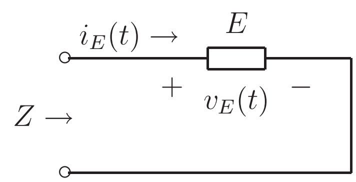
Impedance and Loudspeaker Parameter Measurement

1 Impedance Measurement

Many elements from which electrical circuits are built are two-terminal elements. Some simple examples are resistors, inductors, and capacitors. A more complex element that many people use on a daily basis is the loudspeaker (or a headphone). One way to analyze and characterize two-terminal elements is by looking at their impedance $Z(\omega)$ or Z for short, as shown in the following figure.

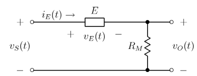


To determine Z for an element E at a given frequency ω, a sinusoidal input voltage $v_E(t) \Leftrightarrow \mathbf{V}_E$ with frequency ω is applied. If the element E is linear, then a sinusoidal current $i_E(t) \Leftrightarrow \mathbf{I}_E$ at frequency ω will flow through it and $Z(\omega)$ can be computed as

$$Z(\omega) = \frac{\mathbf{V}_E}{\mathbf{I}_E} .$$

In general, $Z(\omega)$ is a complex-valued quantity with magnitude $|Z(\omega)|$ and phase $\angle Z(\omega)$.

To measure $Z(\omega)$, e.g., using the sound card probe with amplifier, use the circuit shown below with a suitably chosen reference resistor R_M.



Let $v_S(t)$ be a sinusoid with radian frequency ω and assume that the element E is linear. The voltage across E and the current through E are

$$v_E(t) = v_S(t) - v_O(t) \,, \qquad \text{and} \qquad i_E(t) = \frac{v_O(t)}{R_M} \,,$$

respectively. Without loss of generality, assume that

$$v_E(t) = v_S(t) - v_O(t) = V_E \, \cos\omega t \quad \Longleftrightarrow \quad \mathbf{V}_E = V_E \,,$$

and

$$i_E(t) = \frac{v_O(t)}{R_M} = I_E \, \cos(\omega t + \phi) \quad \Longleftrightarrow \quad \mathbf{I}_E = I_E \, e^{j\phi} \,.$$

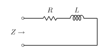
The impedance of the element E at frequency ω is then found as

$$Z = \frac{\mathbf{V}_E}{\mathbf{I}_E} = \frac{V_E}{I_E} \, e^{-j\phi} \qquad \Longrightarrow \qquad |Z| = \frac{V_E}{I_E} \,, \text{and} \quad \angle Z = -\phi \,.$$

If this is measured for different values of ω, then $Z(\omega)$ can be plotted versus ω.

2 Inductor Impedance

A real inductor operated at room temperature will have some nonzero resistance and thus be lossy. This can be modeled as a series connection of an (ideal) R and an (ideal) L component as shown in the following figure.



The impedance of this (real) inductor is

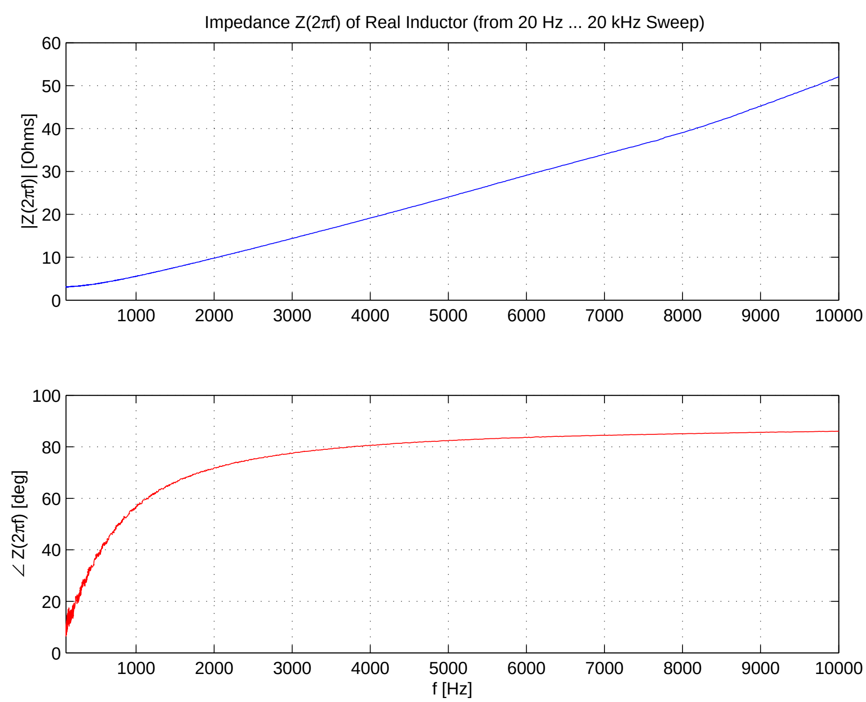
$$Z(\omega) = R + j\omega L \qquad \Longrightarrow \qquad |Z(\omega)| = R\sqrt{1 + (\frac{\omega L}{R})^2}\,, \quad \text{and} \quad \angle Z(\omega) = \tan^{-1}\frac{\omega L}{R}\,.$$

Let $\omega_0 = R/L$. Then

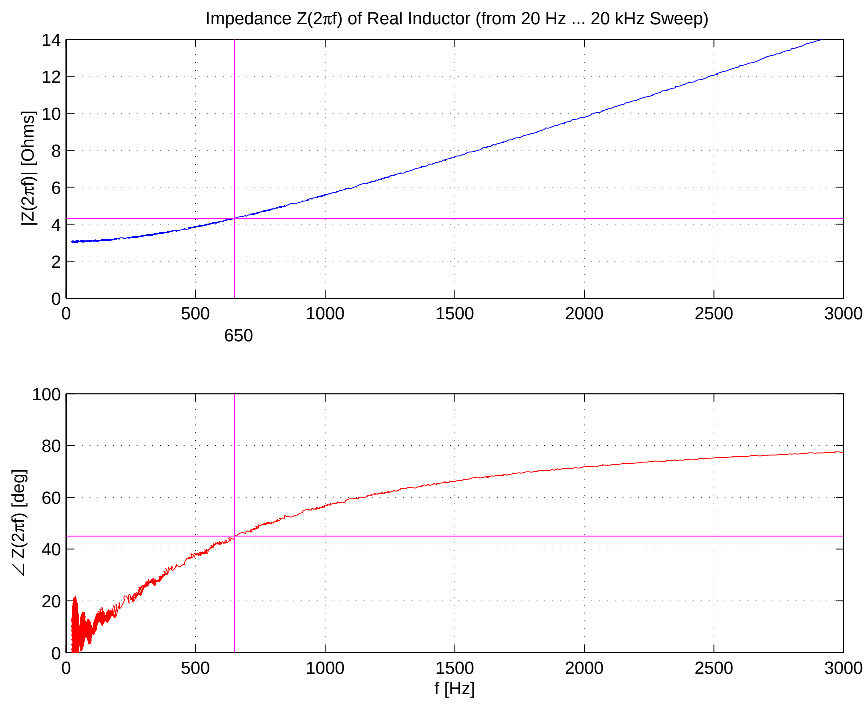
$$|Z(\omega_0)| = \sqrt{2}\,R\,, \qquad \text{and} \qquad \angle(\omega_0) = 45^\circ\,.$$

Thus, ω_0 can be found from the frequency at which the value of the phase is 45° and R can be found from the value of $|Z|$ at ω. Finally, $L = R/\omega_0$. Another way to find L is to look at $|Z(\omega)|/\omega$ for $\omega \gg \omega_0$.

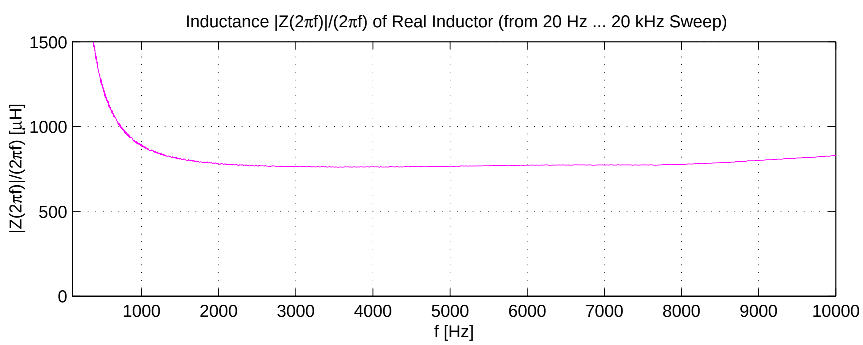
Example: The impedance of an unknown inductance was measured using a reference resistor of $R_M = 47\,\Omega$ and the following plots were obtained.



Zooming in on the region where $\angle Z \approx 45^\circ$ yields $f_0 = 650$ Hz and $|Z(2\pi f0)| = 4.3\,\Omega$ as shown in the next two plots.

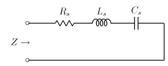


Thus, $R = 4.3/\sqrt{2} = 3.04\,\Omega$ and $L = R/(2\pi f_0) = 744\,\mu$H. The following graph that shows $|Z(\omega)|/\omega$ confirms this result.



3 Impedance of Resonant Circuit

A **series resonant circuit** consisting of R_s, L_s, and C_s is shown in the following schematic.



The impedance of the series resonant circuit is

$$Z(\omega) = R_s \, \frac{\omega_0^2 - \omega^2 + 2j\zeta\omega_0\omega}{2j\zeta\omega_0\omega} \,, \qquad \text{where} \qquad \omega_0 = \frac{1}{\sqrt{L_s C_s}} \,, \quad 2\zeta\omega_0 = \frac{R_s}{L_s} \,.$$

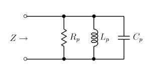
At $\omega = \omega_0$

$$|Z(\omega_0)| = R_s \,, \quad \angle Z(\omega_0) = 0^\circ \,.$$

To find ζ determine $\omega_{x^-} < \omega_0$ and $\omega_{x^+} > \omega_0$ such that

$$|Z(\omega_{x^-})| = |Z(\omega_{x^+})| = \sqrt{2}\, R_s \qquad \Longrightarrow \qquad \omega_{x^+} - \omega_{x^-} = 2\zeta\omega_0 \,.$$

A **parallel resonant circuit** with elements R_p, L_p, and C_p is shown in the next circuit.



The impedance of the parallel resonant circuit is

$$Z(\omega) = R_p \, \frac{2j\zeta\omega_0\omega}{\omega_0^2 - \omega^2 + 2j\zeta\omega_0\omega} \,, \qquad \text{where} \qquad \omega_0 = \frac{1}{\sqrt{L_p C_p}} \,, \quad 2\zeta\omega_0 = \frac{1}{R_p C_p} \,.$$

At $\omega = \omega_0$

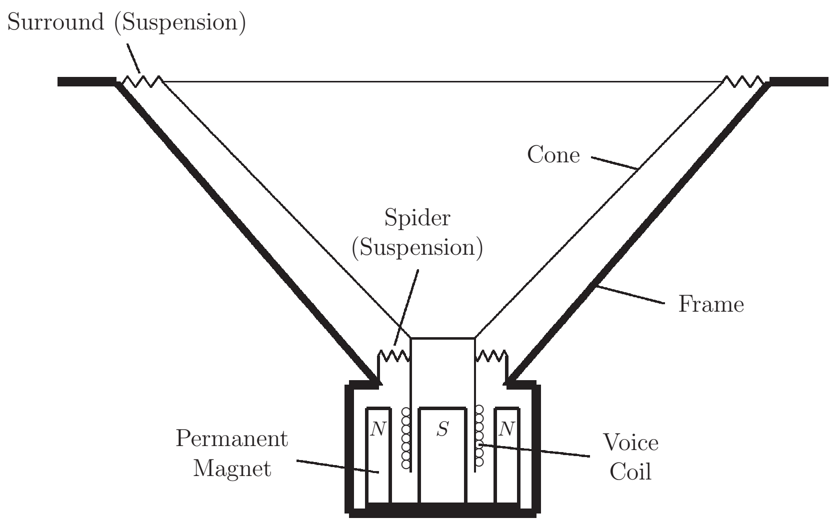
$$|Z(\omega_0)| = R_p \,, \quad \angle Z(\omega_0) = 0^\circ \,.$$

To find ζ determine $\omega_{x^-} < \omega_0$ and $\omega_{x^+} > \omega_0$ such that

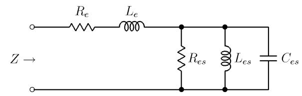
$$|Z(\omega_{x^-})| = |Z(\omega_{x^+})| = \frac{R_p}{\sqrt{2}} \qquad \Longrightarrow \qquad \omega_{x^+} - \omega_{x^-} = 2\zeta\omega_0 \,.$$

4 Loudspeaker Equivalent Circuit

The following figure shows a cross-section of a standard electrodynamic loudspeaker.



When a voltage is applied to the voice coil, the current through the wire causes a force between the voice coil and the permanent magnet. This moves the cone membrane which leads to a change in air pressure. If this occurs at frequencies within the audible range ($\approx 20 \ldots 16000$ Hz) then the changes in air pressure are perceived as sound. An electrodynamic loudspeaker thus converts electrical energy first into mechanical energy, and then into acoustical energy. A simplified equivalent electrical circuit that models a dynamic loudspeaker in open air or in a closed box is shown in the following schematic.



The quantities R_e and L_e in the equivalent circuit model the resistance and the inductance of the voice coil. The other three elements in the circuit are electrical equivalents for the mechanical losses (R_{es}), the compliance (L_{es}) and the moving mass (C_{es}) of the speaker system. The s subscript refers to the speaker. The main contribution to R_{es} is friction in the suspension of the speaker cone. Compliance is the inverse of stiffness and, in open air, L_{es} is an inverse measure of the stiffness of the speaker suspension (surround and spider). If the speaker is mounted in a closed box then L_{es} also includes the (inverse of the) stiffness of the air trapped in the box. The quantity C_{es}, finally, is the electrical equivalent of the mass of the cone, the voice coil, and the air mass that is moved by the cone.

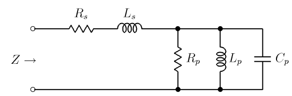
The impedance of the electrical equivalent circuit can be computed as

$$Z_e(s) = \frac{R_{es}L_eL_{es}C_{es}\, s^3 + (R_eR_{es}L_{es}C_{es} + L_eL_{es})\, s^2 + (R_eL_{es} + R_{es}L_e + R_{es}L_{es})\, s + R_eR_{es}}{R_{es}L_{es}C_{es}\, s^2 + L_{es}\, s + R_{es}}\,.$$

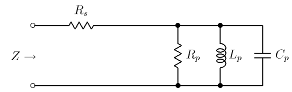
It is not difficult to compute $Z_e(j\omega)$ when R_e, L_e, R_{es}, L_{es}, and C_{es} are given. The interesting question, however, is how to find R_e, L_e, R_{es}, L_{es}, and C_{es} from the measured impedance $Z(\omega)$ of a speaker under test. This type of problem is called **system identification**.

5 Loudspeaker Parameter Estimation

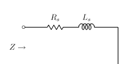
Consider the following 3'rd order circuit.



At low frequencies (including the resonance peak due to L_p and C_p) it is well approximated by the following 2'nd order circuit.



At high frequencies C_p acts like a short circuit, and the following 1'st order RL circuit is a good approximation.



As shown earlier

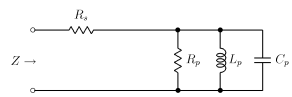
$$Z(\omega) = R_s + j\omega L_s \qquad \Longrightarrow \qquad |Z(\omega)| = \sqrt{R_s^2 + (L\omega)^2}\,.$$

If R_s is known (e.g., from a dc measurement with a multimeter), then

$$L = \frac{\sqrt{|Z(\omega)|^2 - R_s^2}}{\omega} ,$$

for a suitably high value of ω.

Parameter Estimation at Low Frequencies. In this case the equivalent circuit is:



The impedance Z seen between the two terminals is

$$Z(\omega) = \frac{R_s\,(\omega_0^2 - \omega^2) + (R_s + R_p)\, j2\zeta\omega_0\omega}{\omega_0^2 - \omega^2 + j2\zeta\omega_0\omega} , \qquad \text{where} \qquad \omega_0 = \frac{1}{\sqrt{L_p C_p}} , \quad 2\zeta\omega_0 = \frac{1}{R_p C_p} .$$

At $\omega = 0$ and $\omega = \omega_0$

$$|Z(0)| = R_s , \quad \angle Z(0) = 0^\circ , \qquad \text{and} \qquad |Z(\omega_0)| = R_s + R_p , \quad \angle Z(\omega_0) = 0^\circ ,$$

and thus ω_0, R_s, and R_p can be easily found from plots of $|Z(\omega)|$ and $\angle Z(\omega)$ versus ω. To find ζ, proceed as follows. Find $\omega_{x^-} < \omega_0$ and $\omega_{x^+} > \omega_0$ such that

$$|Z(\omega_{x^-})| = |Z(\omega_{x^+})| = \sqrt{\frac{R_s^2 + (R_s + R_p)^2}{2}} \qquad \Longrightarrow \qquad \omega_{x^+} - \omega_{x^-} = 2\zeta\omega_0 .$$

Since $2\zeta\omega_0 = 1/(R_p C_p)$ and R_p is already known, this determines C_p and, through ω_0, L_p. Thus, all parameters of this circuit can be determined easily from knowledge of $Z(\omega)$ over a suitable range (which must include ω_0) of ω.

