-fed port can be replaced by an equal magnitude of \( z \)-directed, fictitious, equivalent electric current density at the port location.

The equivalent \( z \)-directed current \( i_j \) fed at the \( j^{th} \) port of width \( w_j \) as shown in Figure 3.1 is given by:

\[
i_j(x_0, y_0) = \int_{w_j} J_s(x_0, y_0) d\bar{r}_0
\]

(3.9)

where \( J_s(x_0, y_0) \) is the input current over the port width \( w_j \). Thus, from Equations (3.5), (3.6), and (3.9), and with the assumption that the current densities are constant over the port widths \( w_i \) and \( w_j \), the elements of the impedance matrix of the planar circuit can be expressed as:

\[
Z_{ij} = \frac{1}{w_i w_j} \int_{w_i} \int_{w_j} G(x_i, y_i \mid x_j, y_j) \, d\bar{r}_i \, d\bar{r}_j
\]

(3.10)

where \( d\bar{r}_i \) and \( d\bar{r}_j \) are the incremental distances over the port widths \( w_i \) and \( w_j \) respectively.

The expressions for finding the impedance matrix elements of a rectangular segment, three different triangular segments, a circular sector segment,
and an annular sector segment are given in Chapter 4, Chapter 5, and Appendix C.

3.2 Segmentation and Desegmentation Methods

One of the methods used to analyze a more complex planar circuit (i.e., the circuit is made up of two or more regular segments) is known as the segmentation method [6], [7], [8], [9], [10], [12]. A complementary procedure is known as the desegmentation method [7], [8], [10], [11]. The segmentation and desegmentation methods using impedance matrices [6], [11] are discussed in this section.

3.2.1 Segmentation method. The segmentation process involves breaking up a more complex planar circuit configuration into segments of regular shapes. Then, each segment is characterized by its impedance matrix. The combination of the impedance matrices of all the segments is used to evaluate the characterization of the whole circuit. To illustrate this segmentation process, consider the right-angle bend discontinuity shown in Figure 3.3. The discontinuity is broken up into two multiport segments (see Figure 3.3(b)). The impedance matrices for these two segments are $[Z_A]$ and $[Z_B]$ respectively. The two segments are connected together by interconnected ports ($q_i$ for segment $A$ and $r_i$ for segment $B$). The computational effort is reduced if $q_1$ is connected to $r_1$, $q_2$ to $r_2$, etc. [6]. The external ports for each segment ($p_a$ for segment $A$ and $p_b$ for segment $B$) are numbered first before the interconnection ports. The number of the external ports of segment $A$ is $p_a$ and that of segment $B$ is $p_b$. There are $q (=r)$ interconnected ports. The $Z$-matrices of
Figure 3.3. Right-angle bend discontinuity: (a) original overall configuration; (b) the configuration is broken up into two multiport segments.

Both segments $A$ and $B$ can be written together as follow:

\[
\begin{bmatrix}
V_p \\
V_q \\
V_r
\end{bmatrix} = 
\begin{bmatrix}
Z_{pp} & Z_{pq} & Z_{pr} \\
Z_{qp} & Z_{qq} & 0 \\
Z_{rp} & 0 & Z_{rr}
\end{bmatrix} 
\begin{bmatrix}
I_p \\
I_q \\
I_r
\end{bmatrix}
\]

(3.11)

where $p = p_a + p_b$, $V_p$ and $I_p$ are the voltages and currents at the external ports $p_a$ and $p_b$; and $V_q$, $V_r$ and $I_q$, $I_r$ are the voltages and currents at the interconnected ports $q$ and $r$. The conditions for interconnecting $q_i$ to $r_i$ are:

\[
V_q = V_r \tag{3.12}
\]

and,

\[
I_q + I_r = 0 \tag{3.13}
\]
The $Z$-matrix of the overall configuration shown in Figure 3.3(a) can be computed by substituting Equations (3.12) and (3.13) into Equation (3.11) and eliminating $V_s$ and $I_s$:

$$Z_c = Z_{pp} + (Z_{pq} - Z_{pr})(Z_{qq} + Z_{rr})^{-1}(Z_{rp} - Z_{qp})$$  \hspace{1cm} (3.14)

The voltages at the interconnected port, related to the currents flowing in the external ports $p$, can be computed by substituting Equations (3.12) and (3.13) into Equation (3.11) and eliminating $V_p$ and $I_s$:

$$V_q = [Z_{qp} + Z_{qq}(Z_{qq} + Z_{rr})^{-1}(Z_{rp} - Z_{qp})]I_p$$  \hspace{1cm} (3.15)

Finally, the $Z$-matrix can be converted into an $S$-matrix representation by the following relation:

$$S = \sqrt{Y_0}(Z_c - Z_0)(Z_c + Z_0)^{-1}\sqrt{Z_0}$$  \hspace{1cm} (3.16)

where $Z_0$, $\sqrt{Z_0}$, and $\sqrt{Y_0}$ are diagonal matrices with the diagonal elements given by $Z_{01}$, $Z_{02}$, $\cdots$, $Z_{0n}$; $\sqrt{Z_{01}}$, $\sqrt{Z_{02}}$, $\cdots$, $\sqrt{Z_{0n}}$; and $\frac{1}{\sqrt{Z_{01}}}$, $\frac{1}{\sqrt{Z_{02}}}$, $\cdots$, $\frac{1}{\sqrt{Z_{0n}}}$ respectively. The diagonal matrix elements $Z_{01}$, $\cdots$, $Z_{0n}$ are the normalizing impedances at the various ports of the circuit.

3.2.2 Desegmentation method. The desegmentation procedure is complementary to the segmentation method. It is used for two reasons: first, for analyzing circuit configurations which can not be solved by the segmentation method (see Figure 3.4); second, some circuit configurations are easier or simpler to analyze using desegmentation rather than segmentation method (see Figure 3.5). The desegmentation process involves adding a new segment or segments to the original configuration. Then, the resulting new configuration and the added segments are characterized by their impedance.
Figure 3.4. Circuit configuration for which desegmentation is needed: (a) original circuit; (b) a circular segment is added to perform the analysis.

matrices. The characterization of the original circuit is carried out by removing or de-embedding the impedance matrix of the added segments from the impedance matrix of the new configuration. To illustrate this desegmentation procedure, consider the chamfered right-angle bend discontinuity configuration shown in Figure 3.5(a,c).

If a triangular segment $\beta$ is added to the original configuration $\alpha$, the resulting new configuration $\gamma$ is a right-angle bend which was analyzed in Section 3.2.1 (see also Figure 3.3). The impedance matrices for these two configurations are $[Z_\beta]$ and $[Z_\gamma]$ respectively. The two configurations are connected together by interconnected ports ($q_i$ for configuration $\alpha$ and $r_i$ for configuration $\beta$). As in the segmentation process, $q_1$ is connected to $r_1$, $q_2$ to $r_2$, etc. The $d$-ports are the external ports of segment $\beta$ and the number of $d$-ports must be at least equal to $q$ or $r$. The $p$-ports are the external ports of segment $\alpha$. Thus, the external ports of configuration $\gamma$ are $p$ and $d$. The conditions for interconnecting $q_i$ to $r_i$ are given by Equations (3.12) and (3.13). Therefore, the impedance matrix of the original configuration $\alpha$, with $d = q = r$, can be
Figure 3.5. Chamfered right-angle bend discontinuity: (a) original circuit; (b) segmentation process involves analyzing four segments; (c) desegmentation process involves analyzing three segments only.
expressed in terms of the impedance matrices of the $\beta$ and $\gamma$ configurations as:

$$Z_{(p \times p)}^\alpha = Z_{pp}^\gamma - Z_{pd}(Z_{dd}^\gamma - Z_{dd}^\beta)^{-1}Z_{dp}$$  \hspace{1cm} (3.17)

It may be noted that Equation (3.17) valid only for $d = q = r$. Thus, the scattering matrix of the original configuration (i.e. configuration $\alpha$) can be determined using Equation (3.16) with $Z_c$ replaced by $Z^\alpha$ found from above.
CHAPTER 4

IMPEDEANCE MATRIX EXPRESSIONS FOR 45°-90°- 45°
TRIANGULAR SEGMENT

In this chapter the single infinite series formulation for the Z-matrix of an isosceles triangular segment is derived. Then this new single series formulation is compared with the double infinite series formulation. But, first the motivation behind the derivation of the single infinite series formulation is presented.

4.1 Motivation for the Derivation of the Single Infinite Series Formulation

One of the basic requirements for a formulation to be useful in an interactive Computer Aided Design environment is that the formulation should be computationally inexpensive. The available double infinite series impedance matrix formulation for isosceles triangular segment is not suitable for this interactive environment due to the large computational time required. A typical computer time required by this double infinite series formulation to characterize a 50 % chamfered microstrip bend is about 1 hour of cpu time on the HP-9000 model 319C workstation. This computation is carried out using 100 terms for each series and 16 total interconnected ports (i.e., the triangular segment involved has 16 ports). This large computational requirement of the double series formulation makes it unattractive for CAD applications. Thus, a better and faster impedance matrix formulation for isosceles triangular segment is needed.
Figure 4.1. The right-angle isosceles triangular segment configuration and the port locations.

4.2 Single Infinite Series Impedance Matrix Expressions for Isosceles Triangular Segment

The computation of the impedance matrix elements for an isosceles triangular segment can be improved by converting the double series expression for the Green’s function into a single series expression. Using this single series Green’s function in the Equation (3.10) gives the single series impedance matrix expressions.

4.2.1 Single series Green’s function. A faster impedance matrix formulation for an isosceles triangular segment can be obtained by carrying out one of the infinite summations analytically. The details of this derivation are given in the Appendix A. The geometry of the right-angle isosceles triangular segment is shown in Figure 4.1. The double infinite series Green’s function expression for an isosceles triangular segment is given in [19] as:

\[ G(x_i, y_i | x_j, y_j) = \frac{j \omega \mu_0 \hbar}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sigma_m \sigma_n. \]
\[
\frac{[\cos(k_x x_i) \cos(k'_y y_i) + (-1)^{m+n} \cos(k'_x x_i) \cos(k_x y_i)]}{\cos(k_x x_j) \cos(k'_y y_j) + (-1)^{m+n} \cos(k'_x x_j) \cos(k_x y_j)}} \frac{(m^2 + n^2)\pi^2 - a^2 k^2}{(m^2 + n^2)\pi^2 - a^2 k^2} \] 

(4.1)

where \( h \) is the substrate height and

\[
k_x = \frac{m \pi}{a} \tag{4.2}
\]

\[
k'_y = \frac{n \pi}{a} \tag{4.3}
\]

\[
\sigma_l = \begin{cases} 
1 & \text{for } l = 0 \\
2 & \text{for } l \neq 0
\end{cases} \tag{4.4}
\]

\[
k^2 = \omega^2 \mu \epsilon_0 \epsilon_r (1 - j \delta_e) \tag{4.5}
\]

where \( \delta_e \) is the effective loss tangent given by Equation (2.17). The inner summation can be carried out analytically using trigonometric Fourier Series [20] as shown in the Appendix A.1. The resulting single infinite series Green’s function is shown below:

\[
G(x_i, y_i|x_j, y_j) = -\frac{j\omega \mu_0 h}{2a} \{G_1 + G_2\} \tag{4.6}
\]

For the \( G_1 \) term, if \((x_i + x_j) \geq (y_i + y_j)\) then

\[
G_1 = \sum_{m=0}^{\infty} 2D_m \cos(k_x x_i) \cos(k_x x_j) \cos(\gamma_m(y_1+ - a)) \cos(\gamma_m y_1+ ) \tag{4.7}
\]

and if \((y_i + y_j) > (x_i + x_j)\) then

\[
G_1 = \sum_{m=0}^{\infty} 2D_m \cos(k_x y_i) \cos(k_x y_j) \cos(\gamma_m(y_4+ - a)) \cos(\gamma_m y_4+ ) \tag{4.8}
\]

For the \( G_2 \) term, if \((x_i + y_j) \geq (y_i + x_j)\) then

\[
G_2 = \sum_{m=0}^{\infty} 2D_m (-1)^m \cos(k_x x_i) \cos(k_x y_j) \cos(\gamma_m y_2+ ) \cos(\gamma_m y_2- ) \tag{4.9}
\]
and if \((y_i + x_j) > (x_i + y_j)\) then

\[
G_2 = \sum_{m=0}^{\infty} 2D_m (-1)^m \cos(k_x y_i) \cos(k_x x_j) \cos(\gamma_m y_{3>} \cos(\gamma_m y_{3<})
\]  \hspace{1cm} (4.10)

where,

\[
D_m = \frac{\sigma_m}{\gamma_m \sin(\gamma_m a)} \hspace{1cm} (4.11)
\]

\[
y_{1>} = \max(y_i, y_j) \hspace{1cm} (4.12)
\]

\[
y_{1<} = \min(y_i, y_j) \hspace{1cm} (4.13)
\]

\[
y_{2>} = \max(y_i, x_j) \hspace{1cm} (4.14)
\]

\[
y_{2<} = \min(y_i, x_j) \hspace{1cm} (4.15)
\]

\[
y_{3>} = \max(x_i, y_j) \hspace{1cm} (4.16)
\]

\[
y_{3<} = \min(x_i, y_j) \hspace{1cm} (4.17)
\]

\[
y_{4>} = \max(x_i, x_j) \hspace{1cm} (4.18)
\]

\[
y_{4<} = \min(x_i, x_j) \hspace{1cm} (4.19)
\]

\[
\gamma_m = \pm \sqrt{k^2 - k_x^2} \hspace{1cm} (4.20)
\]

where, \(k^2\) is given by Equation (4.5); \(k_x\) is given by Equation (4.2); the sign of \(\gamma_m\) is chosen such that \(\text{Im}(\gamma_m)\) is negative; and \(\sigma_m\) is given by Equation (4.4).

**4.2.2 Z-matrix when port \(i\) and port \(j\) are located along \(OA\) or along \(OB\).** When both ports are located along \(OA\) (see Figure 4.1), the integration in Equation (3.10) is with respect to the variable \(x\) (i.e., \(d\tau_i = dx\) and \(d\tau_j = dx\)) and \((x_i + x_j) > (y_i + y_j)\). Hence, the expression given in Equation (4.7) is chosen for \(G_1\). On the other hand, when both the ports are located along \(OB\), then the integration is with respect to the variable \(y\) (i.e. \(d\tau_i = dy\) and \(d\tau_j = dy\)) and \((y_i + y_j) > (x_i + x_j)\). Hence, the expression given
in Equation (4.8) is chosen for $G_1$. These choices ensure the convergence of the series for the impedance matrix element $Z_{ij}$. Thus, using Equations (3.10) and (4.6), the expression for $Z_{ij}$ when both ports $i$ and $j$ are located either along $OA$ or $OB$ becomes:

$$Z_{ij} = -\frac{j\omega \mu_0 h}{2a} \{Z_1 + Z_2\} \quad (4.21)$$

where

$$Z_1 = \sum_{m=0}^{\infty} 2D_m \cos(\gamma_m a) \cos(k_z u_i) \text{sinc}(\frac{k_z w_i}{2}) \cos(k_z u_j) \text{sinc}(\frac{k_z w_j}{2}) \quad (4.22)$$

The expression for $Z_2$, if $u_i \geq u_j$, becomes

$$Z_2 = \sum_{m=0}^{\infty} 2D_m (-1)^m \cos(k_z u_i) \text{sinc}(\frac{k_z w_i}{2}) \cos(\gamma_m u_j) \text{sinc}(\frac{k_z w_j}{2}) \quad (4.23)$$

and if $u_j > u_i$ then

$$Z_2 = \sum_{m=0}^{\infty} 2D_m (-1)^m \cos(k_z u_j) \text{sinc}(\frac{k_z w_j}{2}) \cos(\gamma_m u_i) \text{sinc}(\frac{k_z w_i}{2}) \quad (4.24)$$

where

$$(u_i, u_j) = \begin{cases} (x_i, x_j) & \text{for } i, j \text{ along } OA \\ (y_i, y_j) & \text{for } i, j \text{ along } OB \end{cases} \quad (4.25)$$

and $D_m$ is given by Equation (4.11); $\gamma_m$ is given by Equation (4.20); and $k_z$ is given by Equation (4.2).

As $m$ becomes large, the imaginary part of the arguments of the complex trigonometric functions $\sin(\gamma_m \cdot)$ and $\cos(\gamma_m \cdot)$ in the Equations (4.11), (4.22), (4.23), and (4.24) will become very large and give rise to a numerical problem. To avoid this numerical problem, the trigonometric functions are replaced by their large argument approximations:

$$\sin(\gamma_m \cdot) = \frac{1}{2j} \exp(j\gamma_m \cdot) \quad (4.26)$$
\[
\cos(\gamma_m \cdot) = \frac{1}{2} \exp(j \gamma_m \cdot) \tag{4.27}
\]

Thus, using Equations (4.26) and (4.27), the single series expression for \( Z_{ij} \) in Equations (4.22) to (4.24) can be rewritten as:

\[
Z_1 = \sum_{m=0}^{L} 2D_m \cos(\gamma_m a) \cos(k_x u_i) \text{sinc}(\frac{k_x u_i}{2}) \cos(k_x u_j) \text{sinc}(\frac{k_x w_j}{2}) \\
+ \sum_{m=L+1}^{\infty} \frac{2j\sigma_m}{\gamma_m} \cos(k_x u_i) \text{sinc}(\frac{k_x u_i}{2}) \cos(k_x u_j) \text{sinc}(\frac{k_x w_j}{2}) \tag{4.28}
\]

The expression for \( Z_2 \), if \( u_i \geq u_j \), is

\[
Z_2 = \sum_{m=0}^{L} 2D_m (-1)^m \cos(k_x u_i) \text{sinc}(\frac{k_x u_i}{2}) \cos(\gamma_m u_j) \text{sinc}(\frac{k_x w_j}{2}) \\
+ \sum_{m=L+1}^{\infty} \frac{2\sigma_m (-1)^m}{\gamma_m^2 w_j} \exp[-j\gamma_m (a - x_j - \frac{w_j}{2})] \\
\cos(k_x u_i) \text{sinc}(\frac{k_x w_i}{2}) \tag{4.29}
\]

and if \( u_j > u_i \), then

\[
Z_2 = \sum_{m=0}^{L} 2D_m (-1)^m \cos(k_x u_j) \text{sinc}(\frac{k_x w_j}{2}) \cos(\gamma_m u_i) \text{sinc}(\frac{k_x w_i}{2}) \\
+ \sum_{m=L+1}^{\infty} \frac{2\sigma_m (-1)^m}{\gamma_m^2 w_i} \exp[-j\gamma_m (a - x_i - \frac{w_i}{2})] \\
\cos(k_x u_j) \text{sinc}(\frac{k_x w_j}{2}) \tag{4.30}
\]

where \( D_m \) is given by Equation (4.11); \( \gamma_m \) is given by Equation (4.20); \( k_x \) is given by Equation (4.2); and \( u_i \) and \( u_j \) are given by Equation (4.25). The integer \( L \) in Equations (4.28), (4.29), and (4.30) is chosen so that \( (\gamma_m a) \) is less than or equal to 50; this choice is a compromise between computational speed and accuracy.

4.2.3 Z-matrix when one port is located along OA and the other along OB. When one port is located along OA and the other is
located along $OB$ (see Figure 4.1), then the integration in Equation (3.10) is with respect to both $x$ and $y$. In this case the expressions for the Green's function $G_1$ and $G_2$ given in Equations (4.7) to (4.10) can be simplified by interchanging the terms inside the two summations in Equation (4.1). The simplified expressions for $G_1$ and $G_2$ are shown below:

$$G_1 = \sum_{m=0}^{\infty} 2D_m \cos(k_x u_i) \cos(\gamma_m (u_j - a))$$

(4.31)

$$G_2 = \sum_{m=0}^{\infty} 2D_m (-1)^m \cos(k_x u_i) \cos(k_x u_j)$$

(4.32)

where

$$(u_i, u_j) = \begin{cases} (x_i, y_j) & \text{if } i \text{ along } OA, j \text{ along } OB \\ (x_j, y_i) & \text{if } i \text{ along } OB, j \text{ along } OA \end{cases}$$

(4.33)

and $D_m$ is given by Equation (4.11); $\gamma_m$ is given by Equation (4.20); $k_x$ is given by Equation (4.2). Thus, the expression for the impedance matrix elements, making use of the large argument approximations for the complex trigonometric functions given by Equations (4.26) and (4.27), can be written as:

$$Z_{ij} = -\frac{C}{a} \left\{ \sum_{m=0}^{L} 2D_m \cos(k_x u_i) \text{sinc} \left( \frac{k_x w_i}{2} \right) \cos(\gamma_m (u_j - a)) \text{sinc} \left( \frac{k_x w_j}{2} \right) \right. + \sum_{m=L+1}^{\infty} \frac{2\sigma_m}{\gamma_m^{2} u_j} \exp[-j\gamma_m (u_j - \frac{w_j}{2})] \cos(k_x u_i) \text{sinc} \left( \frac{k_x w_i}{2} \right) \\
+ \sum_{m=0}^{L} 2D_m (-1)^m \cos(k_x u_i) \text{sinc} \left( \frac{k_x w_i}{2} \right) \cos(k_x u_j) \text{sinc} \left( \frac{k_x w_j}{2} \right) \\
+ \sum_{m=L+1}^{\infty} \frac{4j\sigma_m (-1)^m}{\gamma_m} \exp(-j\gamma_m a) \cos(k_x u_i) \text{sinc} \left( \frac{k_x w_i}{2} \right) \cdot \cos(k_x u_j) \text{sinc} \left( \frac{k_x w_j}{2} \right) \}$$

(4.34)

where

$$C = \frac{\mu_0 h}{2}$$

(4.35)

and $w_i$ is the width of the port located along $OA$ while $w_j$ is the width of the port located along $OB$; $u_i$ and $u_j$ are given in Equation (4.25).
4.2.4 Z-matrix when port $i$ and port $j$ are located along $AB$. When both ports are located along $AB$ (see Figure 4.1), the integration in Equation (3.10) can be carried out with respect to the variable $x$ by transforming the ports coordinates to $x$ using the following transformation:

$$y = -x + a$$  \hspace{1cm} (4.36)$$

In this case, the expression for the Green's function given by Equation (4.6) can not be used, since the series in both $G_1$ and $G_2$, which are given by Equations (4.7) to (4.10), blow up individually. Hence, both $G_1$ and $G_2$ have to be combined so that the divergent terms cancel out each other. Thus, from the Appendix A.2, the expression for the single series Green's function can be written as:

$$G(x_i, y_i; x_j, y_j) = -\frac{C}{a} \sum_{m=0}^{\infty} 4D_m \cos(k_x x_i) \cos(k_x x_j) \cdot \cos(\gamma_m(y_\succ - a)) \cos(\gamma_m y_\prec)$$  \hspace{1cm} (4.37)$$

where

$$y_\succ = \max(x_i, x_j)$$  \hspace{1cm} (4.38)$$

$$y_\prec = \min(x_i, x_j)$$  \hspace{1cm} (4.39)$$

and $D_m$ is given by Equation (4.11); $\gamma_m$ is given by Equation (4.20); $k_x$ is given by Equation (4.2); and $C$ is given by Equation (4.35). If the locations of port $i$ and $j$ are not the same (i.e., $i \neq j$), then the expression for the impedance matrix elements, making use of the large argument approximations for the complex trigonometric functions given by Equations (4.26) and (4.27), can be written as:

$$Z_{ij} = -\frac{C}{a} \sum_{m=0}^{L} D_m [\cos((\gamma_m - k_x) y_\succ - \gamma_m a) \text{sinc}((\gamma_m - k_x) \frac{w_\succ}{2 \sqrt{2}})]$$
\[ + \cos((\gamma_m + k_x)y_\to - \gamma_m a)\text{sinc}((\gamma_m + k_x)\frac{w_\to}{2\sqrt{2}}) \cdot \\
(\cos((\gamma_m - k_x)y_\leq)\text{sinc}((\gamma_m - k_x)\frac{w_\leq}{2\sqrt{2}}) + \\
\cos((\gamma_m + k_x)y_\leq)\text{sinc}((\gamma_m + k_x)\frac{w_\leq}{2\sqrt{2}})) \\
+ \sum_{m=L+1}^{\infty} \frac{-j\sigma_m}{\gamma_m} \exp(-j\gamma_m(y_\to - \frac{w_\to}{2\sqrt{2}} - y_\leq - \frac{w_\leq}{2\sqrt{2}})) \cdot \\
\left[ \frac{\exp(jk_x(y_\to - \frac{w_\to}{2\sqrt{2}}))}{w_\to(\gamma_m - k_x)} + \frac{\exp(-jk_x(y_\to - \frac{w_\to}{2\sqrt{2}}))}{w_\to(\gamma_m + k_x)} \right] \cdot \\
\left[ \frac{\exp(jk_x(y_\leq - \frac{w_\leq}{2\sqrt{2}}))}{w_\leq(\gamma_m - k_x)} + \frac{\exp(-jk_x(y_\leq - \frac{w_\leq}{2\sqrt{2}}))}{w_\leq(\gamma_m + k_x)} \right] \right] 
\]

(4.40)

where \( D_m \) is given by Equation (4.11); \( \gamma_m \) is given by Equation (4.20); \( k_x \) is given by Equation (4.2); \( C \) is given by Equation (4.35); and \( y_\to \) and \( y_\leq \) are given by Equations (4.38) and (4.39). The terms \( w_\to \) corresponds to the width of the port \( y_\to \) and \( w_\leq \) corresponds to the width of the port \( y_\leq \).

If the locations of port \( i \) and \( j \) are the same (i.e., \( i = j \)), then the integration in Equation (3.10) has to be carried out using the absolute values for the arguments of the trigonometric functions as shown in the Appendix A.3. Thus, the expression for the impedance matrix elements when \( i = j \), making use of the large argument approximations for the complex trigonometric functions given by Equations (4.26) and (4.27), can be written as:

\[ Z_{ij} = -\frac{C}{a}\{Z_1 + Z_2\} \]

(4.41)

where

\[ Z_1 = \sum_{m=0}^{L} 2D_m \left\{ \frac{1}{4} \text{sinc}^2((\gamma_m - k_x)\frac{w}{2\sqrt{2}}) \cos((\gamma_m - k_x)2x - \gamma_m a) \\
+ \frac{1}{2} \text{sinc}((\gamma_m - k_x)\frac{w}{2\sqrt{2}}) \text{sinc}((\gamma_m + k_x)\frac{w}{2\sqrt{2}}) \cos(\gamma_m(2x - a)) \\
+ \frac{1}{4} \text{sinc}^2((\gamma_m + k_x)\frac{w}{2\sqrt{2}}) \cos((\gamma_m + k_x)2x - \gamma_m a) \right\} \]
\[
+ \frac{\sqrt{2}}{2w(\gamma_m - k_x)} \sin((\gamma_m - k_x)\frac{w}{2\sqrt{2}} - \gamma_m a) \text{sinc}((\gamma_m - k_x)\frac{w}{2\sqrt{2}})
\]
\[
+ \frac{\sqrt{2}}{2w(\gamma_m - k_x)} \sin((\gamma_m - k_x)\frac{w}{2\sqrt{2}} - \gamma_m a) \text{sinc}((\gamma_m + k_x)\frac{w}{2\sqrt{2}}) \cos(2k_z x)
\]
\[
+ \frac{\sqrt{2}}{2w(\gamma_m - k_x)} \sin((\gamma_m + k_x)\frac{w}{2\sqrt{2}} - \gamma_m a) \text{sinc}((\gamma_m - k_x)\frac{w}{2\sqrt{2}}) \cos(2k_z x)
\]
\[
+ \frac{\sqrt{2}}{2w(\gamma_m - k_x)} \sin((\gamma_m + k_x)\frac{w}{2\sqrt{2}} - \gamma_m a) \text{sinc}((\gamma_m + k_x)\frac{w}{2\sqrt{2}})
\]
\[
+ \frac{2\gamma_m}{w(\gamma_m^2 - k_z^2)} \sin(\gamma_m a)[1 + \cos(2k_z x) \text{sinc}(\frac{k_z w}{\sqrt{2}})]
\] (4.42)

and for the $Z_2$ term, if $(2x - a) > 0$ then

\[
Z_2 = \sum_{m=L+1}^{\infty} \frac{-j\sigma_m}{\gamma_m w^2} \exp[-j\gamma_m(2a - 2x - \frac{w}{\sqrt{2}})] \cdot \frac{\exp(-jk_z(2x + \frac{w}{\sqrt{2}}))}{(\gamma_m - k_z)^2} + \frac{\exp(jk_z(2x + \frac{w}{\sqrt{2}}))}{(\gamma_m + k_z)^2}
\]
\[
+ \frac{-2j\sigma_m}{\gamma_m w^2(\gamma_m^2 - k_z^2)} \exp[-j\gamma_m(2a - 2x - \frac{w}{\sqrt{2}})]
\]
\[
+ \frac{2j\sigma_m}{\gamma_m w^2(\gamma_m - k_z)^2} + \frac{1}{(\gamma_m + k_z)^2}
\]
\[
+ \frac{j\sigma_m}{\gamma_m w^2} \left[ \frac{\exp(jk_z(2x + \frac{w}{\sqrt{2}}))}{(\gamma_m^2 - k_z^2)} + \frac{\exp(jk_z(2x - \frac{w}{\sqrt{2}}))}{(\gamma_m^2 + k_z^2)} \right]
\]
\[
+ \frac{2\sqrt{2}\sigma_m}{w(\gamma_m^2 - k_z^2)} [1 + \cos(2k_z x) \text{sinc}(\frac{k_z w}{\sqrt{2}})]
\] (4.43)

whereas, if $(2x - a) < 0$ then

\[
Z_2 = \sum_{m=L+1}^{\infty} \frac{-j\sigma_m}{\gamma_m w^2} \exp[-j\gamma_m(2x - \frac{w}{\sqrt{2}})] \cdot \frac{\exp(jk_z(2x - \frac{w}{\sqrt{2}}))}{(\gamma_m - k_z)^2} + \frac{\exp(-jk_z(2x - \frac{w}{\sqrt{2}}))}{(\gamma_m + k_z)^2}
\]
\[
+ \frac{-2j\sigma_m}{\gamma_m w^2(\gamma_m^2 - k_z^2)} \exp[-j\gamma_m(2x - \frac{w}{\sqrt{2}})]
\]
\[
+ \frac{2j\sigma_m}{\gamma_m w^2(\gamma_m - k_z)^2} + \frac{1}{(\gamma_m + k_z)^2}
\]
\[
+ \frac{j\sigma_m}{\gamma_m w^2} \left[ \frac{\exp(jk_z(2x + \frac{w}{\sqrt{2}}))}{(\gamma_m^2 - k_z^2)} + \frac{\exp(jk_z(2x - \frac{w}{\sqrt{2}}))}{(\gamma_m^2 + k_z^2)} \right]
\]
\[
+ \frac{2\sqrt{2}\sigma_m}{w(\gamma_m^2 - k_z^2)} [1 + \cos(2k_z x) \text{sinc}(\frac{k_z w}{\sqrt{2}})]
\]
\[ + \frac{2\sqrt{2} \sigma_m}{w(\gamma_m^2 - k^2)} \left[ 1 + \cos(2k_x x) \text{sinc} \left( \frac{k_x w}{\sqrt{2}} \right) \right] \]

where \( D_m \) is given by Equation (4.11); \( \gamma_m \) is given by Equation (4.20); \( k_x \) is given by Equation (4.2); \( C \) is given by Equation (4.35); the ports location is \( x = x_i = x_j \); and the ports width is \( w = w_i = w_j \).

**4.2.5 Z-matrix when one port is located along OA and the other along AB.** When one port is located along OA and the other is located along AB (see Figure 4.1), then the integration in Equation (3.10) can be carried out with respect to the variable \( z \) by transforming the coordinates of the ports located along AB to \( x - axis \) using Equation (4.36). In this case, the expression for the Green's function given by Equation (4.6) can not be used, since the series in \( G_2 \), which is given by Equations (4.9) and (4.10), diverges. Hence, both \( G_1 \) and \( G_2 \) have to be combined so that the divergent terms cancel out each other. Thus, from the Appendix A.4, the expression for the single series Green's function can be written as:

\[
G(x_i, y_i | x_j, y_j) = -\frac{C}{a} \sum_{m=0}^{\infty} 4D_m \cos(k_x u_i) \cos(k_x u_j) \cos(\gamma_m u_j) \tag{4.45}
\]

where

\[
(u_i, u_j) = \begin{cases} 
(x_i, x_j) & \text{if } i \text{ along } OA, \text{ j along } AB \\
(x_j, x_i) & \text{if } i \text{ along } AB, \text{ j along } OA 
\end{cases} \tag{4.46}
\]

and \( D_m \) is given by Equation (4.11); \( \gamma_m \) is given by Equation (4.20); \( k_x \) is given by Equation (4.2); and \( C \) is given by Equation (4.35). Thus, the expression for the impedance matrix elements, making use of the large argument approximations for the complex trigonometric functions given by Equations (4.26) and (4.27), can be written as:

\[
Z_{ij} = -\frac{C}{a} \left\{ \sum_{m=0}^{L} 2D_m \cos(k_x u_i) \text{sinc} \left( \frac{k_x w_i}{2} \right) \right\} \cdot
\]
\[ \cos((\gamma_m - k_x)u_j) \text{sinc}((\gamma_m - k_x) \frac{w_j}{2\sqrt{2}}) + \cos((\gamma_m + k_x)u_j) \text{sinc}((\gamma_m + k_x) \frac{w_j}{2\sqrt{2}}) \]

\[ + \sum_{m=L+1}^{\infty} \frac{2\sqrt{2}\sigma_m}{\gamma_m w_j} \exp[-j\gamma_m(a - u_j - \frac{w_j}{2\sqrt{2}})] \cos(k_x u_i) \cdot \text{sinc}(\frac{k_x w_i}{2})[\frac{\exp(jk_x(u_j - \frac{w_j}{2\sqrt{2}}))}{(\gamma_m - k_x)} + \frac{\exp(-jk_x(u_j - \frac{w_j}{2\sqrt{2}}))}{(\gamma_m + k_x)}] \]  (4.47)

where \( D_m \) is given by Equation (4.11); \( \gamma_m \) is given by Equation (4.20); \( k_x \) is given by Equation (4.2); \( C \) is given by Equation (4.35); \( u_i \) and \( u_j \) are given by Equation (4.46); and \( w_i \) is the width of the port located along \( OA \) while \( w_j \) is the width of the port located along \( AB \).

4.2.6 Z-matrix when one port is located along \( OB \) and the other along \( AB \). When one port is located along \( OB \) and the other is located along \( AB \) (see Figure 4.1), then the integration in Equation (3.10) can be carried out with respect to the variable \( x \) by transforming the coordinates of the ports located along \( AB \) to \( x - axi \) using Equation (4.36). In this case, the expression for the Green’s function given by Equation (4.6) can not be used, since the series in \( G_2 \), which is given by Equations (4.9) and (4.10), diverges. Hence, both \( G_1 \) and \( G_2 \) have to be combined together so that the divergent terms cancel out each other. This case is similar to the case discussed in Section 4.2.5. Thus, the expression for the single series Green’s function can be written as:

\[ G(x_i; y_i | x_j; y_j) = -\frac{C}{a_m} \sum_{m=0}^{\infty} 4D_m(-1)^m \cos(k_x u_i) \cos(k_x u_j) \cos[\gamma_m(u_j - a)] \]  (4.48)

where

\[(u_i; u_j) = \begin{cases} (y_i; x_j) & \text{if } i \text{ along } OB, j \text{ along } AB \\ (y_j; x_i) & \text{if } i \text{ along } AB, j \text{ along } OB \end{cases} \]  (4.49)
and $D_m$ is given by Equation (4.11); $\gamma_m$ is given by Equation (4.20); $k_x$ is given by Equation (4.2); and $C$ is given by Equation (4.35). Thus, the expression for the impedance matrix elements, making use of the large argument approximations for the complex trigonometric functions given by Equations (4.26) and (4.27), can be written as:

$$
Z_{ij} = -\frac{C}{a} \left\{ \sum_{m=0}^{L} 2D_m(-1)^m \cos(k_x u_i) \text{sinc}(\frac{k_x w_i}{2}) \cdot \right. \\
\left. \left[ \cos((\gamma_m - k_x)u_j - \gamma_m a)\text{sinc}((\gamma_m - k_x)\frac{w_j}{2\sqrt{2}}) \right.ight.
\left.
{\left. + \cos((\gamma_m + k_x)u_j - \gamma_m a)\text{sinc}((\gamma_m + k_x)\frac{w_j}{2\sqrt{2}}) \right] \\
+ \sum_{m=L+1}^{\infty} \frac{2\sqrt{2}\sigma_m(-1)^m}{\gamma_m w_j} \exp[-\gamma_m(u_j - \frac{w_j}{2\sqrt{2}})] \cos(k_x u_i) \cdot \\
\text{sinc}(\frac{k_x w_i}{2}) \left[ \frac{\exp(jk_x(u_j - \frac{w_j}{2\sqrt{2}}))}{(\gamma_m - k_x)} + \frac{\exp(-jk_x(u_j - \frac{w_j}{2\sqrt{2}}))}{(\gamma_m + k_x)} \right] \right\} (4.50)$$

where $D_m$ is given by Equation (4.11); $\gamma_m$ is given by Equation (4.20); $k_x$ is given by Equation (4.2); $C$ is given by Equation (4.35); $u_i$ and $u_j$ are given by Equation (4.49); and $w_i$ is the width of the port located along $OB$ while $w_j$ is the width of the port located along $AB$.

### 4.3 Comparison Between Single Series and Double Series Formulations

A comparison between the single infinite series and the double infinite series expressions for the impedance matrix elements is carried out by considering a nine-port isosceles triangular segment as shown in Figure 4.2. The parameters used in this comparison is given in Table 4.1. The percentage errors in the return loss at the first port, $|Z_{11}|$, versus the time of computations in $CPU$ seconds is shown in Figure 4.2.
Figure 4.2. Comparison between single series and double series impedance matrix formulations for isosceles triangular segment. The numbers on the curves represent the number of terms used in each summation.
Table 4.1. Parameters for isosceles triangular segment used in the comparison between single series and double series formulations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>substrate height</td>
<td>125 $\mu m$</td>
</tr>
<tr>
<td>strip thickness</td>
<td>5 $\mu m$</td>
</tr>
<tr>
<td>dielectric constant</td>
<td>12.9</td>
</tr>
<tr>
<td>loss tangent delta</td>
<td>0.0009</td>
</tr>
<tr>
<td>conductivity</td>
<td>$5.8 \times 10^7$ mho/meter</td>
</tr>
<tr>
<td>frequency</td>
<td>60 GHz</td>
</tr>
</tbody>
</table>

The figure shows that the single infinite series impedance matrix formulation gives considerable improvements in both the computational time required and the rate of convergence of the series. For instance, to achieve an error in $Z_{11}$ about 0.5%, the single series formulation requires only 5 terms; whereas the double series formulation requires 150 terms. The computation times are 1.0 cpu-seconds and 3303.1 cpu-seconds respectively. This comparison is carried out on the HP-9000 model 319C workstation.

For most practical cases, carrying out the single series formulation up to 50 terms will give good accuracy.
CHAPTER 5

IMPEDEANCE MATRIX EXPRESSIONS FOR CIRCULAR SECTOR AND ANNULAR SECTOR SEGMENTS

In this chapter the expressions for the impedance matrix elements for circular sector and annular sector segments are derived. The derivations of the expressions for the impedance matrix elements for both cases are based on the double infinite series Green’s functions given by [21].

5.1 Impedance Matrix Expressions for Circular Sector Segment

The available Green’s function for circular sector is in double infinite series expression. Thus, the expressions for the impedance matrix elements derived are in double infinite series also.

5.1.1 Green’s function for a circular sector. The geometry of the circular sector segment is specified in Figure 5.1. The double infinite series Green’s function for circular sector segment is given in [21] as:

$$G(\rho_i, \phi_i; \rho_j, \phi_j) = \frac{2lh}{j\omega \pi a^2} + 2j\omega \mu h \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\sigma_n J_{n_l}(k_{mn_l} \rho_i) J_{n_l}(k_{mn_l} \rho_j) \cos(n_i \phi_i) \cos(n_j \phi_j)}{\pi [a^2 - \frac{n_l^2}{k_{mn_l}^2}](k_{mn_l}^2 - k^2) J_{n_l}^2(k_{mn_l} a)}$$ (5.1)

where

$$\sigma_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n \neq 0 \end{cases}$$ (5.2)

$$n_l = nl \text{ where, } l = \frac{\pi}{\alpha}$$ (5.3)

$$k^2 = \omega^2 \mu \epsilon_0 \epsilon_r (1 - j\delta_e)$$ (5.4)
Figure 5.1: The circular sector segment configuration and the port locations.

and \( \delta_e \) is the effective loss tangent given by Equation (2.17). The factors \( J_{n_l}(\bullet) \) represent the Bessel’s function of the first kind of integer order \( n_l \). The term \( k_{mn_l} \) is the solution of:

\[
\frac{\partial}{\partial \rho} [J_{n_l}(k_{mn_l} \rho)] \big|_{\rho=a} = 0
\] (5.5)

The subscript \( m \) in \( k_{mn_l} \) denotes the \( m^{th} \) root of the Equation (5.5). For the case when \( m = 0 \) and \( n_l = 0 \), the value of \( k_{mn_l} \) is zero, which implies that the Bessel’s function is unity throughout the sectoral segment. Thus, for zeroth order Bessel’s function \( (n_l = 0) \), the first root of Equation (5.5) is taken to be the non-zero root.

The expression for the elements of the impedance matrix for various port locations can be derived by using the Green’s function given by Equation (5.1) in the integration given by Equation (3.10). The details of this derivations are given in the Appendix B.1.

5.1.2 Z-matrix when both ports \( i \) and \( j \) are located along the radial edges. There are three possible cases depending on the port locations: both ports are located along \( \phi = 0 \) edge; both ports are located
along $\phi = \alpha$; and one port is located along $\phi = 0$ and the other is along $\phi = \alpha$.

In all of these cases, the integration given by Equation (3.10) is with respect to the variable $\rho$. The expressions for the elements of the impedance matrix for all of these cases can be written as (from Appendix B.1.1):

$$Z_{ij} = \frac{2\gamma \mu h}{\alpha k^2 a^2} + \frac{2\gamma \mu h}{\alpha w_1 w_j} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\sigma_n \cos(n_1 \phi_i) \cos(n_1 \phi_j) I(i)I(j)}{k_{mn}^2 (a^2 - \frac{n_1^2}{k_{mn}^2}) (k_{mn}^2 - k^2) J_{n_1}^2 (k_{mn} a)}$$

(5.6)

where $h$ is the substrate height, $a$ is the radius of the circular sector, $\alpha$ is the circular sector angle, $\sigma_n$ is given by Equation (5.2), $n_1$ is given by Equation (5.3), and $k^2$ is given by Equation (5.4). From Appendix B.1.1, if $n_1$ is even then the terms $I(i)$ and $I(j)$ are defined as:

$$I(\cdot) = \int_{t_1}^{t_2} J_0(t) dt + 2 \sum_{k=0}^{\frac{1}{2}(n_1-2)} \{J_{2k+1}(t_1) - J_{2k+1}(t_2)\}$$

(5.7)

and if $n_1$ is odd then the terms $I(i)$ and $I(j)$ are defined as:

$$I(\cdot) = J_0(t_1) - J_0(t_2) + 2 \sum_{k=1}^{\frac{1}{2}(n_1-1)} \{J_{2k}(t_1) - J_{2k}(t_2)\}$$

(5.8)

where

$$t_1 = k_{mn_1}(\rho - \frac{w}{2})$$

(5.9)

$$t_2 = k_{mn_1}(\rho + \frac{w}{2})$$

(5.10)

where $\rho$ is the radial location of the port; $w$ is the linear width of the port; and $k_{mn_1}$ is the solution of Equation (5.5).

5.1.3 Z-matrix when both ports $i$ and $j$ are located along the curved edge. In this case, the radial locations for both ports are the same (i.e. $\rho_i = \rho_j = a$). The integration given by Equation (3.10) is with respect to the variable $\phi$. Thus, from Appendix B.1.2, the expressions for the
elements of the impedance matrix can be written as:

\[
Z_{ij} = -\frac{2j\omega \mu h}{\alpha k^2 a^2} + \frac{2j\omega \mu h}{\alpha} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\sigma_n \cos(n_l \phi_i) \cos(n_l \phi_j) \sin\left(\frac{n_j w}{2a}\right)}{(a^2 - \frac{n_j^2}{k_{mn_i}^2})(k_{mn_i}^2 - k^2)}
\]

\[(5.11)\]

where \(h\) is the substrate height; \(a\) is the radius of the circular sector; \(\alpha\) is the circular sector angle; \(\sigma_n\) is given by Equation (5.2); \(n_l\) is given by Equation (5.3); \(k^2\) is given by Equation (5.4); \(\phi\) is the angular location of the associated port and \(w\) is the curvilinear width of the associated port; and \(k_{mn_i}\) is the solution of Equation (5.5).

5.1.4 Z-matrix when one port is located along the radial edge and the other along the curved edge. In this case, there are two possible combinations depending on the locations of the ports. One combination is when one port is located along \(\phi = 0\) edge and the other is along the curved edge \((\rho = a)\). The second combination is when one port is located along \(\phi = \alpha\) edge and the other is along the curved edge \((\rho = a)\). In these cases, the integration given by Equation (3.10) is with respect to both the variables \(\rho\) and \(\phi\). Thus, from Appendix B.1.1 and Appendix B.1.2, the expressions for the elements of the impedance matrix for both cases can be written as:

\[
Z_{ij} = -\frac{2j\omega \mu h}{\alpha k^2 a^2} + \frac{2j\omega \mu h}{\alpha \omega r} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\sigma_n \cos(n_l \phi_i) \cos(n_l \phi_j) \sin\left(\frac{n_j w}{2a}\right) I'}{(a^2 - \frac{n_j^2}{k_{mn_i}^2})(k_{mn_i}^2 - k^2) J_n(k_{mn_i} a)}
\]

\[(5.12)\]

where \(h\) is the substrate height; \(a\) is the radius of the circular sector; \(\alpha\) is the circular sector angle; \(\sigma_n\) is given by Equation (5.2); \(n_l\) is given by Equation (5.3); and \(k^2\) is given by Equation (5.4).

In the expression for the elements of the impedance matrix given by Equation (5.12), if port \(i\) is located along the radial edge and port \(j\) is along
the curved edge then \( w'_i = w_i \), \( w'_c = w_j \), and \( I' = I(i) \); where \( I(i) \) is defined by Equations (5.7) and (5.8). On the other hand, if port \( i \) is located along the curved edge and port \( j \) is along the radial edge then \( w'_i = w_j \), \( w'_c = w_i \) and \( I' = I(j) \); where \( I(j) \) is also defined by Equations (5.7) and (5.8). In both cases \( w'_i \) denotes the linear width of the associated port; \( w'_c \) denotes the curvilinear width of the associated port; \( \phi \) denotes the angular location of the associated port; and \( k_{mn} \) is the solutions of Equation (5.5).

### 5.2 Impedance Matrix Expressions for Annular Sector Segment

The available Green's function for an annular sector is a double infinite series expression. Thus, the expressions for the impedance matrix elements derived are in a double infinite series also.

#### 5.2.1 Green's function for an annular sector. The geometry of the annular sector segment is specified in Figure 5.2. The double infinite series Green's function for a circular sector segment is given in [21] as:

\[
G(\rho_i, \phi_i|\rho_j, \phi_j) = \frac{2lh}{j\omega \pi (b^2 - a^2)} + \frac{2j\omega \mu h}{\pi} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\sigma_n}{(k_{mn}^2 - k^2)}
\]
\[
\frac{F_{mn_i}(\rho_i)F_{mn_i}(\rho_j) \cos(n_l\phi_i) \cos(n_l\phi_j)}{[(b^2 - \frac{n_l^2}{k_{mn_i}^2})F_{mn_i}^2(b) - (a^2 - \frac{n_l^2}{k_{mn_i}^2})F_{mn_i}^2(a)]}
\]

(5.13)

where

\[
\sigma_n = \begin{cases} 
1 & \text{for } n = 0 \\
2 & \text{for } n \neq 0 
\end{cases}
\]

(5.14)

\[
n_l = nl \quad \text{where}, \quad l = \frac{\pi}{\alpha}
\]

(5.15)

\[
k^2 = \omega^2 \mu \varepsilon_0 \varepsilon_r (1 - j \delta_e)
\]

(5.16)

here, \(\delta_e\) is the effective loss tangent given by Equation (2.17). The terms \(F_{mn_i}(\cdot)\)
are defined as:

\[
F_{mn_i}(\cdot) = N'_{n_l}(k_{mn_i} a)J_{n_l}(k_{mn_i} \cdot) - J'_{n_l}(k_{mn_i} a)N_{n_l}(k_{mn_i} \cdot)
\]

(5.17)

and the terms \(k_{mn_i}\) are the solutions of:

\[
J'_{n_l}(k_{mn_i} a)N'_{n_l}(k_{mn_i} b) - J'_{n_l}(k_{mn_i} b)N'_{n_l}(k_{mn_i} a) = 0
\]

(5.18)

The terms \(J_{n_l}(\cdot)\) represent the Bessel's function of the first kind of integer order \(n_l\). The terms \(N_{n_l}(\cdot)\) represent the Bessel's function of the second kind (Neumann function) of integer order \(n_l\). The factor \(J'_{n_l}(k_{mn_i} \cdot)\) is the derivative of the Bessel's function with respect to the argument \((\cdot)\). Similarly, the factor \(N'_{n_l}(k_{mn_i} \cdot)\) is the derivative of the Neumann function with respect to the argument \((\cdot)\). The subscript \(m\) in \(k_{mn_i}\) denotes the \(m^{th}\) root of the Equation (5.18).

For the case when \(n_l = 0\) the index \(m\) can be zero which makes the value of \(k_{mn_i}\) equals to zero, this implies that the function \(F_{mn_i}\) is unity throughout the sectoral segment, and hence the first term in Equation (5.13) is obtained.

The expression for the elements of the impedance matrix for various ports locations can be derived by using the Green's function given by Equation (5.13) in the integration given by Equation (3.10). The details of this derivations are given in the Appendix B.2.
5.2.2 Z-matrix when both ports \( i \) and \( j \) are located along the radial edges. There are three possible cases depending on the ports locations: both ports are located along \( \phi = 0 \) edge; both ports are located along \( \phi = \alpha \); and one port is located along \( \phi = 0 \) and the other is along \( \phi = \alpha \). In all of these cases, the integration given by Equation (3.10) is with respect to the variable \( \rho \). The expressions for the elements of the impedance matrix for all of these cases can be written as (from Appendix B.2.1):

\[
Z_{ij} = \frac{2j\omega\mu h}{\alpha k^2 (b^2 - a^2)} + \frac{2j\omega\mu h}{\alpha w_i w_j} \sum_{n=0}^\infty \sum_{m=1}^\infty \frac{\sigma_n \cos(n_i \phi_i) \cos(n_j \phi_j)}{(k^2_{mn} - k^2)k^2_{mn}} \frac{W(i) W(j)}{[(b^2 - \frac{n_i^2}{k^2_{mn}})F^2_{mn}(b) - (a^2 - \frac{n_i^2}{k^2_{mn}})F^2_{mn}(a)]}
\]

where

\[
W(\cdot) = \{N'_{ni}(k_{mn},a)I^J(\cdot) - J'_{ni}(k_{mn},a)I^N(\cdot)\}
\]

and \( h \) is the substrate height; \( a \) is the radius of the annular sector; \( \alpha \) is the annular sector angle; \( \sigma_n \) is given by Equation (5.2); \( n_i \) is given by Equation (5.3); \( k^2 \) is given by Equation (5.4); and \( I^J(\cdot) \) is defined by Equation (5.7) and (5.8).

From Appendix B.2.1, if \( n_i \) is even then the terms \( I^N(i) \) and \( I^N(j) \) are defined as:

\[
I^N(\cdot) = \int_{t_1}^{t_2} N_0(t)dt + 2 \sum_{k=0}^{\frac{1}{2}(n_i-2)} \{N_{2k+1}(t_1) - N_{2k+1}(t_2)\}
\]

and if \( n_i \) is odd then the terms \( I^N(i) \) and \( I^N(j) \) are defined as:

\[
I^N(\cdot) = N_0(t_1) - N_0(t_2) + 2 \sum_{k=1}^{\frac{1}{2}(n_i-1)} \{N_{2k}(t_1) - N_{2k}(t_2)\}
\]

where

\[
t_1 = k_{mn}(\rho - \frac{w}{2})
\]

\[
t_2 = k_{mn}(\rho + \frac{w}{2})
\]
where $\rho$ is the radial location of the port and $w$ is the linear width of the port, and $k_{mn_i}$ is the solutions of Equation (5.5).

### 5.2.3 Z-matrix when both ports $i$ and $j$ are located along the curved edges.

There are three possible cases depending on the locations of the ports: both ports are located along the $\rho = a$ curved edge; both ports are located along the $\rho = b$ curved edge; and one port is located along the $\rho = a$ edge and the other along the $\rho = b$ edge. In all of these cases, the integration given by Equation (3.10) is with respect to the variable $\phi$. From the Appendix B.2.2, the expressions for the elements of the impedance matrix can be written as:

$$Z_{ij} = -\frac{2\omega \mu h}{\alpha k^2 (b^2 - a^2)} + \frac{2\omega \mu h}{\alpha \sum_{n=0}^{\infty} \sum_{m=1}^{\infty}} \frac{\sigma_n F_{mn_i}(\rho_i) F_{mn_i}(\rho_j)}{(k_{mn_i}^2 - k^2)} \times \frac{\cos(n_i \phi_i) \text{sinc}(\frac{n_i w_i}{2a}) \cos(n_j \phi_j) \text{sinc}(\frac{n_j w_j}{2a})}{[(b^2 - \frac{n_i^2}{k_{mn_i}^2}) F_{mn_i}^2(b) - (a^2 - \frac{n_i^2}{k_{mn_i}^2}) F_{mn_i}^2(a)]}$$

(5.25)

where $h$ is the substrate height; $a$ is the radius of the annular sector; $\alpha$ is the annular sector angle; $\sigma_n$ is given by Equation (5.14); $n_i$ is given by Equation (5.15); $k^2$ is given by Equation (5.16); $\phi_i$ or $\phi_j$ is the angular location of the associated port and $w_i$ or $w_j$ is the curvilinear width of the associated port; $F_{mn_i}(\cdot)$ is given by Equation (5.17); and $k_{mn_i}$ is the solutions of Equation (5.18).

### 5.2.4 Z-matrix when one port is located along the radial edge and the other along the curved edge.

In this case, there are four possible combinations depending on the locations of the ports. The first two combinations are when one port is located along the $\phi = 0$ edge and the other is along one of the curved edges ($\rho = a$ or $\rho = b$). The last two combinations are when one port is located along the $\phi = \alpha$ edge and the other is along one of the curved edges ($\rho = a$ or $\rho = b$). In these cases, the integration given
by Equation (3.10) is with respect to both the variables \( \rho \) and \( \phi \). Thus, from Appendix B.2.1 and Appendix B.2.2, the expressions for the elements of the impedance matrix can be written as:

\[
Z_{ij} = -\frac{2\omega \mu h}{\alpha k^2(b^2 - a^2)} + \frac{2\omega \mu h}{\alpha w_r'} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\sigma_n W'F'}{(k_{mn}^2 - k^2)k_{mn}} \cdot \frac{\cos(n_1\phi_i)\cos(n_1\phi_j)\text{sinc}(\frac{nw_r'}{2a})}{[(b^2 - \frac{n_1^2}{k_{mn}^2})F_{mn}^2(b) - (a^2 - \frac{n_1^2}{k_{mn}^2})F_{mn}^2(a)]}
\]

(5.26)

where \( h \) is the substrate height; \( a \) is the radius of the annular sector; \( \alpha \) is the annular sector angle; \( \sigma_n \) is given by Equation (5.14); \( n_1 \) is given by Equation (5.15); and \( k^2 \) is given by Equation (5.16).

In the expression for the elements of the impedance matrix given by Equation (5.26), if port \( i \) is located along the radial edge (along \( \phi = 0 \) or \( \phi = \alpha \)) and port \( j \) is along the curved edge (\( \rho = a \) or \( \rho = b \)) then \( w_r' = w_i \); \( w_c' = w_j \); \( W' = W(i) \), where \( W(i) \) is defined by Equation (5.20); and \( F' = F_{mn}(j) \), where \( F_{mn}(j) \) is defined by Equation (5.17). On the other hand, if port \( i \) is located along the curved edge and port \( j \) is along the radial edge then \( w_r' = w_j \); \( w_c' = w_i \); \( W' = W(j) \), where \( W(j) \) is also defined by Equation (5.20); and \( F' = F_{mn}(i) \), where \( F_{mn}(i) \) is defined by Equations (5.17). In both cases \( w_r' \) denotes the linear width of the associated port; \( w_c' \) denotes the curvilinear width of the associated port; \( \phi \) denotes the angular location of the associated port; and \( k_{mn} \) is the solution of Equation (5.18).
CHAPTER 6

NUMERICAL IMPLEMENTATION

The first section of this chapter describes the planar analysis procedure for discontinuity characterization. A computer program for implementation of planar analysis is discussed in the second section of this chapter. The three parameters that influence the convergence of the results are discussed next. The third section discusses the computations of the real zeros of Bessel's functions; this computation is required when analyzing circular sector and annular sector segments. In the last section, a procedure for shifting the reference planes is described.

6.1 Analysis Procedure for Discontinuity Characterization

The planar analysis discussed in Chapter 3 can be used for discontinuity characterization. The summary of the analysis procedure is shown in Figure 6.1:

- The first step is to model the discontinuity configuration using the planar waveguide model discussed in Chapter 2. The line width is replaced by its effective width to account for the fringing field at the edges. The dielectric constant is replaced by the effective dielectric constant to account for the fields being partially in the substrate and partially in air. The fringing fields in the discontinuity region is accounted for by extending the corner section, similar to the extension for the line. The effective dielectric constant for the discontinuity region is taken to be
Figure 6.1: Analysis procedure for discontinuity characterization.
the same as for the line.

- The next step is to break the discontinuity configuration into multi-port regular segments. Each regular segment has external ports and interconnected ports along its periphery. A shorter computational time is achieved by minimizing the number of regular segments.

- Each of the regular segment is characterized by its impedance matrix. The expressions used in computing the impedance matrix of each segment depend on the shape of the segment. The expression for various shape of segments are given in Chapter 4, Chapter 5, and Appendix C. The dimension of the impedance matrix of each segment shown in Figure 6.1 are: \((1 + n_1 + n_2) \times (1 + n_1 + n_2)\) for the first segment, \((n_2 + n_3 + n_4) \times (n_2 + n_3 + n_4)\) for segment two, \((n_4 + n_5) \times (n_4 + n_5)\) for the third segment, and \((1 + n_1 + n_3 + n_5) \times (1 + n_1 + n_3 + n_5)\) for the fourth segment.

- The impedance matrix and the scattering matrix of the overall discontinuity configuration is computed using the formulas given in Section 3.2. The process of combining the regular segments are carried out by combining two segments at a time. The combination of the first two segments is combined with the third segment. The combination of these three segments is combined with the fourth, etc. This process is carried out for \((\text{number of segments} - 1)\) times. The dimensions of the overall impedance matrix and scattering matrix are \((2 \times 2)\).

- The last step in the analysis is to shift the reference plane back to the physical discontinuity plane. The procedure for shifting the reference plane is described in Section 6.5.
6.2 Planar Circuit Analysis Program

A computer program for analyzing microstrip circuits/discontinuities has been developed. The formulas implemented in the computer program are those given in Chapter 2, Chapter 3, Chapter 4, Chapter 5, and in the Appendix C. A flowchart of the code implemented is provided in Figure 6.2. The input to the program can either be physical dimensions or line impedances. In both cases the effective parameters are computed from the formulas given in Chapter 2. Once the effective dimensions are obtained, then the configuration is broken up into multiport segments and the port coordinates are computed. The impedance matrix of each segment is found by using the appropriate expression for a particular shape of the segment given in Chapter 4, Chapter 5, and Appendix C. The impedance matrix and the scattering matrix of the overall discontinuity configuration are determined from the formulas given in Section 3.2. Finally, the reference planes are shifted using the procedures outlined in Section 6.5.

6.3 Convergence of Results

There are three parameters that influence the convergence, hence the accuracy, of the results:

- **The number of terms used in computing the impedance matrix elements.** The number of terms needed in the impedance matrix computation depends on the location and orientation of the ports involved in the computation. For the step-in-width discontinuity where all the ports are oriented in the same direction the number of terms needed is about ten as seen from Figure 6.3. It may be noted that the Z-matrices in this case are computed using the single summation
Figure 6.2: Flowchart of the Planar Circuit Analysis Program.
Figure 6.3. Variation of the number of terms (\textit{NTERMS}) in step-in-width discontinuity analysis.
Figure 6.4. Variation of the number of interconnected ports (NC) in step-in-width discontinuity analysis.
Figure 6.5. Variation of the length of the wider segment normalized by its width ($L2/W2$) in step-in-width discontinuity analysis.
expressions given in the Appendix C.1. However, larger number of terms are needed (about thirty) if the discontinuity analyzed is either right-angle bend or tee-junction. The number of terms needed is found by iterative computations. In the code developed 100 terms are used.

- **The number of interconnected ports used in connecting two or more multi-port segments.** The number of interconnected ports needed depends on the size of the segments involved and on the variation of fields along the interface. The values of $|S_{11}|$ obtained as a function of the number of interconnected ports in a symmetric step discontinuity is shown in Figure 6.4. From these results one can conclude that at least three interconnected ports are required. It might be noted that this result is valid only for this particular case. Again an iterative process is needed to find the optimum number of interconnected ports. In the code developed the number of interconnected ports is taken to be eight over the effective width of the line.

- **The choice of reference plane for evaluating the impedance matrices.** The choice of the reference plane means choosing the length of the segment that contains the external ports. There is a minimum length required for a segment in order to make sure that the higher order mode excited by the discontinuity decays to negligible value at the location of the external ports. In Figure 6.5, the value of $|S_{11}|$ for a symmetric step discontinuity is plotted as a function of the length of the wider segment normalized by its width. From the plot one can conclude that the length of the wider segment must be at least about 0.15 times its width. However, segment lengths of $\frac{A}{4}$ and $\frac{A}{2}$ are to be
avoided in order to avoid any numerical error since $Z_{11} = Z_{22} \to 0$ for $\frac{3}{4}$ segments and $Z_{11} = Z_{22} \to \infty$ for $\frac{3}{2}$ segments.

6.4 Computing Real Zeros of Bessel's Functions

One of the many computations required in analyzing a circular sector and annular sector segments is finding the real zeros of Bessel's functions. In this Section the procedures and expressions for finding these zeros are presented.

6.4.1 Computing real zeros of the derivative of the Bessel's function. In computing the impedance matrix elements of a circular sector segment the values of $k_{mn}$ is determined by computing the zeros of Equation (5.5). The zeros of Equation (5.5) are independent of the physical dimensions of the circular sector being analyzed; thus, the computation of these zeros is very straightforward. To compute these zeros, the expressions given in [22] and [23] can be used. For $z \geq 3$, the $z^{th}$ zero of order $n$ can be computed using the expression given below [22]:

$$j''_{n,z} \sim \beta - \frac{\mu + 3}{8\beta} - \frac{4(7\mu^2 + 82\mu - 9)}{3(8\beta)^3} - \frac{32(83\mu^3 + 2075\mu^2 - 3039\mu + 3537)}{15(8\beta)^5} - \frac{64(6949\mu^4 + 296492\mu^3 - 1248002\mu^2 + 7414380\mu - 5853627)}{105(8\beta)^7} - \ldots$$

(6.1)

where

$$\mu = 4n^2$$

(6.2)

$$\beta = z + \frac{1}{2}n - \frac{3}{4}$$

(6.3)

For $n \neq 0$ and $z = 1$ (i.e. the first zero), the expression below can be used [22]:

$$j''_{n,1} \sim n + 0.8086165n^{\frac{1}{3}} + 0.072490n^{-\frac{1}{3}} - 0.05808n^{-1} + 0.0094n^{-\frac{3}{2}} + \ldots$$

(6.4)
For $n > 2$ and $z = 2$, then the $z^{th}$ zero can be computed using the expression given below [23]:

$$j_{n,2}^* \sim n + 2.5780961n^{\frac{1}{3}} + 1.955186n^{-\frac{1}{3}} - 0.08925n^{-1} - 0.2941n^{-\frac{5}{3}} + \ldots \quad (6.5)$$

When $n = 0$, the first root is zero.

6.4.2 Computing real zeros of cross products of the Bessel's functions. In computing the impedance matrix elements of an annular sector segment, the values of $k_{mn}$ are determined by finding the zeros of the cross products of the Bessel's functions given in Equation (5.18). In this case, the location of the zeros depend upon the parameter $\lambda = \frac{b}{a}$ where $b$ is the outer radius of the annular sector and $a$ is the inner radius of the annular sector (see Figure 5.2). As $\lambda$ gets larger the zeros becomes smaller and the "period" decreases. When $\lambda \to \infty$ then all the zeros of this Bessel's equation becomes zero. On the other hand, as $\lambda$ gets smaller then the zeros becomes larger and the "period" increases. Finally, the zeros blows up when $\lambda = 1$, which means that the outer radius is the same as the inner radius ($a = b$). These zeros can be determined using any commercially available zero finding routines and a good initial guess. The initial guesses for the first zeros ($z = 1$) are given below:

$$
\begin{align*}
\frac{\pi}{\lambda-1} & \quad \text{if } n = 0 \\
\frac{n}{\lambda} & \quad \text{if } 0 < n \leq 20 \\
\frac{n+\pi}{\lambda} & \quad \text{if } n > 20
\end{align*}
$$

(6.6)

For the second zero ($z = 2$), the initial guesses are:

$$
\begin{align*}
\frac{2\pi}{\lambda-1} & \quad \text{if } n = 0 \\
\frac{\pi}{\lambda-1} + \frac{n}{\lambda} & \quad \text{if } n > 0
\end{align*}
$$

(6.7)
When $z$ is large the expression for the asymptotic expansion given in [22] can be used as an initial guess. The large $z$ can be defined as:

$$\begin{align*}
\frac{n}{\lambda - 1} & \quad \text{if } (\lambda - 1) < 1 \\
n + \frac{n}{\lambda} & \quad \text{if } (\lambda - 1) \geq 1
\end{align*}$$  \tag{6.8}

and the asymptotic expansion given in [22] is expressed as follow:

$$\beta + \frac{p}{\beta} + \frac{q - p^2}{\beta^3} + \frac{r - 4pq + 2p^3}{\beta^5} + \ldots$$  \tag{6.9}

where

$$\beta = \frac{z\pi}{(\lambda - 1)}$$  \tag{6.10}

$$p = \frac{\mu + 3}{8\lambda}$$  \tag{6.11}

$$q = \frac{(\mu^2 + 46\mu - 63)(\lambda^3 - 1)}{6(4\lambda)^3(\lambda - 1)}$$  \tag{6.12}

$$r = \frac{(\mu^3 + 185\mu^2 - 2053\mu + 1899)(\lambda^5 - 1)}{5(4\lambda)^5(\lambda - 1)}$$  \tag{6.13}

and $\mu$ is defined by Equation (6.2). The values of the zeros for $3 \leq z \leq \text{large } z$ can be determined using the periodicity of the cross products given in Equation (5.18).

### 6.5 Shift of Reference Planes

In the analysis of microstrip discontinuities, the reference planes for specifying the scattering parameters are normally taken at the physical junction. However, the results obtained from segmentation and desegmentation methods are with respect to the reference planes at the edges containing the external ports of the discontinuity configuration. Thus, the reference planes given by these methods need to be shifted to the physical discontinuity planes.
The situation for a 2-ports network is shown in Figure 6.6. For an \(N\)-ports network, the shift of the reference planes can be carried out as follows:

\[
S'_{ii} = S_{ii} \exp(2\gamma_i L_i) \quad \text{for } i = 1, 2, \cdots, N
\]  

(6.14)

for \(i = j\); and for \(i \neq j\) the following is used:

\[
S'_{ij} = S_{ij} \exp(\gamma_i L_i + \gamma_j L_j) \quad \text{for } \begin{cases} i = 1, 2, \cdots, N \\ j = 1, 2, \cdots, N \end{cases}
\]

(6.15)

where \(S'_{ii}\) and \(S'_{ij}\) are the shifted \(|S|\) matrix; \(S_{ii}\) and \(S_{ij}\) are the unshifted \(|S|\) matrix; \(L_i\) and \(L_j\) are the lengths between the old and the new reference planes at external ports \(i\) and \(j\) respectively; and \(\gamma_i\) and \(\gamma_j\) are the complex propagation constant of the microstrip lines connected to the external ports \(i\) and \(j\) respectively. The propagation constant \(\gamma_i\) may be written as:

\[
\gamma_i = \alpha_i + j\beta_i = \frac{\pi \tan \delta_e \sqrt{\varepsilon_r(f)}}{\lambda_0} + j \frac{2\pi f \sqrt{\varepsilon_r(f)}}{c}
\]

(6.16)

for \(i = 1, 2, \cdots, N\) and where \(\tan \delta_e\) is the effective loss tangent as given by Equation (2.17); \(\varepsilon_r(f)\) is the effective dielectric constant at port \(i\) given by Equation (2.11); \(\lambda_0\) is the free space wavelength; \(f\) is the operating frequency; and \(c\) is the speed of light (\(= 3 \times 10^8\) meter/second).

Thus, using Equations (6.14) and (6.15) the reference plane of any \(N\)-ports networks/circuits can be located anywhere inside or outside the circuit. If \(L_i\) is positive then the reference plane is shifted inward (from the external port to the circuit inside). On the other hand, if \(L_i\) is negative then the reference plane is shifted outward (external to the circuit).
Figure 6.6: Shifting of reference plane in S-matrix for a 2-ports network.
CHAPTER 7

COMPENSATED MICROSTRIP RIGHT-ANGLE BENDS

In this chapter various ways of compensating microstrip right-angle bends by modifying the discontinuity configuration are presented. All of these ways of modifying the discontinuity configuration, discussed in this chapter, are carried out by removing the excess discontinuity capacitance associated with the unmodified right-angle bend; this procedure results in configurations such as chamfered bend, bend rounded at the outside corner, bend rounded at both the outside and inside corners, and outer recess bend. The last section of this chapter discusses and compares all of these various ways of compensating microstrip bends.

7.1 Chamfered Bends

One of the ways to eliminate or reduce the unwanted fringing capacitance is by chamfering the outside corner of the bend as shown in Figure 2.3(b). Thus, the compensation of right-angle bend discontinuity in this case involves finding the optimum amount (percentage) of chamfering needed to get zero or minimum reflection coefficient. In general, the percent chamfer for the bend shown in Figure 7.1 is defined as:

\[
\% \text{chamfer} = (1 - \frac{b}{w} \cos(\frac{\phi}{2})) \times 100\% \quad (7.1)
\]

For right-angle bend discontinuity, \(\phi = 90^\circ\), the \% chamfer becomes:

\[
\% \text{chamfer} = (1 - \frac{b}{w\sqrt{2}}) \times 100\% \quad (7.2)
\]
Figure 7.1: Definition of chamfering in microstrip bend discontinuity.

In general, there are three criteria for determining an optimum discontinuity configuration:

- 1. minimum reflection (i.e. minimum | S_{11} |)
- 2. minimum variation of the equivalent electrical length with frequency
- 3. minimum radiation loss from the discontinuity

The planar analysis, described in this thesis, allows one to satisfy the first and the second criteria. The equivalent length \( \Delta l \) normalized by the substrate height can be computed as follows:

\[
\frac{\Delta l}{h} = \frac{\phi_{21}}{\beta h}
\]  

(7.3)

where \( h \) is the substrate height; \( \phi_{21} \) is the phase of the transmission coefficient (in radians); and \( \beta = \frac{2\pi f \sqrt{\varepsilon_r(f)}}{c} \) is the propagation or phase constant, with \( f \) is the operating frequency, \( \varepsilon_r(f) \) is the effective dielectric constant given by Equation (2.11).

The chamfered right-angle bend can be modelled in four different
ways, as shown in Figure 7.2. The first model, shown in Figure 7.2(a), is obtained by connecting the points c and c' after replacing the physical width of the lines by its effective width. This gives a smaller extension at the corner compared to the extension of the line (\( \Delta c = \Delta w/\sqrt{2} \)). The second model, shown in Figure 7.2(b), is obtained by keeping the same extension at the corner as the extension of the line, i.e. \( \Delta c = \Delta w \). The third model, shown in Figure 7.2(c), is obtained by keeping the lengths of the chamfered corner the same for both the physical and effective configurations. This model gives a larger corner extension compared to the line extension (\( \Delta c = w\sqrt{2} \)). The last model, shown in Figure 7.2(d), is obtained by treating the corner (triangular section) as a transmission line of average width \( b/2 \). Thus, the extension of the corner section is the extension of this transmission line of width \( b/2 \).

The parameters chosen for these chamfered right-angle bends are those used in the experiments reported in [3] and are shown in Table 7.1. The line impedance is 50 ohms. The optimum chamfer for each one of these four planar models are shown in Figure 7.3. The optimum chamfer for the first model is 56 % and the corresponding VSWR is 1.003. The optimum chamfer for the other three models are 65 %, 77 %, and 59 % respectively; and the corresponding VSWR are 1.003, 1.002, and 1.005 respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substrate height</td>
<td>3.175 millimeter</td>
</tr>
<tr>
<td>Strip thickness</td>
<td>5 μm</td>
</tr>
<tr>
<td>Dielectric constant</td>
<td>2.62</td>
</tr>
<tr>
<td>Loss tangent delta</td>
<td>0.0001</td>
</tr>
<tr>
<td>Conductivity</td>
<td>( 5.8 \times 10^7 ) mho/meter</td>
</tr>
<tr>
<td>Frequency</td>
<td>1.5 GHz</td>
</tr>
</tbody>
</table>
Figure 7.2. Four models for chamfered right-angle bends: a.) model 1, $\Delta c < \Delta w$; b.) model 2, $\Delta c = \Delta w$; c.) model 3, $\Delta c > \Delta w$; d.) model 4, $\Delta c = \text{extension of a line of width } b/2$. 

\[
\Delta w = \frac{w_e - w}{2}
\]

\[
\Delta c = \frac{\Delta w}{\sqrt{2}}
\]

\[
\Delta w = \frac{w_e - w}{2}
\]

\[
\Delta c = \Delta w = \frac{w_e - w}{2}
\]

\[
\Delta w = \frac{w_e - w}{2}
\]

\[
\Delta c = \Delta w \sqrt{2}
\]

\[
\Delta w = \frac{w_e - w}{2}
\]

\[
\Delta c = \Delta w \sqrt{2}
\]
Figure 7.3: Optimum chamfer for four models of chamfered microstrip bends.

Consider a 59% chamfered microstrip right-angle bend discontinuity with the same parameters shown in Table 7.1; the line impedance is 50 ohms. This chamfered bend is simulated at "high" frequency (i.e., close to cut-off frequency of the next higher mode) using the four planar models shown in Figure 7.2. The variation of the reflection coefficients of the four planar models with frequency is shown in Figure 7.4. The high values of the reflection coefficients as the frequency approaches 6 GHz is related to the cut-off of the next higher mode (at 6.3 GHz). The plot of the variation of $|S_{11}|$ with frequency shown in Figure 7.4 suggests that the percent chamfered needed is a function
Figure 7.4. Variation of the reflection coefficients ($|S_{11}|$) of the four planar models with frequency.

of frequency. Consider model 2 of the chamfered bend, this model gives an optimum chamfer of 65% at 1.5 GHz; while a 59% chamfer for this model gives optimum performance at 4.2 GHz. All four models suggest that the amount of chamfering needed at higher frequency is less than the amount of chamfering needed at lower frequency.

The equivalent length ($Δl/h$) for all four models are computed using Equation (7.3). The variation of these equivalent lengths with frequency is shown in Figure 7.5. A figure of merit, $FM$ of these chamfered bends can
Figure 7.5. Variation of the equivalent lengths ($\Delta l/h$) of the four planar models with frequency.

be computed as follows:

$$FM = \frac{\Delta l/h \bigg|_{f_1} - \Delta l/h \bigg|_{f_2}}{f_1 - f_2}$$

(7.4)

where, $f_1$ and $f_2$ are the upper and lower bound of the frequency range being considered. This figure of merit describes the dispersion of the associated discontinuity configuration; the ideal value is $FM = 0$. Thus, from Figure 7.5, the $FM$ of the unchamfered bend is 0.5657 $(GHz)^{-1}$. The $FM$ for the four models of the chamfered bends (model 1 to model 4) are 0.1814 $(GHz)^{-1}$, 0.2158 $(GHz)^{-1}$, 0.2378 $(GHz)^{-1}$, 0.4582 $(GHz)^{-1}$ respectively. These results show that model 1, model 2, and model 3 have relatively the same dispersion
which is much lower than the dispersion of the unchamfered bend. However, model 4 shows relatively the same dispersion behavior as the unchamfered bend. Comments on the selection of the appropriate model are included in Section 7.5.

7.2 Bends Rounded at the Outside Corner

The fringing capacitance at the corner of the right-angle bend microstrip discontinuity can also be eliminated or reduced by rounding off the outside corner. There are three ways to model this rounded bend as shown in Figure 7.6. The first model, shown in Figure 7.6(a), is obtained by using the same extension for the rounded corner as the extension of the line. The second model, shown in Figure 7.6(b), is obtained by using a variable extension. The extension is shortest at the corner, and becomes equal to the extension of the line at the planes $T\cdot T$. The third model, shown in Figure 7.6(c), is obtained by taking the corner extension to be the same as the extension of a circular disc.

The extension of a circular disc can be determined using the static fringing capacitance expression for a circular disc given in [24]. The static capacitance for microstrip patch of arbitrary shape given in [24] can be written as follows:

$$C \simeq \frac{\varepsilon_0 \varepsilon_r S}{h} + \frac{\varepsilon_0 P}{\pi} \left\{ \ln \left( \frac{2P}{\pi h} \right) + 1 + \frac{J_P}{2} + \varepsilon_r \left[ \ln(2\pi) - 2Q_0(\delta_c) \right] \right\} \quad (7.5)$$

where

$$\delta_c = \frac{1 - \varepsilon_r}{1 + \varepsilon_r} \quad (7.6)$$

$$Q_0(\delta_c) = \sum_{m=1}^{\infty} (\delta_c)^m \ln(m) \quad (7.7)$$
Figure 7.6. Three models for bends rounded at the outside corner: a.) model 1, $\Delta c = \Delta w$; b.) model 2, variable $\Delta c$; c.) model 3, $\Delta c =$ the extension of a circular disc.
For circular disc of radius \( r \): \( P = 2\pi r \), \( S = \pi r^2 \). and \( J_p = -4 \). Hence, from Equation (7.5) the fringing capacitance becomes:

\[
C_f \simeq 2\pi \epsilon_0 \{ \ln \left( \frac{4\pi}{h} \right) - 1 + \epsilon_r [\ln(2\pi) - 2Q_0(\epsilon_r)] \}
\]  

(7.8)

Thus, the physical radius of the circular disc can be replaced by its effective radius such that the fringing capacitance is accounted for. The expression for this effective radius can be written as follow:

\[
r_e = r \left( 1 + \frac{hC_f}{\pi \epsilon_0 \epsilon_r r^2} \right)^{\frac{1}{2}}
\]  

(7.9)

where \( h \) is the substrate height; \( r \) is the physical radius; \( \epsilon_r \) is the dielectric constant; \( \epsilon_0 \) is the free space permittivity (\( = [1/(36\pi)] \times 10^{-9} \text{ F/m} \)); and \( C_f \) is given by Equation (7.8).

The parameters used are the same as those given in Table 7.1; the line impedance is 50 \text{ ohms}. The results for all three models are given in Table 7.2. The variation of the reflection coefficients with frequency and the variation of the equivalent lengths with frequency for model 2 are shown in Table 7.3. The \textit{figure of merit} of this outside rounded bend is \( FM = 0.2531 \text{ (GHz)}^{-1} \) compared to the unchamfered bend which is \( FM = 0.5657 \text{ (GHz)}^{-1} \). Thus, model 2 of the outside rounded bend shows relatively similar dispersion behavior as the first three models of the chamfered bend discussed in Section 7.1.

Selection of the appropriate model can be based on evaluation of total capacitance for the rounded bend by using Equation (7.5) and comparison of
Table 7.3: Variations of $|S_{11}|$ and $\Delta l/h$ with frequency

| Type            | $\Delta l/h_{15GHz}$ | $\Delta l/h_{6GHz}$ | $|S_{11}|_{15GHz}$ | $|S_{11}|_{6GHz}$ |
|-----------------|-----------------------|----------------------|-------------------|-------------------|
| rounded bend    | 1.8136                | 2.9527               | 0.00073           | 0.37168           |
| uncompensated   | 1.8029                | 4.3485               | 0.17501           | 0.99311           |

this value with the parallel plate capacitances of the three models.

7.3 Bends Rounded at Both the Outside and Inside Corners

An extension of compensation by using rounded bend, presented in the previous Section, is by rounding off both the outside and the inside corners. This configuration is shown in Figure 7.7. The physical geometry shown in Figure 7.7 is a function of the inner radius of the rounded corner. The larger the inner radius the larger the space requirement. This larger space requirement is not suitable for microwave integrated circuit purposes. For the bend rounded at both the outside and the inside corners discussed in this Section, the inner radius is taken to be 3.67 $mm$, which turns out to be the extension of the microstrip line connected to this rounded corner. The planar model shown in Figure 7.7 shows that the outside corner is modelled similar to model 2 (Figure 7.6(b)) in the case where only the outside corner of the bend is rounded. Since the center of both the outside and the inside corner have to be the same, the modeling of the inside corner is dictated by the modeling of the outside corner. The parameters used for this case are given in Table 7.1 and the line impedance is 50 $ohms$. The results are given in Table 7.4.

Table 7.4: Results for bend rounded at both the outside and inside corners

| $S_{11}$ | 0.0178 |
| $VSWR$   | 1.0362 |
Figure 7.7: Bend rounded at both the outside and inside corners.

7.4 Outer Recess Bends

Another way to remove the excess discontinuity capacitance associated with the unmodified right-angle bend is by taking a square cutout at the outside corner [5]. This configuration generates some interest due to the simpler process of artwork generation compared to the artwork generations for the other three configurations discussed in the previous three Sections. The planar model of this outer recess bend is shown in Figure 7.8. The percent recess for the bend shown in Figure 7.8 is defined as:

\[
\% \text{recess} = \frac{s}{w} \times 100 \%
\]  

(7.10)

where \(w\) is the physical width of the bend and \(s\) is the length of the square cutout. The parameters used for this case are given in Table 7.1 and the line impedance is 50 ohms. The results are given in Table 7.5.
Table 7.5: Results for outer recess bend

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>optimum recess</td>
<td>91 %</td>
</tr>
<tr>
<td>$S_{11}$</td>
<td>0.0376</td>
</tr>
<tr>
<td>VSWR</td>
<td>1.0782</td>
</tr>
</tbody>
</table>

\[ \Delta w = \frac{w_g - w}{2} \]

Figure 7.8: Outer recess bend.

7.5 Discussions and Comparisons

The planar model for the unmodified microstrip right-angle bend is shown in Figure 2.3(a). Using the same parameters as given in Table 7.1 and the same 50 ohms line impedance, this unmodified right-angle bend gives $|S_{11}| = 0.175$ and $VSWR = 1.4243$. Comparing all the modified discontinuity configurations and the unmodified one, the following observations can be made:

- As expected, all of the compensated bends give considerable improvements in performance compared to the unmodified bend.
- For the chamfered bends, model 3 gives the best result. The results given by model 3 agree with the experimental results given by [3]. An
empirical expression for optimum chamfer is given by [4] as:

\[
\% \text{chamfer} = 52 + 65 \exp(-1.35 \frac{w}{h})
\]  

(7.11)

where \(w\) is the physical width of the line and \(h\) is the substrate height. This expression valid for \(w/h \geq 0.25\) and \(\epsilon_r \leq 25\); the accuracy is better than 4\%. In this case, for \(w/h = 2.8\), the optimum chamfer predicted by this empirical formula is \(\sim 53\%\). Thus, based on this empirical formula, model 1 agrees with the results given by [4]. It may be noted that the experimental results obtained by [3] do not agree well with the experimental results given by [4].

- Based on the second criteria discussed in Section 7.1 (i.e. minimum variation of the equivalent length with frequency), one can conclude from Figure 7.5 that the first model yields the best results for the chamfered bend.

- The variation of the reflection coefficient with frequency shown in Figure 7.4 shows that all four models give considerable improvement in performance up to the cut-off frequency of the next higher mode. However, no decision can be made on which one of the four models is the correct model. Thus, it is difficult to decide which one of these four models is a (more nearly) correct model. The key to an improved modeling of the chamfered bend configurations is by determining the correct extension at the corner. This can be accomplished by determining the static capacitance of the chamfered bend configurations by using the approach contained in Equation (7.5). Also, the calculation of the quasi-static inductance will help in improving the planar model for the discontinuity region, as this will give an approximation to the
value of the effective dielectric constant to be used for this region. Another way to verify these four models is by comparing the results of these chamfered bend configurations with the results obtained from full-wave analysis (which are not currently available).

- For the bends rounded at the outside corner, model 2 gives the best results. This model is believed to be the correct model for bend rounded at the outside corner. The extension at the corner is very close to the extension of a circular disc (2.149 mm as supposed to 2.143 mm for the circular disc); and the extension gradually approaching the extension of the line as it gets closer to the transmission line section. Model 3 is not a good model, even though the extension is taken to be the same as the extension of a circular disc, this is because of the abrupt change at the intersection between the transmission line and the discontinuity regions.

- For the bend rounded at both the outside and inside corners, the results are not as good as the results given by model 2 of the bend rounded at the outside corner only. The modeling of the outside corner in this case is the same as for the second model (see Figure 7.6(b)) of a bend rounded at the outside corner only. However, in this configuration the inside corner is rounded too, this introduces an additional capacitance in the discontinuity region. Better results are expected when the inner radius is increased. Of course, this increases the physical space required.

- Taking a square cutout at the outside corner does give a better result compared to the unmodified bend. However, the improvement over
the unmodified bend is not as good as the other three modified bend configurations discussed in Sections 7.1, 7.2, and 7.3. In addition, the high percentage of recess required might lead to fabrication problems.
CHAPTER 8

CONCLUSIONS

The planar waveguide model for microstrip lines and the segmentation/desegmentation methods for planar circuit analysis have been used to develop a code for analysis of microstrip discontinuities. There are three parameters that influence the convergence of the results. They are: the number of terms used in computing the impedance matrix elements; the number of interconnected ports used in connecting two multiport segments; and the length of the segment (in case of a rectangular segment). The optimum values for all of these parameters are determined through iterative processes.

A faster algorithm for computing the elements of the impedance matrix for isosceles triangular segment has been achieved by converting the double infinite series formulation to a single infinite series formulation. This conversion is carried out by summing up one of the series analytically using the trigonometric Fourier series. This single series formulation is much faster computationally therefore, it is more suitable for Computer Aided Design purposes.

The derivations and implementations of the expressions for the elements of the impedance matrix for circular sector and annular sector segments allows one to analyze more variety of discontinuity configurations. One of the essential steps in the analysis of circular sector and annular sector is the determination of the real zeros of Bessel’s functions. These zeros can be determined using any commercially available zero finding routines and a good initial guess.
The initial guesses for these purposes are presented in the Chapter on numerical implementation Section 6.4. It may be noted that the analysis of these two sectoral segments as presented in Chapter 5 is computationally very expensive. Therefore, it is necessary to convert these formulations into single series formulations.

Various ways of compensating microstrip right-angle bends are presented. All of the compensations performed are carried out by removing the excess discontinuity capacitance associated with the unmodified bend. The results show that the compensated bends give considerable improvements in performance compared to the unmodified bend. There are four planar models that can be derived for a chamfered bend. Upon comparing the results of these four models with the experimental results given by [3] and [4], no decision can be made on which one of the four models is the correct model. This is because the results given in [4] do not agree with the results given by [3]. The small variations in \(|S_{11}|\) around the optimum chamfer region, as shown by the results presented in Chapter 7, is believed to be the reason for the discrepancies between the experimental results given in [3] and [4]. One way to decide which one of the four models is more correct is by determining the extension at the corner which can be accomplished by determining the static capacitance of the chamfered bend configuration. For this purposes, one might be able to use the expression for the static capacitance for microstrip patch of arbitrary shape [24] given by Equation (7.5).

For the case considered, compensation by rounding off the outside corner only is found to perform better than compensation by rounding off both
the outside and inside corners. This is due to the additional capacitance introduced by the rounded inside corner. Another way of removing the discontinuity capacitance associated with the unmodified bend is by taking a square cutout at the outside corner of the bend. This procedure is found to be unattractive due to the high percentage of recess needed. This high percentage of recess might lead to fabrication problems.

Many planar models of compensated right-angle bends can be derived because there is no expression available for determining the extension at the discontinuity region (the corner of the bend). The results of various models of various compensation techniques presented in this report can be verified by experiment. By carrying out experiments one might be able to choose which one of the many models is correct. The experiment involves the fabrication of a square microstrip resonator as shown in Figure 8.1. The four corners are made up of the bend configurations to be analyzed for instance, chamfered corner, rounded corner, and outside and inside rounded corner. The input and output lines are very lightly coupled such that they do not influence (or cause minimum influence to) the resonant frequency of the square resonator. If necessary, a circular reference loop of the same circumference can be fabricated in addition to the square resonator. The theoretical shift in the resonant frequency can be computed knowing the equivalent length $\Delta l/h$. If the experiment were to be carried out at 6 $GHz$ (for the parameters specified in Table 7.1 and for a 59 % chamfer), the resonant frequencies based on different models would differ by approximately 300 $MHz$. By comparing the measured and theoretical resonant frequencies one can decide which of the models is the correct model.

As discussed in Section 7.5, the key to an improved modeling of the
Figure 8.1: Square microstrip resonator for experimental verification.
compensated bend configurations is by determining the correct extension at the discontinuity region. This can be accomplished by determining the static capacitance of the compensated bend configurations by using the approach contained in Equation (7.5). Also, the calculation of the quasi-static inductance will help in improving the planar model for the discontinuity region, as this will give an approximation to the value of the effective dielectric constant to be used for this region.

Finally, another way of compensating microstrip right-angle bends is by introducing an inductance at the discontinuity region such that the inductance compensates the discontinuity capacitance. This approach results in configurations such as inner recess bends [5]. The configuration of the inner recess bend is shown in Figure 8.2. The % recess for this configuration is defined by Equation (7.10). The planar analysis method described in this report can also be used to analyze this type of compensation.
BIBLIOGRAPHY


APPENDIX A

DERIVATION OF SINGLE SERIES EXPRESSIONS FOR ISOSCELES TRIANGULAR SEGMENT

This Appendix consists of four sections. The first Section contains the detailed derivation of a single series Green's function for the case when the ports are located along OA or OB. The second and the third Sections describe similar procedures for the case when the ports are located along AB. The last section is for the case when the ports are along OA and AB. The geometry of the right-angle isosceles triangular segment is shown in Figure 4.1 in Chapter 4.2.

A.1 Single Series Green's Function When Both Ports i and j are Along OA or Along OB

The double series Green's function given by Equation (4.1) can be rewritten as:

\[
G(x_i, y_i | x_j, y_j) = C \left\{ \sum_{m=0}^{\infty} \sigma_m \cos(k_x x_i) \cos(k_x x_j) \cdot \right. \\
\left. \sum_{n=0}^{\infty} \frac{\sigma_n \cos(k_y y_i) \cos(k_y y_j)}{(m^2 + n^2) \pi^2 - a^2 k^2} \right. \\
+ \sum_{m=0}^{\infty} \left[ \sigma_m (-1)^n \cos(k_x x_i) \cos(k_x y_j) \cdot \right. \\
\left. \sum_{n=0}^{\infty} \frac{\sigma_n (-1)^n \cos(k_y y_i) \cos(k_y x_j)}{(m^2 + n^2) \pi^2 - a^2 k^2} \right] \\
+ \sum_{m=0}^{\infty} \left[ \sigma_m (-1)^m \cos(k_x y_i) \cos(k_x x_j) \cdot \right. \\
\left. \sum_{n=0}^{\infty} \frac{\sigma_n (-1)^m \cos(k_y x_i) \cos(k_y y_j)}{(m^2 + n^2) \pi^2 - a^2 k^2} \right] \\
\left. \sum_{n=0}^{\infty} \frac{\sigma_n (-1)^n \cos(k_y x_i) \cos(k_y y_j)}{(m^2 + n^2) \pi^2 - a^2 k^2} \right] \\
\]
\[ G(x_i, y_i | x_j, y_j) = 2C \{ G_1 + G_2 \} \]  

(A.2)

where if \((x_i + x_j) \geq (y_i + y_j)\) then

\[ G_1 = \sum_{m=0}^{\infty} [\sigma_m \cos(k_z x_i) \cos(k_z x_j) \sum_{n=0}^{\infty} \frac{\sigma_n \cos(k'_y y_i) \cos(k'_y y_j)}{(m^2 + n^2) \pi^2 - a^2 k'^2}] \]  

(A.3)

whereas if \((y_i + y_j) > (x_i + x_j)\) then

\[ G_1 = \sum_{m=0}^{\infty} [\sigma_m \cos(k_z x_i) \cos(k_z x_j) \sum_{n=0}^{\infty} \frac{\sigma_n \cos(k'_y x_i) \cos(k'_y x_j)}{(m^2 + n^2) \pi^2 - a^2 k'^2}] \]  

(A.4)

and for the \(G_2\) term, if \((x_i + y_j) \geq (y_i + x_j)\) then

\[ G_2 = \sum_{m=0}^{\infty} [\sigma_m(-1)^m \cos(k_z x_i) \cos(k_z x_j) \sum_{n=0}^{\infty} \frac{\sigma_n(-1)^n \cos(k'_y y_i) \cos(k'_y x_j)}{(m^2 + n^2) \pi^2 - a^2 k'^2}] \]  

(A.5)

whereas if \((y_i + x_j) > (x_i + y_j)\) then

\[ G_2 = \sum_{m=0}^{\infty} [\sigma_m(-1)^m \cos(k_z y_i) \cos(k_z x_j) \sum_{n=0}^{\infty} \frac{\sigma_n(-1)^n \cos(k'_y x_i) \cos(k'_y y_j)}{(m^2 + n^2) \pi^2 - a^2 k'^2}] \]  

(A.6)

Consider \(G_1\) as given by Equation (A.3), separating the \(n = 0\) terms gives:

\[ G_1 = \sum_{m=0}^{\infty} \frac{\sigma_m \cos(k_z x_i) \cos(k_z x_j)}{m^2 \pi^2 - a^2 k'^2} + \sum_{m=0}^{\infty} \sigma_m \cos(k_z x_i) \cos(k_z x_j) \cdot S_1(m) \]  

(A.7)

where

\[ S_1(m) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{\cos(k'_y y_i) \cos(k'_y y_j)}{k_z^2 + k'_y^2 - k^2} \]  

(A.8)
The $S_1(m)$ summation given by Equation (A.8) can be carried out analytically using trigonometric Fourier Series [20]. Thus, $S_1(m)$ becomes:

$$S_1(m) = \frac{-1}{\pi^2 \alpha_m^2} + \frac{1}{2\pi \alpha_m} \cosh(\alpha_m(\pi - x_1)) + \cosh(\alpha_m(\pi - x_2)) \sinh(\alpha_m \pi)$$  \hspace{1cm} (A.9)

where

$$\alpha_m = \pm \frac{a}{\pi \sqrt{k^2 - k_1^2}}$$  \hspace{1cm} (A.10)

$$x_1 = \frac{\pi (y_+ + y_-)}{a}$$  \hspace{1cm} (A.11)

$$x_2 = \frac{\pi (y_+ - y_-)}{a}$$  \hspace{1cm} (A.12)

$$y_+ = \max(y_i, y_j)$$  \hspace{1cm} (A.13)

$$y_- = \min(y_i, y_j)$$  \hspace{1cm} (A.14)

Substituting $S_1(m)$ into Equation (A.3) and redefining $\alpha_m$ as:

$$\alpha_m = \pm \frac{a}{\pi \gamma_m}$$  \hspace{1cm} (A.15)

gives the expression for the single series $G_1$ term as shown below:

$$G_1 = \sum_{m=0}^{\infty} \frac{-\sigma_m}{a \gamma_m \sin(\gamma_m a)} \cos(k_x x_i) \cos(k_x x_j) \cos(\gamma_m(y_+ - a)) \cos(\gamma_m y_-)$$  \hspace{1cm} (A.16)

Similarly, following the same procedures for Equations (A.4), (A.5), and (A.6) gives the expressions for the single series Green’s function below:

$$G(x_i, y_j | x_j, y_j) = -\frac{C}{a} \{G_1 + G_2\}$$  \hspace{1cm} (A.17)

where for the $G_1$ term, if $(x_i + x_j) \geq (y_i + y_j)$, then

$$G_1 = \sum_{m=0}^{\infty} 2D_m \cos(k_x x_i) \cos(k_x x_j) \cos(\gamma_m(y_1 - a)) \cos(\gamma_m y_1 \gamma)$$  \hspace{1cm} (A.18)

whereas if $(y_i + y_j) > (x_i + x_j)$ then

$$G_1 = \sum_{m=0}^{\infty} 2D_m \cos(k_x y_i) \cos(k_x y_j) \cos(\gamma_m(y_4 - a)) \cos(\gamma_m y_4 \gamma)$$  \hspace{1cm} (A.19)
and for the $G_2$ term, if $(x_i + y_j) \geq (y_i + x_j)$ then

$$G_2 = \sum_{m=0}^{\infty} 2D_m (-1)^m \cos(k_x x_i) \cos(k_y y_j) \cos(\gamma_m y_2,_) \cos(\gamma_m y_2,_)$$  \hspace{1cm} (A.20)

whereas if $(y_i + x_j) > (x_i + y_j)$ then

$$G_2 = \sum_{m=0}^{\infty} 2D_m (-1)^m \cos(k_x y_i) \cos(k_y x_j) \cos(\gamma_m y_3,_) \cos(\gamma_m y_3,_)$$  \hspace{1cm} (A.21)

where

$$D_m = \frac{\sigma_m}{\gamma_m \sin(\gamma_m a)}$$  \hspace{1cm} (A.22)

$$\sigma_m = \begin{cases} 1 & \text{for } m = 0 \\ 2 & \text{for } m \neq 0 \end{cases}$$  \hspace{1cm} (A.23)

$$\gamma_m = \pm \sqrt{k^2 - k_x^2}$$  \hspace{1cm} (A.24)

$$y_{1,>_<} = \max,\min(y_i, y_j)$$  \hspace{1cm} (A.25)

$$y_{2,>_<} = \max,\min(y_i, x_j)$$  \hspace{1cm} (A.26)

$$y_{3,>_<} = \max,\min(x_i, y_j)$$  \hspace{1cm} (A.27)

$$y_{4,>_<} = \max,\min(x_i, x_j)$$  \hspace{1cm} (A.28)

and $C$ is given by Equation (4.35). The sign of $\gamma_m$ is chosen such that $Im\gamma_m$ is negative.

### A.2 Single Series Green’s Function When Both Ports $i$ and $j$ are Along $AB$

When both ports are along $AB$ the series in Equation (A.17) blows up individually. To avoid this, one has to go back to the double series expression for the Green’s function in Equation (A.2) and combine all four series into one. This can be accomplished by transforming all the $y_i$ and $y_j$ to $x_i$ and $x_j$...
respectively using the transformation \( y = -x + a \). Thus, the simplified double series Green’s function becomes:

\[
G(x_i, y_i | x_j, y_j) = 4C \sum_{m=0}^{\infty} \left[ \sigma_m \cos(k_x x_i) \cos(k_x x_j) \right. \\
\left. \sum_{n=0}^{\infty} \frac{\sigma_n \cos(k'_y x_i) \cos(k'_y x_j)}{(m^2 + n^2)\pi^2 - a^2 k^2} \right]
\]  
(A.29)

where \( C \) is given by Equation (4.35); \( k_x \) is given by Equation (4.2); \( k'_y \) is given by Equation (4.3); \( \sigma_m \) and \( \sigma_n \) are given by Equation (4.4); and \( k^2 \) is given by Equation (4.5). Now, separating the \( n = 0 \) term gives:

\[
G(x_i, y_i | x_j, y_j) = 4C \sum_{m=0}^{\infty} \frac{\sigma_m \cos(k_x x_i) \cos(k_x x_j)}{m^2\pi^2 - a^2 k^2} + \\
\sum_{m=0}^{\infty} \sigma_m \cos(k_x x_i) \cos(k_x x_j)S(m)
\]  
(A.30)

where

\[
S(m) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{\cos(k_x x_i) \cos(k_x x_j)}{k_x^2 + k_y'^2 - k^2}
\]  
(A.31)

Similar to the derivations carried out in the Section A.1, the \( S(m) \) summation given by Equation (A.31) can be carried out analytically using trigonometric Fourier Series [20]. Thus, \( S(m) \) becomes:

\[
S(m) = \frac{-1}{\pi^2 \alpha_m^2} + \frac{1}{2\pi \alpha_m \sinh(\alpha_m \pi)} \cosh(\alpha_m (\pi - x_1)) + \cosh(\alpha_m (\pi - x_2))
\]  
(A.32)

where

\[
\alpha_m = \pm \frac{a}{\pi} \sqrt{k_x^2 - k^2}
\]  
(A.33)

\[
x_1 = \frac{\pi (y_+ + y_-)}{a}
\]  
(A.34)

\[
x_2 = \frac{\pi (y_+ - y_-)}{a}
\]  
(A.35)

\[
y_{+, -} = \max, \min(x_i, x_j)
\]  
(A.36)
Substituting $S(m)$ into Equation (A.30) and redefining $\alpha_m$ as:

$$\alpha_m = \pm \frac{a}{\pi} \gamma_m$$  \hspace{1cm} (A.37)

gives the expression for the single series Green's function for the case when both ports are located along $AB$ as shown below:

$$G(x_i, y_i | x_j, y_j) = -\frac{C}{a} \sum_{m=0}^{\infty} 4D_m \cos(k_x x_i) \cos(k_x x_j) \cos(\gamma_m(y_> - a)) \cos(\gamma_m y_<)$$  \hspace{1cm} (A.38)

where $D_m$ is given by Equation (A.22); $\gamma_m$ is given by Equation (A.24); $y_>$ and $y_<$ are given by Equation (A.36); and $C$ is given by Equation (4.35). The sign of $\gamma_m$ is chosen such that $\Im \gamma_m$ is negative.

A.3 Single Series Impedance Matrix when Both Ports are Along $AB$ and $i = j$

When both ports are located along $AB$ and they are indentical port (i.e., $i = j$) then the integration given by Equation (3.10) has to be carried out as an absolute integration. First, the single series Green's function given by Equation (A.38) is rewritten as:

$$G(x_i, y_i | x_j, y_j) = -\frac{C}{a} \left\{ \sum_{m=0}^{\infty} 2D_m \cos(k_x x_i) \cos(k_x x_j) \cdot \cos(\gamma_m(y_> - a + y_<)) + \sum_{m=0}^{\infty} 2D_m \cos(k_x x_i) \cdot \cos(k_x x_j) \cos(\gamma_m(y_> - a - y_<)) \right\}$$  \hspace{1cm} (A.39)

Then using Equation (A.39) in the integration in Equation (3.10) gives:

$$Z_{ij} = -\frac{C}{a} \left( \sum_{m=0}^{\infty} 2D_m \frac{\sqrt{2}}{\pi} \frac{\sqrt{2}}{w} \int_{z-\frac{w}{\sqrt{2}}}^{z+\frac{w}{\sqrt{2}}} \int_{x-\frac{w}{\sqrt{2}}}^{x+\frac{w}{\sqrt{2}}} \cos(k_x x_>) \cdot \cos(k_x x_<) \cos(\gamma_m x_>) + \gamma_m x_< - \gamma_m a \right) dx_> dx_< + \sum_{m=0}^{\infty} 2D_m$$
\[ \sqrt{2} \int_{x-\frac{w}{\sqrt{2}}}^{x+\frac{w}{\sqrt{2}}} \sqrt{2} \int_{x-\frac{w}{\sqrt{2}}}^{x+\frac{w}{\sqrt{2}}} \cos(k_x x_<) \cdot \cos(k_x x_<) \cos(\gamma_m (a - |x_> - x_<|)) \, dx_> \, dx_< \]  \hspace{1cm} (A.40)

Since the ports are identical hence, \( x_i = x_j = x \) and \( w_i = w_j = w \). Evaluating the first double integration term in Equation (A.40) gives:

\[ I_1 = \frac{1}{4} \text{sinc}[(\gamma_m - k_x) \frac{w}{2\sqrt{2}}] \{ \cos[(\gamma_m - k_x)2x - \gamma_m a] \cdot \\
\text{sinc}[(\gamma_m - k_x) \frac{w}{2\sqrt{2}}] + \cos(2\gamma_m x - \gamma_m a) \text{sinc}[(\gamma_m + k_x) \frac{w}{2\sqrt{2}}] \} + \\
+ \frac{1}{4} \text{sinc}[(\gamma_m + k_x) \frac{w}{2\sqrt{2}}] \{ \cos(2\gamma_m x - \gamma_m a) \text{sinc}[(\gamma_m - k_x) \frac{w}{2\sqrt{2}}] \} \\
\{ \cos[(\gamma_m + k_x)2x - \gamma_m a] \text{sinc}[(\gamma_m + k_x) \frac{w}{2\sqrt{2}}] \} \]  \hspace{1cm} (A.41)

The second double integration term in Equation (A.40) involves an absolute values. To handle this absolute value, the integration is broken up into two integrations as shown below:

\[ I_2 = \int_{x-\frac{w}{\sqrt{2}}}^{x+\frac{w}{\sqrt{2}}} \frac{\sqrt{2}}{w} \cos(k_x x_<) \cdot \\
\{ \frac{\sqrt{2}}{w} \int_{x_<}^{x_>} \cos(\gamma_m x_> - \gamma_m a - \gamma_m x_<) \cos(k_x x_> \, dx_> + \\
\frac{\sqrt{2}}{w} \int_{x_<}^{x_>} \cos(\gamma_m x_> + \gamma_m a - \gamma_m x_<) \cos(k_x x_> \, dx_> \} \, dx_< \]  \hspace{1cm} (A.42)

Evaluating \( I_2 \) in Equation (A.42) gives:

\[ I_2 = \frac{\sqrt{2}}{2w(\gamma_m - k_x)} \sin[(\gamma_m - k_x) \frac{w}{2\sqrt{2}}] - \gamma_m a] \{ \text{sinc}[(\gamma_m - k_x) \frac{w}{2\sqrt{2}}] + \\
\cos(2k_x x) \text{sinc}[(\gamma_m + k_x) \frac{w}{2\sqrt{2}}] + \frac{\sqrt{2}}{2w(\gamma_m + k_x)} \sin[(\gamma_m + k_x) \frac{w}{2\sqrt{2}}] - \gamma_m a \} \cdot \\
\{ \cos(2k_x x) \text{sinc}[(\gamma_m - k_x) \frac{w}{2\sqrt{2}}] + \text{sinc}[(\gamma_m - k_x) \frac{w}{2\sqrt{2}}] \} \\
+ \frac{\sqrt{2}\gamma_m}{w(\gamma_m^2 - k_x^2)} \sin(\gamma_m a) \{ 1 + \cos(2k_x x) \text{sinc}(\frac{k_x w}{\sqrt{2}}) \} \]  \hspace{1cm} (A.43)
Thus, the single series impedance matrix expression when both ports are located along $AB$ and $i = j$ becomes:

$$Z_{ij} = -\frac{C}{a} \sum_{m=0}^{\infty} 2D_m \{I_1 + I_2\}$$ (A.44)

where $C$ is given by Equation (4.35); $I_1$ is given by Equation (A.41); and $I_2$ is given by Equation (A.43).

### A.4 Single Series Green’s Function when One Port is Along $OA$ and the Other is Along $AB$

When one port is along $OA$ and the other is along $AB$ the series in Equation (A.20) and (A.21) blows up individually. To avoid this, one has to go back to the double series expression for the Green’s function in Equation (A.2) and combine all four series into one. This can be accomplished by transforming all the $y$ - coordinates, either $y_i$ to $x_i$ or $y_j$ to $x_j$, using the transformation $y = -x + a$ and by interchanging the index of summations. Thus, the simplified double series Green’s function becomes:

$$G(x_i, y_i|x_j, y_j) = 4C \sum_{m=0}^{\infty} \sigma_m \cos(k_x x_i) \cos(k_x x_j) \cdot \sum_{n=0}^{\infty} \sigma_n (-1)^n \cos(k'_y x_j)$$ (A.45)

where $C$ is given by Equation (4.35); $k_x$ is given by Equation (4.2); $k'_y$ is given by Equation (4.3); $\sigma_m$ and $\sigma_n$ are given by Equation (4.4); and $k^2$ is given by Equation (4.5). Now, separating the $n = 0$ term gives:

$$G(x_i, y_i|x_j, y_j) = 4C \sum_{m=0}^{\infty} \sigma_m \cos(k_x x_i) \cos(k_x x_j) \frac{\cos(k'_y x_j)}{m^2 \pi^2 - a^2 k'^2} + \sum_{m=0}^{\infty} \sigma_m \cos(k_x x_i) \cos(k_x x_j) S(m)$$ (A.46)

where

$$S(m) = \frac{2}{a^2} \sum_{n=1}^{\infty} (-1)^n \frac{\cos(k'_y x_j)}{k_x^2 + k'_y^2 - k^2}$$ (A.47)
Similar to the derivations carried out in the Appendix Section A.1, the \( S(m) \) summation given by Equation (A.47) can be carried out analytically using trigonometric Fourier Series [20]. Thus, \( S(m) \) becomes:

\[
S(m) = \frac{-1}{\pi^2 \alpha_m^2} + \frac{1}{\pi \alpha_m \sinh(\alpha_m \pi)}
\]  
(A.48)

where

\[
\alpha_m = \pm \frac{a}{\pi} \sqrt{k_x^2 - k^2}
\]
(A.49)

\[
x_1 = \frac{\pi x_j}{a}
\]
(A.50)

Substituting \( S(m) \) into Equation (A.46) and redefining \( \alpha_m \) as:

\[
\alpha_m = \pm \frac{a}{\pi} \gamma_m
\]
(A.51)

gives the expression for the single series Green’s function for the case when port \( i \) is along \( OA \) and port \( j \) is along \( AB \) below:

\[
G(x_i, y_i | x_j, y_j) = -\frac{C}{a} \sum_{m=0}^{\infty} 4D_m \cos(k_x x_i) \cos(k_x x_j) \cos(\gamma_m x_j)
\]
(A.52)

where \( D_m \) is given by Equation (A.22); \( \gamma_m \) is given by Equation (A.24); and \( C \) is given by Equation (4.35). The sign of \( \gamma_m \) is chosen such that \( \Im \gamma_m \) is negative. When port \( i \) is located along \( AB \) and port \( j \) is located along \( OA \), the single series Green’s function given by Equation (A.52) can still be used by interchanging the port locations \( (x_i \text{ and } x_j) \).
APPENDIX B

DERIVATIONS FOR CIRCULAR SECTOR AND ANNULAR SECTOR SEGMENTS

In this appendix the derivations of the double infinite series expressions for the impedance matrix elements of circular and annular sector segments are carried out.

B.1 Circular Sector Segment

Two cases are to be considered when applying the integration given by Equation (3.10) to the double infinite series Green’s function given by Equation (5.1) [21]. The first case is when the integration is along the radial edge, i.e., the integration is with respect to the variables \( \rho \). The second case is when the integration is along the curved edge, i.e., the integration is with respect to the variables \( \phi \).

B.1.1 Integration along the radial edge. When the integration is with respect to the variables \( \rho \), the \( \cos(\cdot) \) term is treated as constant. Hence, the integration to be considered is:

\[
I_{cs} = \frac{1}{w} \int_{\rho - \frac{w}{2}}^{\rho + \frac{w}{2}} \cos(n_{1}\phi)J_{n_{1}}(k_{mn_{1}}\rho) \, d\rho
\]

\[
= \frac{\cos(n_{1}\phi)}{wk_{mn_{1}}} \int_{t_{1}}^{t_{2}} J_{n_{1}}(k_{mn_{1}}\rho) \, d(k_{mn_{1}}\rho) \quad (B.1)
\]

where

\[
t_{1} = k_{mn_{1}} (\rho - \frac{w}{2}) \quad (B.2)
\]

\[
t_{2} = k_{mn_{1}} (\rho + \frac{w}{2}) \quad (B.3)
\]
The integration can be carried out using the relations given in [22], if \( n \) is even then Equation (B.1) becomes:
\[
I = \frac{\cos(n \phi)}{w k_{mn}} \left[ \int_{t_1}^{t_2} J_0(t) dt + 2 \sum_{k=0}^{\frac{1}{2}(n-2)} \{J_{2k+1}(t_1) - J_{2k+1}(t_2)\} \right] \tag{B.4}
\]
and if \( n \) is odd the Equation (B.1) becomes:
\[
I = \frac{\cos(n \phi)}{w k_{mn}} \left[ J_0(t_1) - J_0(t_2) + 2 \sum_{k=0}^{\frac{1}{2}(n-1)} \{J_{2k}(t_1) - J_{2k}(t_2)\} \right] \tag{B.5}
\]

**B.1.2 Integration along the curved edge.** In this case, the integration is with respect to the variables \( \phi \) and the \( J_n(\cdot) \) term is treated as constants. Hence, the integration becomes:
\[
I_{cs} = \frac{a}{w} \int_{\phi - \frac{n \pi}{2a}}^{\phi + \frac{n \pi}{2a}} J_{n_i}(k_{mn}a) \cos(n_i \phi) \, d\phi \\
= J_{n_i}(k_{mn}a) \cos(n_i \phi) \text{sinc} \left( \frac{n_i w}{2a} \right) \tag{B.6}
\]
where \( w \) is the curvilinear width of the port being considered.

**B.2 Annular Sector Segment**

Two cases are to be considered when applying the integration given by Equation (3.10) to the double infinite series Green’s function given by Equation (5.13) [21]. The first case is when the integration is along the radial edge, i.e., the integration is with respect to the variables \( \rho \). The second case is when the integration is along the curved edge, i.e., the integration is with respect to the variables \( \phi \).

**B.2.1 Integration along the radial edge.** When the integration is along the radial edge, then the \( \cos(\cdot) \) term is treated as constant. Hence, the integration to be considered is:
\[
I_{as} = \frac{1}{w} \int_{\rho - \frac{w}{2}}^{\rho + \frac{w}{2}} \cos(n \phi) F_{mn}(\rho) \, d\rho
\]
\[
I^N = \frac{\cos(n_l\phi)}{w k_{mn_i}} \left\{ \int_{t_1}^{t_2} N_{n_i}(k_{mn_i}, \rho) d(k_{mn_i}, \rho) - J_{n_i}'(k_{mn_i}, a) \int_{t_1}^{t_2} N_{n_i}(k_{mn_i}, \rho) d(k_{mn_i}, \rho) \right\}
\]

(B.7)

where \( F_{mn_i} \) is defined by Equation (5.17), and \( t_1 \) and \( t_2 \) are given by Equations (B.2) and (B.3). The result of the first integration in Equation (B.7) is given by Equations (B.4) and (B.5). The second integration in Equation (B.7) is an integration of the Neumann function. This integration can be carried out using the relation given in [20]. Thus, if \( n_l \) is even then the second integration in Equation (B.7) becomes:

\[
I^N = \frac{\cos(n_l\phi)}{w k_{mn_i}} \int_{t_1}^{t_2} N_0(t) dt + 2 \sum_{k=0}^{\frac{1}{2}(n_l-2)} \{N_{2k+1}(t_1) - N_{2k+1}(t_2)\}
\]

(B.8)

and if \( n_l \) is odd the Equation (B.7) becomes:

\[
I^N = \frac{\cos(n_l\phi)}{w k_{mn_i}} [N_0(t_1) - N_0(t_2) + 2 \sum_{k=0}^{\frac{1}{2}(n_l-1)} \{N_{2k}(t_1) - N_{2k}(t_2)\}]
\]

(B.9)

Thus, the integration in Equation (B.7) becomes:

\[
I_{as} = \frac{\cos(n_l\phi)}{w k_{mn_i}} \left\{ N_{n_i}'(k_{mn_i}, a) I^J - J_{n_i}'(k_{mn_i}, a)I^N \right\}
\]

(B.10)

where \( I^J \) is given by Equations (B.4) and (B.5).

**B.2.2 Integration along the curved edge.** In this case, the integration is with respect to the variables \( \phi \) and the \( F_{mn_i} \) term is treated as constant. Hence, the integration becomes:

\[
I_{as} = \frac{a}{w} \int_{\phi_-}^{\phi_+} F_{mn_i}(k_{mn_i}, \rho) \cos(n_l\phi) \, d\phi
\]

\[= F_{mn_i}(k_{mn_i}, \rho) \cos(n_l\phi) \text{sinc}(\frac{n_lw}{2a})
\]

(B.11)

where \( w \) is the curvilinear width of the port being considered.
APPENDIX C

EXPRESSIONS FOR IMPEDANCE MATRIX ELEMENTS FOR
RECTANGULAR AND TRIANGULAR SEGMENTS

Knowing the Green's function of a particular shape of a segment, the impedance matrix elements for the segment can be computed using Equation (3.10). The impedance matrix elements for rectangular and triangular planar segments with open boundaries are presented in this appendix.

C.1 Z-matrix Elements for a Rectangular Segment

In case of a rectangular segment the impedance matrix elements can be expressed in a single infinite series [25]. The geometry of the rectangular segment is specified in Figure C.1

![Diagram of a rectangular segment with port locations](image)

Figure C.1: The rectangular segment configuration and the port locations.
C.1.1 Port $i$ and port $j$ are oriented in the same direction ($x$ or $y$). The $Z$-matrix elements $Z_{ij}$ can be expressed as:

\[
Z_{ij} = -CF \sum_{l=0}^{L} \sigma_l \cos(k_u u_i) \cos(k_u u_j) \cos(\gamma_l z_>) \cos(\gamma_l z_<) \\
\times \frac{\text{sinc}(\frac{k_u w_i}{2}) \text{sinc}(\frac{k_u w_i}{2})}{\gamma_l \sin(\gamma_l F)} - jCF \sum_{l=L+1}^{\infty} \cos(k_u u_j) \cos(k_u u_i) \\
\times \text{sinc}(\frac{k_u w_i}{2}) \text{sinc}(\frac{k_u w_j}{2}) \exp(-j\gamma_l (v_> - v_<)) \frac{1}{\gamma_l}
\]

(C.1)

where

\[
(\gamma_l, v_<) = \begin{cases} 
(y_>, y_<) & \text{for } l = m \\
(x_>, x_<) & \text{for } l = n 
\end{cases}
\]

(C.2)

\[
C = \frac{j \omega \mu_0 h}{ab}
\]

(C.3)

\[
F = \begin{cases} 
\frac{b}{a} & \text{for } l = m \\
\frac{a}{a} & \text{for } l = n 
\end{cases}
\]

(C.4)

\[
(u_i, u_j) = \begin{cases} 
(x_i, x_j) & \text{for } l = m \\
(y_i, y_j) & \text{for } l = n 
\end{cases}
\]

(C.5)

\[
\gamma_l = \pm \sqrt{k^2 - k_u^2}
\]

(C.6)

\[
k^2 = \omega^2 \mu \varepsilon_0 \varepsilon_r (1 - j\delta_e)
\]

(C.7)

here, $\delta_e$ is the effective loss tangent given by Equation (2.17).

\[
k_u = \begin{cases} 
\frac{m}{a} & \text{for } l = m \\
\frac{m}{b} & \text{for } l = n 
\end{cases}
\]

(C.8)

\[
(z_>, z_<) = \begin{cases} 
(y_>- b, y_<) & \text{for } l = m \\
(x_>- a, x_<) & \text{for } l = n 
\end{cases}
\]

(C.9)

\[
y_> = \max(y_i, y_j)
\]

(C.10)
\[ y_\angle = \min(y_i, y_j) \quad (C.11) \]
\[ x_\angle = \max(x_i, x_j) \quad (C.12) \]
\[ x_\triangle = \min(x_i, x_j) \quad (C.13) \]
\[ \sigma_l = \begin{cases} 
1 & \text{for } l = 0 \\
2 & \text{for } l \neq 0 
\end{cases} \quad (C.14) \]

If the ports \( i \) and \( j \) are oriented along \( x \)-direction then \( l = m \) is chosen. On the other hand, if the ports \( i \) and \( j \) are oriented along \( y \)-direction \( l = n \) is chosen. The sign of \( \gamma_l \) is chosen such that \( \text{Im}(\gamma_l) \) is negative. The integer \( L \) in Equation (C.1) is chosen so that \( (\gamma_l F) \) is less than or equal to 50; this choice is a compromise between computational speed and accuracy.

**C.1.2 Port \( i \) and port \( j \) are oriented in different directions.**

For this case, the \( Z \)-matrix elements can be expressed as:

\[ Z_{ij} = -CF \sum_{l=0}^{L} \sigma_l \cos(k_u u_i) \cos(k_u u_j) \cos(\gamma_l z_\angle) \cos(\gamma_l z_\angle) \]
\[ \frac{sinc(\frac{k_u w_i}{2})sinc(\frac{w_i}{2})}{\gamma_l \sin(\gamma_l F)} - CF \sum_{l=L+1}^{\infty} \cos(k_u u_i) \cos(k_u u_j) \]
\[ sinc(\frac{k_u w_i}{2}) \exp(-\gamma_l(v_\angle - v_\angle - \frac{w_j}{2})) \frac{1}{\gamma_l^2 w_j} \quad (C.15) \]

The integer \( l \) is chosen so that it satisfies the following condition:

\[ (v_\angle - v_\angle - \frac{w_j}{2}) > 0 \quad (C.16) \]

Thus, the integer \( l \) is chosen according to the following conditions:

\[ l = m \quad \text{if } \{ \max(y_i, y_j) - \min(y_i, y_j) - \frac{w_j}{2} \} > 0 \quad (C.17) \]
\[ l = n \quad \text{if } \{ \max(x_i, x_j) - \min(x_i, x_j) - \frac{w_j}{2} \} > 0 \quad (C.18) \]

If \( l = m \) then \( w_i \) corresponds to the port oriented in the \( x \)-direction and \( w_j \) corresponds to the port oriented in the \( y \)-direction. If \( l = n \) then \( w_i \) corresponds
to the port oriented in the y-direction and \( u_j \) corresponds to the port oriented in the x-direction. The notations given in Equations (C.2) to (C.14) are applicable in this case also.

### C.2 Z-matrix Elements for Triangular Segments

There are three different triangular configurations for which their Green's functions are available. The three triangular configurations are: right-angle isosceles triangle, 30° - 90° - 60° triangle, and equilateral triangle [19]. Thus, the impedance matrix elements for these three triangular configurations can be computed. The expressions for the impedance matrix elements for right-angle isosceles triangular segment is discussed in Chapter 4. The expressions for the impedance matrix elements for the other two triangular segments are given in this Section.

#### C.2.1 Z-matrix elements for a 30°-90°-60° triangular segment.

The geometry of the 30°-90°-60° triangular segment is specified in Figure C.2.

The impedance matrix elements \( Z_{ij} \) can be expressed as:

\[
Z_{ij} = 8 \gamma \mu_0 h \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{I_{T_1}(i)I_{T_1}(j)}{16\sqrt{3}\pi^2(m^2 + mn + n^2) - 9\sqrt{3}a^2 k^2} \tag{C.19}
\]

where \( h \) is the substrate thickness; \( k^2 \) is given by Equation (C.7). For ports located along the side OA:

\[
I_{T_1}(i) = (-1)^l \cos \left( \frac{2\pi l}{\sqrt{3}a} x_i \right) \text{sinc} \left( \frac{2\pi l}{\sqrt{3}a} \frac{w_i}{2} \right) \\
+ (-1)^m \cos \left( \frac{2\pi m}{\sqrt{3}a} x_i \right) \text{sinc} \left( \frac{2\pi m w_i}{\sqrt{3}a} \frac{2}{2} \right) \\
+ (-1)^n \cos \left( \frac{2\pi n}{\sqrt{3}a} x_i \right) \text{sinc} \left( \frac{2\pi n w_i}{\sqrt{3}a} \frac{2}{2} \right) \tag{C.20}
\]
Figure C.2. The 30°-90°-60° triangular segment configuration and the port locations.

For ports located along the side $OB$:

$$ I_{T_1}(i) = (-1)^l \cos\left(\frac{2\pi(m-n)}{3a}y_i\right) \text{sinc}\left(\frac{2\pi(m-n)w_i}{3a}\right) $$

$$ + (-1)^m \cos\left(\frac{2\pi(n-l)}{3a}y_i\right) \text{sinc}\left(\frac{2\pi(n-l)w_i}{3a}\right) $$

$$ + (-1)^n \cos\left(\frac{2\pi(l-m)}{3a}y_i\right) \text{sinc}\left(\frac{2\pi(l-m)w_i}{3a}\right) $$

(C.21)

For ports located along the side $AB$:

$$ I_{T_1}(i) = \cos\left(\frac{4\pi(m-n)}{3a}y_i\right) \text{sinc}\left(\frac{4\pi(m-n)w_i}{3a}\right) $$

$$ \cos\left(\frac{4\pi(n-l)}{3a}y_i\right) \text{sinc}\left(\frac{4\pi(n-l)w_i}{3a}\right) $$

$$ \cos\left(\frac{4\pi(l-m)}{3a}y_i\right) \text{sinc}\left(\frac{4\pi(l-m)w_i}{3a}\right) $$

(C.22)

The integers $l, m, n$ have to satisfy the following conditions:

$$ l + m + n = 0 $$

(C.23)

The Equations (C.20) to (C.23) hold also for $I_{T_1}(j)$. 
C.2.2 Z-matrix elements for a 60°-60°-60° triangular segment. The geometry of the equilateral triangular segment is specified in Figure C.3.

![Equilateral Triangular Segment Diagram](image)

Figure C.3. The equilateral triangular segment configuration and the port locations.

The impedance matrix elements $Z_{ij}$ can be expressed as:

$$Z_{ij} = 4j\omega \mu_0 h \sum_{m=-\infty}^{\infty} \frac{I_{T_1}(i)I_{T_1}(j)}{16\sqrt{3}\pi^2(m^2 + mn + n^2) - 9\sqrt{3}a^2k^2}$$  \hspace{1em} (C.24)

where $h$ is the substrate height and $k^2$ is given by Equation (C.7).

For ports located along the side $BC$ the expressions for $I_{T_1}(i)$ and $I_{T_1}(j)$ are given by Equation (C.21). For ports located along the side $AB$ or $AC$ the expressions for $I_{T_1}(i)$ and $I_{T_1}(j)$ are given by Equation (C.22).
For ports located along the side $BC$:

$$
I_{T_2}(i) = (-1)^l \sin\left(\frac{2\pi(m-n)}{3a} y_i\right) \text{sinc}\left(\frac{2\pi(m-n)w_i}{3a}\right) \\
+ (-1)^m \sin\left(\frac{2\pi(n-l)}{3a} y_i\right) \text{sinc}\left(\frac{2\pi(n-l)w_i}{3a}\right) \\
+ (-1)^n \sin\left(\frac{2\pi(l-m)}{3a} y_i\right) \text{sinc}\left(\frac{2\pi(l-m)w_i}{3a}\right) \tag{C.25}
$$

For ports located along the side $AB$ or $AC$:

$$
I_{T_2}(i) = -\sin\left(\frac{4\pi(m-n)}{3a} y_i\right) \text{sinc}\left(\frac{4\pi(m-n)w_i}{3a}\right) \\
- \sin\left(\frac{4\pi(n-l)}{3a} y_i\right) \text{sinc}\left(\frac{4\pi(n-l)w_i}{3a}\right) \\
- \sin\left(\frac{4\pi(l-m)}{3a} y_i\right) \text{sinc}\left(\frac{4\pi(l-m)w_i}{3a}\right) \tag{C.26}
$$

The Equations (C.25) and (C.26) hold also for $I_{T_2}(j)$. The integers $l$, $m$, and $n$ have to satisfy the condition given in Equation (C.23).