AWGN Channel with Matched Filter Receiver

1 Waveform Data Transmission Model

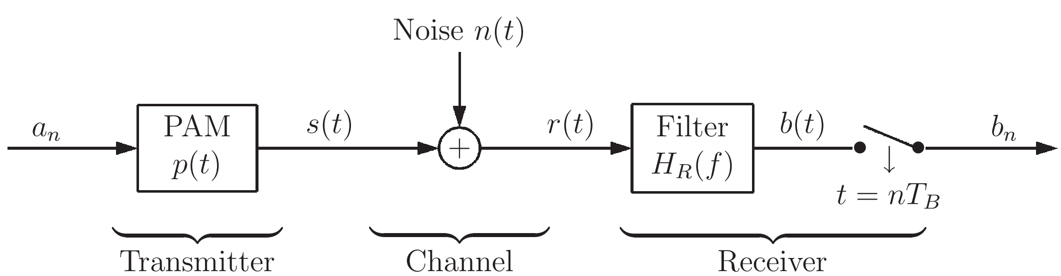
To transmit a discrete data sequence $\{a_n\}$ over a waveform channel, it needs to be converted to a waveform $s(t)$. If pulse amplitude modulation (PAM) with a pulse $p(t)$ is used then

$$s(t) = \sum_{n=-\infty}^{\infty} a_n \, p(t - nT_B) ,$$

where $F_B = 1/T_B$ is the baud or symbol rate of the data sequence $\{a_n\}$. If an additive noise model is used for the channel then the received waveform is

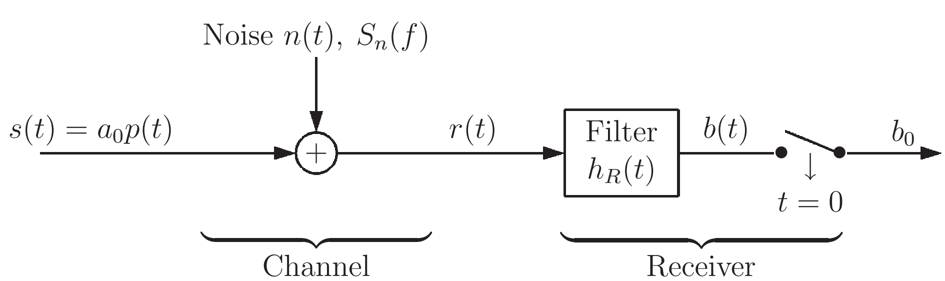
$$r(t) = s(t) + n(t) ,$$

where $n(t)$ is the noise, characterized by an appropriate random process. The task of the receiver is to extract a sequence $\{b_n\}$ from $r(t)$ such that $\{b_n\}$ is as good an estimate of $\{a_n\}$ as possible. The blockdiagram of such a PAM communication system is shown in the following figure.



2 "One Shot" Model

For the purpose of analyzing the performance of a PAM system in the presence of additive white Gaussian noise (AWGN) with mena zero and (2-sided) power spectral density $S_n(f) = \mathcal{N}_0/2$, the following "one shot" model can be used.



A single (randomly chosen) symbol a_0 is transmitted as $s(t) = a_0\, p(t)$ and received as

$$r(t) = s(t) + n(t) = a_0\, p(t) + n(t)\ .$$

The receiver filters $r(t)$ and produces

$$b(t) = r(t) * h_R(t)\ , \qquad \text{and} \qquad b_0 = b(0)\ .$$

One optimality criterion to design the receiver filter $h_R(t)$ is to maximize the signal-to-noise ratio (SNR) at the output of the filter at the sampling time instant $t = 0$. This results in a matched filter for $h_R(t)$ which, for the AWGN case, is

$$h_R(t) = \frac{p^*(-t)}{\int_{-\infty}^{\infty} |p(\mu)|^2\, d\mu}\ , \qquad \Longleftrightarrow \qquad H_R(f) = \frac{P^*(f)}{\int_{-\infty}^{\infty} |P(\nu)|^2\, d\nu}\ ,$$

where * denotes complex conjugation.

The expected value at the output $b(t)$ of the matched filter can be expressed as

$$\begin{aligned} E[b(t)] &= E[a_0] \int_{-\infty}^{\infty} H_R(f)\, P(f)\, e^{j2\pi ft}\, df + \int_{-\infty}^{\infty} h_R(\mu)\, \underbrace{E[n(t-\mu)]}_{=0}\, d\mu \\ &= E[a_0] \int_{-\infty}^{\infty} \frac{P^*(f)}{\int_{-\infty}^{\infty} |P(\nu)|^2\, d\nu}\, P(f)\, e^{j2\pi ft}\, df\ . \end{aligned}$$

Thus, after sampling at $t = 0$,

$$E[b_0] = E[b(0)] = E[a_0]\, \frac{\int_{-\infty}^{\infty} |P(f)|^2\, df}{\int_{-\infty}^{\infty} |P(\nu)|^2\, d\nu} = E[a_0]\ .$$

In the absence of noise

$$\begin{aligned} E[|b(t)|^2] &= E\Big[\big(a_0 \int_{-\infty}^{\infty} H_R(f)\, P(f)\, e^{j2\pi ft}\, df\big)\big(a_0 \int_{-\infty}^{\infty} H_R(\nu)\, P(\nu)\, e^{j2\pi\nu t}\, d\nu\big)^*\Big] \\ &= E[|a_0|^2] \Big| \int_{-\infty}^{\infty} H_R(f)\, P(f)\, e^{j2\pi ft}\, df \Big|^2 = E[|a_0|^2] \Big| \int_{-\infty}^{\infty} \frac{|P(f)|^2}{\int_{-\infty}^{\infty} |P(\nu)|^2\, d\nu}\, e^{j2\pi ft}\, df \Big|^2. \end{aligned}$$

The signal power after sampling at $t = 0$ is therefore

$$E[|b_0|^2] = E[|b(0)|^2] = E[|a_0|^2] \left| \frac{\int_{-\infty}^{\infty} |P(f)|^2\, df}{\int_{-\infty}^{\infty} |P(\nu)|^2\, d\nu} \right|^2 = E[|a_0|^2]\ .$$

In the absence of the signal $s(t)$, i.e., if $r(t) = n(t)$, $b(t)$ has power spectral density

$$S_b(f) = \frac{\mathcal{N}_0}{2}\, |H_R(f)|^2\ ,$$

since $h_R(t)$ is a linear and time-invariant system. This translates into noise variance

$$\sigma_b^2 = R_b(0) = \frac{\mathcal{N}_0}{2} \int_{-\infty}^{\infty} |H_R(f)|^2\, df = \frac{\mathcal{N}_0}{2} \int_{-\infty}^{\infty} \left| \frac{P^*(f)}{\int_{-\infty}^{\infty} |P(\nu)|^2\, d\nu} \right|^2 df = \frac{\mathcal{N}_0}{2}\, \frac{1}{\int_{-\infty}^{\infty} |P(\nu)|^2\, d\nu}\ ,$$

where $R_b(\tau)$ is the autocorrelation function of $b(t)$ when no signal $s(t)$ is present. Note that

$$E_p = \int_{-\infty}^{\infty} |p(\mu)|^2 \, d\mu = \int_{-\infty}^{\infty} |P(\nu)|^2 \, d\nu \qquad \Longrightarrow \qquad \sigma_b^2 = \frac{\mathcal{N}_0}{2E_p} \, ,$$

where E_p is the PAM pulse energy. The average signal energy after sampling at $t = 0$ at the ouptut of the matched filter is $E[|b_0|^2] = E[|a_0|^2]$ and thus the SNR after sampling at $t = 0$ is

$$\frac{S}{N} = \frac{E[|b_0|^2]}{\sigma_b^2} = E[|a_0|^2] \, \frac{2E_p}{\mathcal{N}_0} = \frac{E_s}{\mathcal{N}_0/2} \, ,$$

where the signal energy E_s is defined as

$$E_s = E\Big[\int_{-\infty}^{\infty} |s(\mu)|^2 \, d\mu \Big] = E[|a_0|^2] \int_{-\infty}^{\infty} |p(\mu)|^2 \, d\mu \, .$$

3 Probability of Error

Without loss of generality, assume that the signal $s(t) = a_0 \, p(t)$ is normalized so that $E_p = \int_{-\infty}^{\infty} |p(\mu)|^2 \, d\mu = 1$ and thus $E_s = E[|a_0|^2]$. If a_0 is a binary random variable with values $a_0 \in \{A, B\}$, then b_0 is a Gaussian random variable with mean either A or B and variance $\sigma_b^2 = \mathcal{N}_0/2$, i.e.,

$$f_{b_0}(\beta|a_0{=}A) = \frac{e^{-(\beta-A)^2/\mathcal{N}_0}}{\sqrt{\pi \mathcal{N}_0}} \, , \qquad \text{and} \qquad f_{b_0}(\beta|a_0{=}B) = \frac{e^{-(\beta-B)^2/\mathcal{N}_0}}{\sqrt{\pi \mathcal{N}_0}} \, .$$

The decision rule for a maximum likelihood (ML) receiver upon receiving $b_0 = \beta$ is:

$$\text{Decide } \hat{a}_0{=}A \text{ iff } \quad f_{b_0}(\beta|a_0{=}A) > f_{b_0}(\beta|a_0{=}B), \quad \text{else set } \hat{a}_0{=}B.$$

Substituting the conditional pdfs yields:

$$\text{Decide } \hat{a}_0{=}A \text{ iff } \qquad (\beta - A)^2 < (\beta - B)^2 \qquad \text{or} \qquad \beta < \frac{A+B}{2} \, .$$

Assuming $B > A$, the probability of a decision error given $a_0{=}A$ is

$$P(\mathcal{E}|a_0{=}A) = \int_{(A+B)/2}^{\infty} f_{b_0}(\beta|a_0{=}A) \, d\beta = \frac{1}{\sqrt{\pi \mathcal{N}_0}} \int_{(A+B)/2}^{\infty} e^{-(\beta-A)^2/\mathcal{N}_0} \, d\beta$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{B-A}{2\sqrt{\mathcal{N}_0}}}^{\infty} e^{-\mu^2} \, d\mu = \frac{1}{2} \text{erfc}\Big(\frac{B-A}{2\sqrt{\mathcal{N}_0}} \Big) \, ,$$

where the complementary error function $\text{erfc}(x)$ is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-\mu^2} \, d\mu \qquad \Longrightarrow \qquad \text{erfc}(-x) = 1 - \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-\mu^2} \, d\mu \, .$$

The probability of a decision error given $a_0{=}B$ is computed similarly as

$$P(\mathcal{E}|a_0{=}B) = \int_{-\infty}^{(A+B)/2} f_{b_0}(\beta|a_0{=}B)\,d\beta = \frac{1}{\sqrt{\pi\mathcal{N}_0}}\int_{-\infty}^{(A+B)/2} e^{-(\beta-B)^2/\mathcal{N}_0}\,d\beta$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{A-B}{2\sqrt{\mathcal{N}_0}}} e^{-\mu^2}\,d\mu = \frac{1}{2}\text{erfc}\Big(\frac{B-A}{2\sqrt{\mathcal{N}_0}}\Big)\;. = P(\mathcal{E}|a_0{=}A)$$

The probability of a symbol error for an AWGN channel with matched filter receiver an ML decision rule and $a_0 \in \{A, B\}$ is therefore

$$P_s(\mathcal{E}) = P(\mathcal{E}|a_0{=}A)\,P(a_0{=}A) + P(\mathcal{E}|a_0{=}B)\,P(a_0{=}B) = \frac{1}{2}\text{erfc}\Big(\frac{|B-A|}{2\sqrt{\mathcal{N}_0}}\Big)\;,$$

i.e., it depends only on the distance $|B-A|$ between the possible values of a_0 and the (2-sided) noise power spectral density $\mathcal{N}_0/2$. If antipodal signaling is used (e.g., for BPSK or QPSK modulation of a carrier) then $B = -A$ which implies

$$|B-A| = 2|A| = 2\sqrt{E_b} \qquad \text{where} \qquad E_b = A^2\,P(a_0{=}A) + B^2\,P(a_0{=}B)\;,$$

that is E_b is the bit energy.

Uncoded antipodal ($a_0 \in \{-A, +A\}$) signaling. The probability of bit error on an AWGN channel with SNR $E_b/\mathcal{N}_0$ and a matched filter receiver with ML decision rule is

$$P_b(\mathcal{E}) = \frac{1}{2}\text{erfc}\Big(\frac{2|A|}{2\sqrt{\mathcal{N}_0}}\Big) = \frac{1}{2}\text{erfc}\Big(\sqrt{\frac{E_b}{\mathcal{N}_0}}\Big)\;,$$

where $E_b = A^2$ is th energy per bit.

Coded antipodal ($a_0 \in \{-A, +A\}$) signaling. The probability of error between two binary codewords Hamming distance d apart on an AWGN channel with SNR $E_c/\mathcal{N}_0$ per code bit and a matched filter receiver with ML decision rule is

$$P_d(\mathcal{E}) = \frac{1}{2}\text{erfc}\Big(\frac{2\sqrt{d}|A|}{2\sqrt{\mathcal{N}_0}}\Big) = \frac{1}{2}\text{erfc}\Big(\sqrt{\frac{d\,E_c}{\mathcal{N}_0}}\Big)\;,$$

where $E_c = A^2$ is the energy per coded bit.

Probability of Block Error for Soft Decisions. Let $\mathcal{C}$ be a linear binary $(n, k, d_{\min})$ code with weight distribution $\{A_w\}$. Using a union bound, the probability of block error on an AWGN channel with SNR $E_c/\mathcal{N}_0$ per code bit, antipodal signaling, and a matched filter receiver with soft ML decision rule for codewords (i.e., based on unquantized Euclidean distance between codewords) is bounded as

$$P_B(\mathcal{E}) \leq \sum_{w=d_{\min}}^{n} A_w\,P_w(\mathcal{E})\;, \qquad \text{where} \qquad P_w(\mathcal{E}) = \frac{1}{2}\text{erfc}\Big(\sqrt{\frac{w\,E_c}{\mathcal{N}_0}}\Big)\;.$$

Probability of Bit Error for Soft Decisions. Let $\mathcal{C}$ be a linear binary $(n, k, d_{\min})$ code with extended weight distribution $\{A(w,i)\}$. Using a union bound, the probability of bit error on an AWGN channel with SNR $E_c/\mathcal{N}_0$ per code bit, antipodal signaling, and a matched filter receiver with soft ML decision rule for codewords (i.e., based on unquantized Euclidean distance between codewords) is bounded as

$$P_b(\mathcal{E}) \le \frac{1}{k} \sum_{w=d_{\min}}^{n} P_w(\mathcal{E}) \sum_{i=1}^{k} i\, A(w,i)\,, \qquad \text{where} \qquad P_w(\mathcal{E}) = \frac{1}{2}\text{erfc}\Big(\sqrt{\frac{w\,E_c}{\mathcal{N}_0}}\Big)\,.$$

Example: The probabilities of block and bit error for a binary $(7, 4, 3)$ Hamming code on an AWGN channel with a soft decision matched filter ML receiver are shown in the following graph versus the SNR $E_c/\mathcal{N}_0$ per code bit. The probability of bit error for uncoded antipodal signaling is also shown for comparison.