Consider our usual scalar wave equation. This is valid for TE modes in slabs or approximately valid for weakly guiding structures where the index change is small.

\[ \nabla^2 E + k_0^2 \left[ \varepsilon^0(x, y) + \varepsilon^1(x, y, z) \right] E = 0 \]

We have broken the dielectric distribution in space into an unperturbed part \( \varepsilon^0 \) and a perturbation, \( \varepsilon^1 \). If the perturbation is not present, we know the solutions:

\[ E(x, y, z) = \sum_m a_m E_m(x, y) e^{-j\beta_m z} + \int a_m E_m(x, y) e^{-j\beta_m z} \, dm \]

We’re typically concerned guided modes, so we will drop the integral over radiation modes from here forward. We will use these solutions to get an approximate solution to the “perturbed” wave equation:

\[ \nabla^2 E + k_0^2 \varepsilon^0 E = -k_0^2 \varepsilon^1 E \]

Since there is an “inhomogeneous” right-hand side of the equation, the solution above is wrong. However, an approximate solution can be obtained by assuming that the amplitudes \( a_m \) are no longer constant, but instead vary slowly in \( z \).

\[ E(x, y, z) = \sum_m a_m(z) E_m(x, y) e^{-j\beta_m z} \]
Start with the scalar, inhomogeneous wave equation
\[
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)E + k_0^2 \varepsilon^0 E = -k_0^2 \varepsilon^1 E
\]
and plug in the modal expansion for the first term on the left. Break the Laplacian up into \(xy\) and \(z\) parts:
\[
\sum_m \left\{ a_m(z)e^{-j\beta_m z} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_m(x, y) + E_m(x, y) \frac{d^2}{dz^2} a_m(z)e^{-j\beta_m z} \right\} + k_0^2 \varepsilon^0 E = -k_0^2 \varepsilon^1 E
\]
Take the \(z\) derivative via the chain rule and regroup:
\[
\sum_m \left\{ a_m(z)e^{-j\beta_m z} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E_m(x, y) + E_m(x, y) \left[ \frac{d^2}{dz^2} a_m(z) - 2 j\beta_m \frac{d}{dz} a_m(z) - \beta_m^2 \right] e^{-j\beta_m z} \right\} + k_0^2 \varepsilon^0 E = -k_0^2 \varepsilon^1 E
\]
\[
\sum_m \left\{ a_m(z) \nabla^2 \left[ E_m(x, y) e^{-j\beta_m z} \right] + E_m(x, y) \left[ \frac{d^2}{dz^2} a_m(z) - 2 j\beta_m \frac{d}{dz} a_m(z) \right] e^{-j\beta_m z} \right\} + k_0^2 \varepsilon^0 E = -k_0^2 \varepsilon^1 E
\]
Now plug in the modal expansion for the second \(E\) term on the left side:
\[
\sum_m \left\{ a_m(z) \nabla^2 \left[ E_m(x, y) e^{-j\beta_m z} \right] + k_0^2 \varepsilon^0 E_m(x, y) e^{-j\beta_m z} \right\} = 0
\]
\[
\sum_m \left\{ a_m(z) \left[ \frac{d^2}{dz^2} a_m(z) - 2 j\beta_m \frac{d}{dz} a_m(z) \right] e^{-j\beta_m z} \right\} = -k_0^2 \varepsilon^1 E
\]
The entire upper term is zero since the modes are solutions of the homogeneous wave equation, leaving
\[
\sum_m \left[ \frac{d^2}{dz^2} a_m(z) - 2 j\beta_m \frac{d}{dz} a_m(z) \right] E_m(x, y) e^{-j\beta_m z} = -k_0^2 \varepsilon^1 \sum_m a_m(z) E_m(x, y) e^{-j\beta_m z}
\]
which is still transcendental (\(E\) on both sides) so pretty hopeless.
We will now assume that the perturbation is sufficiently weak that the first term on the LHS is smaller than the second (the “slowly varying envelope assumption”):

\[- \sum_m 2j\beta_m \frac{d}{dz} a_m(z) E_m(x, y)e^{-j\beta_m z} = -k_0^2 \epsilon^1 \sum_m a_m(z) E_m(x, y)e^{-j\beta_m z}\]

Orthogonality of modes now comes to our rescue. Multiply both sides by the profile of a test mode \( n \) and integrate over \( x \) and \( y \). Orthogonality collapses the sum on the left, but NOT the sum on the right since \( \epsilon^1 \) varies in \( x \) and \( y \).

\[-2j\beta_n \langle E_n | E_n \rangle \frac{d}{dz} a_n(z) e^{-j\beta_n z} = -k_0^2 \sum_m a_m(z) \langle E_n | \epsilon^1 | E_m \rangle e^{-j\beta_m z}\]

\[\frac{d}{dz} a_n(z) e^{-j\beta_n z} = \frac{-jk_0^2}{2\beta_n \langle E_n | E_n \rangle} \sum_m a_m(z) e^{-j\beta_m z}\]

\[\frac{d}{dz} a_n(z) e^{-j\beta_n z} = -j \sum_m \kappa_{n,m} a_m(z) e^{-j\beta_m z}\]

where

\[\langle E_n | E_n \rangle = \int |E_n(x, y)|^2 dx \, dy\]

\[\langle E_n | \epsilon^1 | E_m \rangle = \int E_n(x, y) \epsilon^1(x, y, z) E_m^*(x, y) dx \, dy\]

\[\kappa_{n,m} = \frac{k_0^2 \langle E_n | \epsilon^1 | E_m \rangle}{2\beta_n \langle E_n | E_n \rangle} \begin{bmatrix} 1 \\ m \end{bmatrix}\]

\( \kappa_{n,m} \) is the coupling coefficient expressing how mode \( m \) couples to mode \( n \). It is real for a real dielectric perturbation. Note that it can depend on \( z \).
Coupled modes
Alternate form of equation

If the perturbation $\varepsilon^l$ is independent of $z$ then $\kappa_{n,m}$ is a constant. It would be nice to have only ordinary, constant coefficient differential equations in this case. Thus, the coupled mode equations are usually written in terms of the field amplitude with the plane-wave carrier included:

$$u_m(z) \equiv a_m(z)e^{-j\beta_m z}$$

which puts the coupled-mode equations in their FIELD form:

$$\frac{d u_n}{d z} = -j\beta_n u_n(z) - j\sum_m \kappa_{n,m}(z) u_m(z)$$

Interpretation of this equation.

$$\frac{d a_n}{d z} = -j\sum_m a_m(z)\kappa_{n,m}(z)e^{j(\beta_n - \beta_m)z}$$

ENVELOPE FORM

Each modal envelope is driven by all the others through a complex coupling coefficient $\kappa_{n,m}(z)e^{j(\beta_n - \beta_m)z}$

If this coefficient is real and constant in $z$, the envelope $a_n$ will grow linearly with distance, assuming the envelope $a_m$ is undepleted.

To be real and constant, the perturbation must be Bragg matched (aka “conserve momentum”, “phase-matched”)

$$\kappa_{n,m}(z) \propto e^{-j(\beta_n - \beta_m)z}$$

or $\vec{k}_{\text{Grating}} \equiv \vec{k}_{\text{Diff}} - \vec{k}_{\text{Inc}}$ “Bragg matching”

Robert R. McLeod, University of Colorado
Relationship of coupled modes to perturbation method

Starting with the ENVELOPE form of the coupled mode equation, let’s separate out the term \( m = n \)

\[
\frac{d}{dz} a_n(z) e^{-j\beta_n z} = -j\kappa_{n,n} a_n(z) e^{-j\beta_n z} - j \sum_{m \neq n} \kappa_{n,m} a_m(z) e^{-j\beta_m z}
\]

and examine the impact of the perturbation \( \epsilon^1 \) just on mode \( n \) without the coupling of other modes

\[
\frac{d}{dz} a_n(z) e^{-j\beta_n z} = -j\kappa_{n,n} a_n(z) e^{-j\beta_n z}
\]

\( \kappa_{n,n} \equiv \frac{k_0 \langle E_n | \epsilon^1 | E_n \rangle}{2N_n \langle E_n | E_n \rangle} \left[ \frac{1}{m} \right] \)

This last looks strikingly like the perturbation correction to the effective index. Substituting from that expression:

\[
\kappa_{n,n} = \frac{k_0 \langle E_n | (\epsilon^{Total} - \epsilon^0) | E_n \rangle}{2N_n \langle E_n | E_n \rangle} = \frac{k_0 \langle E_n | (n_{Ex}^2 - n^2) | E_n \rangle}{2N_n \langle E_n | E_n \rangle} = \frac{k_0 (N_{Ex}^2 - N_n^2)}{2N_n}
\]

\[
= k_0 \left[ (N_n + \Delta N)^2 - N_n^2 \right] = k_0 \left[ N_n^2 + 2N_n \Delta N + \Delta N^2 - N_n^2 \right]
\]

\[
\approx k_0 \Delta N
\]

The simplified differential equation thus looks like:

\[
\frac{d}{dz} a_n(z) = -j(k_0 \Delta N) a_n(z) \quad \rightarrow \quad a_n(z) = a(0) e^{-j k_0 \Delta N}
\]

Yielding exactly the perturbation solution

\[
E(x, y, z) = a_n(0) E_n(x, y) e^{-j k_0 (N_n + \Delta N) z} = a_n(0) E_n(x, y) e^{-j k_0 N_{Ex} z}
\]

Thus the coupled-mode method contains the perturbation solution.
Coupling between non-orthogonal modes

Coupled mode theory is often applied to the modes of separate guides that are not orthogonal. How can the theory apply then? Starting again part way through the derivation:

$$- \sum_{m} 2 j \beta_m \frac{d}{dz} a_m(z) E_m(x, y) e^{-j \beta_m z} = -k_0^2 \varepsilon \sum_{m} a_m(z) E_m(x, y) e^{-j \beta_m z}$$

but now the modes $m$ can be of separate guides and thus not all orthogonal. Divide the set $m$ into two sets $m_1$ (modes of the guide of interest) and $m_2$ (modes of all other guides) Multiply by the mode shape of a mode $n_1$ chosen from the first set and integrate transversely:

$$- 2 j \beta_{n_1} \langle E_{n_1} | E_{n_1} \rangle \frac{d}{dz} a_{n_1}(z) e^{-j \beta_{n_1} z} - \sum_{m_2} 2 j \beta_{m_2} \langle E_{n_1} | E_{m_2} \rangle \frac{d}{dz} a_{m_2}(z) e^{-j \beta_{m_2} z}$$

$$= -k_0^2 \sum_{m} a_m(z) \langle E_{n_1} | \varepsilon^1 | E_m \rangle e^{-j \beta_m z}$$

Assuming the coupling is weak (which we already assumed),

$$\langle E_{n_1} | E_{n_1} \rangle \gg \langle E_{n_1} | E_{m_2} \rangle$$

Resulting in the same coupled modes equation:

$$- 2 j \beta_{n_1} \langle E_{n_1} | E_{n_1} \rangle \frac{d}{dz} a_{n_1}(z) e^{-j \beta_{n_1} z} = -k_0^2 \sum_{m} a_m(z) \langle E_n | \varepsilon^1 | E_m \rangle e^{-j \beta_m z}$$
Two identical coupled guides with uniform coupling

The equations for the field amplitudes $u_m(z) \equiv a_m(z)e^{-j\beta_m z}$ are

$$\frac{d}{dz} u_0 = -j\beta_0 u_0 - j\kappa u_1$$
$$\frac{d}{dz} u_1 = -j\beta_1 u_0 - j\kappa u_0$$

which have simple solutions:

$$I_0(z) \equiv |u_0(z)|^2 = \frac{\cos^2(gz) + \gamma^2}{1 + \gamma^2}$$
$$I_1(z) \equiv |u_1(z)|^2 = \frac{\sin^2(gz)}{1 + \gamma^2}$$

where

$$g^2 \equiv \kappa^2 + \left(\frac{\beta_0 - \beta_1}{2}\right)^2$$
$$\gamma \equiv \frac{(\beta_0 - \beta_1)/2}{\kappa}$$
Specific case: 2 || guides

Problem

Two slab waveguides of 8 \( \mu m \) width, \( \delta n = .01, n_{clad} = 1.5, \lambda_0 = 1 \mu m \)

\[
\kappa(z) \equiv \frac{k_0}{2N_{eff}} \left[ \frac{\langle E_0(x-5\mu m)\varepsilon'(x)E_0(x+5\mu m) \rangle}{\langle E_0 \mid E_0 \rangle} \right]
\]

\[
\approx = .0002329 \ \mu m^{-1}
\]

\[
Z_c = \frac{\pi}{2\kappa} = 6.745 \ mm
\]

Slight inaccuracy of coupled modes:

modes isolated guides are slightly narrower than those of the 2-guide system, which will overestimate the coupling length.

Assumption of coupled modes is that coupling is weak enough that modes nearly unchanged.
Coupled mode

vs. BPM

Create two appropriate slab guides in 1+1 D BPM. Let’s assume we don’t have a general mode solver around, so we’ll use the analytic result of the variational technique to pick the best Gaussian to launch:

\[ r_0 = \frac{\rho}{\sqrt{V - 1}} = \frac{4}{\sqrt{4.36 - 1}} = 2.18 \mu m \]

Expect \( V/(\pi/2)+1 = 3 \) modes.

“Fuzz” is higher-order excitation.

It’s odd, so excited some \( m=1 \).

Higher order modes are more tightly confined and thus coupling distance is predicted to be longer.

\[ e^{-\left(\frac{x-5 \mu m}{2.18 \mu m}\right)^2} \]


**Coupled mode vs. Eigenmode expansion**

Use single slab waveguide solver to find properties of both guides

\[ E_S(x)e^{-jk_0N_S z} + E_{AS}(x)e^{-jk_0N_{AS} z} \]

\[ k_0N_A Z_c - k_0N_{AS} Z_c = \pi \]

\[ Z_c = \frac{\lambda_0}{2\Delta N} = 6.651 \text{ mm} \]

This solution is exact. Note however it depends on accuracy of \( N \) quite critically.
N+1 nearest neighbor-coupled modes

Coupled mode equations:
\[ \frac{\partial}{\partial z} u_0 = -j \beta_0 u_0 - j \kappa u_1 \]
\[ \frac{\partial}{\partial z} u_n = -j \beta_n u_n - j \kappa (u_{n-1} + u_{n+1}) \]
\[ \frac{\partial}{\partial z} u_N = -j \beta_N u_N - j \kappa u_{N-1} \]

Closed-form solution via Mathematica which is sum of sines and cosines of various periods.

• If this describes Gaussian modes in Gaussian gradient-index guides, we can use the variational solutions to give analytic forms for \( \beta \) and \( \kappa \).
• Similarly, if these coupled modes were plane waves in a volume hologram, we could write \( \beta \) and \( \kappa \) analytically.
Effect of phase mismatch

Perfect phase matching

Imperfect phase matching
Reflection due to perturbation
Set up of problem

Consider a guide with an arbitrary perturbation in the region 
\(-L/2 < z < L/2\)

\[ \varepsilon^0(x) \quad \varepsilon^0(x) + \varepsilon^1(x, z) \quad \varepsilon^0(x) \]

The coupled mode equations for the forward (+) and backwards (-) wave are

\[ \frac{d a_{0+}(z)}{dz} e^{-j\beta_0 z} = -j \kappa_{0,0}(z) a_{0+}(z) e^{-j\beta_0 z} - j \kappa_{0,0}(z) a_{0-}(z) e^{+j\beta_0 z} \]

\[ \frac{d a_{0-}(z)}{dz} e^{+j\beta_0 z} = +j \kappa_{0,0}(z) a_{0-}(z) e^{+j\beta_0 z} + j \kappa_{0,0}(z) a_{0+}(z) e^{-j\beta_0 z} \]

where the mode profile \( E_0(x) \) and the propagation constant \( \beta_0 \) are found for the unperturbed waveguide described by \( \varepsilon^0(x) \). The coupling is given by

\[ \kappa_{0,0}(z) \equiv \frac{k_0^2 \langle E_0 | \varepsilon^1(x, z) | E_0 \rangle}{2\beta_0 \langle E_0 | E_0 \rangle} \]

which is zero outside of \( |z| > L/2 \).

This completely describes the problem. The simplicity of C.M. is one of its main attractions. Next we’ll solve the D.E.s in various cases.
Reflection due to perturbation
Uniform region

If the perturbation doesn’t depend on \( z \), neither does \( \kappa \) in \(|z|<L/2\).

\[
\frac{da_{0+}(z)}{dz} e^{-j\beta_0z} = -j \kappa_{0,0} a_{0+}(z)e^{-j\beta_0z} - j \kappa_{0,0} a_{0-}(z)e^{+j\beta_0z}
\]
\[
\frac{da_{0-}(z)}{dz} e^{+j\beta_0z} = +j \kappa_{0,0} a_{0-}(z)e^{+j\beta_0z} + j \kappa_{0,0} a_{0+}(z)e^{-j\beta_0z}
\]

The equations can be simplified by a change of variables to take into account the expected correction to the propagation constant driven by the first term:

\[
a_{0+}(z) = b_{0+}(z)e^{-j\kappa_{0,0}z} \quad a_{0-}(z) = b_{0-}(z)e^{+j\kappa_{0,0}z}
\]
\[
\frac{db_{0+}(z)}{dz} e^{-j(\beta_0+\kappa_{0,0})z} = -j \kappa_{0,0} b_{0-}(z)e^{+j(\beta_0+\kappa_{0,0})z}
\]
\[
\frac{db_{0-}(z)}{dz} e^{+j(\beta_0+\kappa_{0,0})z} = +j \kappa_{0,0} b_{0+}(z)e^{-j(\beta_0+\kappa_{0,0})z}
\]

Take the \( z \) derivative of the first and substitute in the second:

\[
\frac{d^2b_{0+}(z)}{dz^2} e^{-j(\beta_0+\kappa_{0,0})z} = -j \kappa_{0,0} \left[ \frac{db_{0-}(z)}{dz} + j(\beta_0 + \kappa_{0,0}) b_{0-}(z) \right] e^{+j(\beta_0+\kappa_{0,0})z}
\]
\[
\frac{d^2b_{0+}(z)}{dz^2} - j2(\beta_0 + \kappa_{0,0}) \frac{db_{0+}(z)}{dz} - \kappa_{0,0}^2 b_{0+}(z) = 0
\]

Note the sign. We’ll come back to this.

This is found to oscillate rapidly with small amplitude. Why? The perturbation does not Bragg match the + and − waves.
Reflection due to perturbation
Uniform region – correct solution

The right way to solve this is as a Fresnel reflection problem. The impedance of at TE mode is:

\[ H_x = -\frac{N_0}{\eta_0} E_y \]
\[ H_x = -\frac{N_0 + \kappa_{0,0}/k_0}{\eta_0} E_y = -\frac{N_1}{\eta_0} E_y \]

Applying the EM boundary conditions (continuity of \( E \) and \( H \)):

\[ E_{inc} + E_{refl} = E_{trans} \quad H_{inc} + H_{refl} = H_{trans} \]

Generates the Fresnel reflection and transmission coefficients:

\[ r \equiv \frac{E_{refl}}{E_{inc}} = \frac{N_0 - N_1}{N_0 + N_1} \approx -\frac{\kappa_{0,0}}{2k_0 N_0} \quad R = |r|^2 \]
\[ t \equiv \frac{E_{trans}}{E_{inc}} = \frac{2N_0}{N_0 + N_1} \approx 1 \]

A uniform perturbation of length \( L \) thus forms a Fabry-Perot etalon:

\[ T = \frac{I_{trans}}{I_{inc}} = \frac{1}{1 + F \sin^2\left(\pi \frac{\nu}{\nu_0}\right)} \]
\[ F = \frac{4R}{(1 - R)^2} \quad \text{Finesse} \]
\[ \nu_0 = \frac{c}{S} = \frac{c}{2N_1L} \quad \text{Free spectral range} \]
Reflection due to perturbation
Bragg-matched hologram

Consider a guide with a reflection grating in the region $z = [0, +L]$

Assume the perturbation is a sin wave with period half that of the carrier:

$$\kappa_{0,0}(z) = \frac{k_0^2 \langle E_0 | \mathbf{e}^\dagger(x, z) | E_0 \rangle}{2 \beta_0 \langle E_0 | E_0 \rangle} = \frac{k_0^2 \langle E_0 | \mathbf{e}^\dagger(x) 2 \cos(2\beta_0 z) | E_0 \rangle}{2 \beta_0 \langle E_0 | E_0 \rangle} = \kappa_{0,0}(e^{+j2\beta_0 z} + e^{-j2\beta_0 z})$$

Substituting into the CM equations and retaining only slowly-varying terms

$$\frac{da_{0+}(z)}{dz} = -j \kappa_{0,0} a_{0-}(z)$$
$$\frac{da_{0-}(z)}{dz} = +j \kappa_{0,0} a_{0+}(z)$$

Yields the simple solution:

$$\frac{d^2 a_{0+}(z)}{dz^2} - \kappa_{0,0}^2 a_{0+}(z) = 0$$
$$a_{0+}(z) = C_+ e^{\kappa_{0,0}z} + C_- e^{-\kappa_{0,0}z}$$
$$a_{0-}(z) = j C_+ e^{\kappa_{0,0}z} - j C_- e^{-\kappa_{0,0}z}$$

BCs: $a_{0+}(0) = A$ and $a_{0-}(L)=0$:

$$a_{0+}(z) = A \frac{\cosh[\kappa_{0,0}(z-L)]}{\cosh[\kappa_{0,0}L]}$$
$$a_{0-}(z) = A j \frac{\sinh[\kappa_{0,0}(z-L)]}{\cosh[\kappa_{0,0}L]}$$

$$a_{0+}^2(z) - a_{0-}^2(z) = \text{constant}$$

Forward-Reverse power conserved

$L = \frac{1}{\kappa}$
Reflection due to perturbation
Bragg-mismatched hologram

Now let the hologram not be perfectly Bragg matched:

$$\kappa_{0,0}(z) = \frac{k_0^2 \langle E_0 | e^{i(x)} 2 \cos(k_G z) | E_0 \rangle}{2 \beta_0 \langle E_0 | E_0 \rangle} = \kappa_{0,0}(e^{j k_G z} + e^{-j k_G z})$$

Substitute this into the CM equations and keep the slowly-varying terms:

$$\frac{da_{0+}(z)}{dz} = -j \kappa_{0,0} a_{0-}(z)e^{j(2 \beta_0 - k_G)z} \quad \frac{da_{0-}(z)}{dz} = +j \kappa_{0,0} a_{0+}(z)e^{-j(2 \beta_0 - k_G)z}$$

Combine these two 1st order DEs into a single second order:

$$\frac{d^2 a_{0+}(z)}{dz^2} - j(2 \beta_0 - k_G) \frac{d a_{0+}(z)}{dz} - \kappa_{0,0}^2 a_{0+}(z) = 0$$

Note similarity to uniform perturbation.

The solution versus $z$ is straight-forward but rather complicated, so lets just extract the power reflectivity of the grating, $R = |a_{0+}(0)|^2$:

$$R \equiv \frac{|a_{0-}(0)|^2}{|a_{0+}(0)|^2} = \left\{ \left( \frac{\Delta k}{2 \kappa_{0,0}} \right)^2 + \left[ 1 - \left( \frac{\Delta k}{2 \kappa_{0,0}} \right)^2 \right] \coth \left[ L \kappa_{0,0} \sqrt{1 - \left( \frac{\Delta k}{2 \kappa_{0,0}} \right)^2} \right] \right\}^{-1}$$

Bragg mismatch:

$$\Delta k \equiv k_G - 2 \beta$$

If $\kappa$ is weak, then reflectivity is large only for small Bragg mismatch $\Delta k$ and is very small otherwise.

AKA “fiber Bragg grating” a narrow band reflection filter.

Robert R. McLeod, University of Colorado
Mode conversion due to perturbation

Set up of problem

Consider a multimode guide with a reflection grating in the region \( z = [0, +L] \)

The coupled mode equations for two forward waves \( m \) and \( n \) are:

\[
\frac{da_m(z)}{dz} e^{-ji\beta_mz} = -j \kappa_{m,m}(z)a_m(z)e^{-ji\beta_mz} - j \kappa_{m,n}(z)a_n(z)e^{-ji\beta_nz}
\]

\[
\frac{da_n(z)}{dz} e^{-ji\beta_nz} = -j \kappa_{n,n}(z)a_n(z)e^{-ji\beta_nz} - j \kappa_{n,m}(z)a_m(z)e^{-ji\beta_mz}
\]

where the mode profiles \( E_m(x), E_n(x) \) and the propagation constants \( \beta_m, \beta_n \) are found for the unperturbed waveguide described by \( \mathcal{E}^0(x) \).

The coupling is given by

\[
\kappa_{m,n}(z) \equiv \frac{k_0^2}{2\beta_m} \frac{\langle E_m | \mathcal{E}^1(x,z) | E_n \rangle}{\langle E_m | E_m \rangle} \approx \kappa_{n,m}^* = 0 \text{ if } |z|>L/2.
\]

The perturbation correction to \( \beta \) is given by the \( \kappa_{n,n} \) and \( \kappa_{m,m} \) terms

\[
\kappa_{m,m}(z) \equiv \frac{k_0^2}{2\beta_m} \frac{\langle E_m | \mathcal{E}^1(x,z) | E_m \rangle}{\langle E_m | E_m \rangle} = 0 \text{ if } |z|>L/2.
\]
Mode conversion due to perturbation

Solution

Assume the perturbation is a sine wave of arbitrary period

\[ e^1(x, z) = e^1(x) 2 \cos(k_G z) \]

\[ \kappa_{m,n}(z) \equiv \frac{k_0^2 \langle E_m \rangle \langle e^1(x) 2 \cos(k_G z) \rangle |E_n|}{2 \beta_m \langle E_m | E_m \rangle} = \kappa_{m,n}(e^{+jk_G z} + e^{-jk_G z}) \]

Substitute into the basic CM equation

\[ \frac{d a_m(z)}{dz} = -j \kappa_{m,m} (e^{+jk_G z} + e^{-jk_G z}) a_m(z) - j \kappa_{m,n} (e^{+jk_G z} + e^{-jk_G z}) a_n(z)e^{-j(\beta_n-\beta_m)z} \]

If \( k_G \) is near zero, this term shifts the propagation constant of mode \( m \).
If \( k_G \) is not near zero, the term oscillates rapidly, giving little contribution.
Similarly, if \( \Delta k = k_G - (\beta_n - \beta_m) \) is near zero, this term is efficient at coupling mode \( n \) to mode \( m \), otherwise it oscillates rapidly. We can thus drop all terms except those that vary slowly to get:

\[ \frac{d a_m(z)}{dz} = -j \kappa_{m,n} a_n(z)e^{+j\Delta k z} \]

\[ \frac{d a_n(z)}{dz} = -j \kappa_{m,n}^* a_m(z)e^{-j\Delta k z} \]

Take the derivative of one equation and substitute to get a 2nd-order DE:

\[ \frac{d^2 a_m(z)}{dz^2} - j \Delta k \frac{d a_m(z)}{dz} + |\kappa_{m,n}|^2 a_m(z) = 0 \]

Apply BCs: \( a_m(0) = A \) and \( a_n(0) = 0 \). For \( \Delta k = 0 \), \( \kappa = |\kappa_{m,n}| \)

\[ a_m(z) = A e^{+jz \Delta k/2} \left[ \cos(g z) - j \frac{\Delta k}{2g} \sin(g z) \right] \]

\[ \rightarrow_{\Delta k=0} A \cos(\kappa z) \]

\[ a_n(z) = -j \frac{g}{\kappa} A e^{-jz \Delta k/2} \sin(g z) \left[ 1 - \left( \frac{\Delta k}{2g} \right)^2 \right] \]

\[ \rightarrow_{\Delta k=0} -j A \sin(\kappa z) \]
Guides with weak loss or gain via CM

Recall the definition of the complex dielectric constant:

$$\varepsilon_{\text{complex}} = \varepsilon - \frac{j}{\omega \varepsilon_0} \sigma = \varepsilon - j\frac{\mu_0 \varepsilon_0}{2\pi \varepsilon_0} \lambda_0 \sigma = \varepsilon - j\frac{n_0}{k_0} \sigma$$

If the perturbation to the waveguide is complex, the coupling constant is

$$\kappa_{n,m} \equiv \frac{k_0^2 \langle E_n | \varepsilon^* | E_m \rangle}{2 \beta_n \langle E_n | E_n \rangle} = \frac{k_0}{2N} \frac{\langle E_n | \text{Re}(\varepsilon^*) | E_m \rangle}{\langle E_n | E_n \rangle} - j \frac{n_0}{2N} \frac{\langle E_n | \sigma^* | E_m \rangle}{\langle E_n | E_n \rangle}$$

Assuming only one mode is present and that the perturbation is uniform in \( z \), the coupled mode equation is:

$$\frac{d u_n}{dz} = -j \beta_n u_n(z) - j \kappa_{n,n} u_n(z)$$

CM eq. in FIELD form

$$= -j[\beta_n + \text{Re}(\kappa_{n,n})] u_n(z) - \left[ \frac{n_0}{2N} \frac{\langle E_n | \sigma^* | E_n \rangle}{\langle E_n | E_n \rangle} \right] u_n(z)$$

$$u_n(z) = u_n(0) \exp \left\{ - \left[ \frac{n_0}{2N} \frac{\langle E_n | \sigma^* | E_n \rangle}{\langle E_n | E_n \rangle} \right] z - j[\beta_n + \text{Re}(\kappa_{n,n})] z \right\}$$

The second term is our typical perturbation correction to the propagation constant. The first term gives loss (\( \sigma > 0 \)) or gain (\( \sigma < 0 \)).

Loss: absorption due to allowed electronic transitions.

Gain: stimulated emission due to excited electronic transitions.

E.g.: Laser, semi-conductor amplifier, doped fiber amplifier/laser
Nonlinear coupled modes

Inclusion of a nonlinearily-generated polarization, $P_{nl}$ in the coupled-mode derivation adds a new term of fairly obvious form. Writing the CM envelope equations with normalized modes, for simplicity, this is:

$$
\frac{d}{dz} \hat{a}_n(z) e^{-j \beta_n z} = -j \sum_m \kappa_{n,m} \hat{a}_m(z) e^{-j \beta_m z} - j \frac{k_0^2}{2 \beta_n \varepsilon_0} \oint \hat{E}_n^* \cdot \bar{P}_{nl} \, dx \, dy
$$

$$
\kappa_{n,m} \equiv \frac{k_0^2}{2 \beta_n} \left\langle \hat{E}_n | \hat{E}_m^* \right\rangle \left[ \frac{1}{m} \right]
$$

A third-order nonlinear polarization (for example) is written

$$
\bar{P}(\omega_q) = \varepsilon_0 \chi^3(\omega_q = \omega_m + \omega_n + \omega_p) \hat{E}(\omega_m) \hat{E}(\omega_n) \hat{E}(\omega_p)
$$

Let’s look specifically at stimulated Raman scattering. From the table of nonlinear susceptibilities:

$$
\hat{e}_S^* \cdot \bar{P}_S^{(3)} = 6 \varepsilon_0 \hat{e}_S^* \cdot \chi^{(3)}(\omega_S = \omega_S + \omega_P - \omega_P) \hat{e}_S \hat{e}_P \hat{e}_S^* E_S | E_P |^2 e^{-j k_S z}
$$

$$
\equiv 6 \varepsilon_0 \chi_R E_S | E_P |^2 e^{-j k_S z}
$$

$$
\hat{e}_P^* \cdot \bar{P}_P^{(3)} = 6 \varepsilon_0 \hat{e}_P^* \cdot \chi^{(3)}(\omega_P = \omega_P + \omega_S - \omega_S) \hat{e}_P \hat{e}_S \hat{e}_S^* E_P | E_S |^2 e^{-j k_P z}
$$

$$
\equiv 6 \varepsilon_0 \chi_R E_P | E_S |^2 e^{-j k_P z}
$$

The factor of 6 arises from the isotropic material symmetry and accounts for all of the identical terms in the $\chi$ tensor.
Stimulated vs Spontaneous Raman Scattering

Spontaneous Raman scattering: A short wavelength, higher energy pump photon inelastically scatters off of and creates an “optical phonon”. The photon losses an amount of energy equal to the phonon energy such that the scattered Stokes photon emerges with lower energy and frequency and thus consequently a longer wavelength.

\[
\hbar \omega_p \rightarrow \hbar \omega_S = \hbar (\omega_p - \Omega)
\]

Pump photon destroyed

Stokes photon created

Phonon created

Stimulated Raman scattering: A short wavelength, higher energy pump photon inelastically scatters off of and creates an “optical phonon”. A coincident photon at the Stokes wavelength stimulates the emission of a second, scattered Stokes photon that emerges in phase with the incident Stokes photon.

Incident Stokes photon

\[
\hbar \omega_S \rightarrow \hbar \omega_p \rightarrow \hbar \omega_S
\]

Pump photon destroyed

Stokes photon created

Incident Stokes photon

\[
\hbar \omega_S \rightarrow \hbar \omega_S
\]

Phonon created
Stimulated Raman gain

Setup

Plugging the nonlinear polarization into the nonlinear CM equations and assuming there is no linear coupling between the pump (P) and Stokes (S) waves yields:

\[
\frac{d}{dz} \hat{a}_S = -j \frac{3k_0^2}{\beta_S} \chi_R \hat{a}_S \hat{a}_P^2 \iint \left| \hat{E}_S \right|^2 \left| \hat{E}_P \right|^2 dxdy
\]

\[
\frac{d}{dz} \hat{a}_P = -j \frac{3k_0^2}{\beta_P} \chi_R \hat{a}_S \hat{a}_P^2 \iint \left| \hat{E}_S \right|^2 \left| \hat{E}_P \right|^2 dxdy
\]

It is more common to write these equations in terms of the peak intensity in each mode. Transforming variables and assuming there is a small absorption loss for both waves yields our final equations

\[
\frac{d}{dz} I_S = -\alpha I_S + g_R I_S I_P
\]

\[
\frac{d}{dz} I_P = -\alpha I_P - \frac{\omega_p}{\omega_S} g_R I_S I_P
\]

where \( g \) is the effective Raman gain factor in units of meters per watt. It is >0 when \( \chi_R < 0 \).

\[
g_R = -\frac{3 \omega_s \mu_0}{N_S N_P} \chi_R \iint \left| \hat{E}_S \right|^2 \left| \hat{E}_P \right|^2 dxdy
\]

This term is usually not included in the literature.
Stimulated Raman gain
Weak gain solution

First note that, as advertised, this non-parametric interaction is guaranteed to be phase matched.

Second, not that in the absence of loss

\[
\frac{d}{dz} I_S(\omega_S) = - \frac{d}{dz} I_P(\omega_P)
\]

formally known as a Manly-Rowe relation. AKA conservation of energy.

If the gain is small so that the depletion of the pump can be neglected,

\[
I_S(l) = I_S(0) e^{g_R I_P(0) l_{eff} - \alpha d}
\]

\[
l_{eff} = \frac{1 - e^{-\alpha d}}{\alpha}
\]

The total gain is thus

\[
\frac{I_S(l)}{I_S(0)e^{-\alpha d}} = e^{g_R I_P(0) l_{eff}}
\]

So the exponential gain is proportional to pump power.
Stimulated Raman gain
Weak gain numerical example

Raman amplification is commonly used at telecom wavelengths, $\lambda_s = 1.55$ microns. Consider an amplifier designed to restore a weak signal at -15 dBm to one milliwatt or 0 dBm in 25 km of fiber with an absorption coefficient of .2 dB/km. Let the fiber have an effective area of 50 $\mu$m$^2$.

Taking the Raman frequency shift to be 13.2 THz, the pump wavelength should be

$$\lambda_p = c \left/ \left( \frac{c}{1.55 \times 10^{-6}} + 13.2 \times 10^{12} \right) \right. = 1.4510 \, [\mu m]$$

The Raman gain factor at 1.55 microns is

$$g_R = 6.45 \times 10^{-14} \, [m/W]$$

The effective length of the fiber is

$$\alpha = 0.2 \, [dB/km] = 0.046 \, [l/km]$$

$$l_{eff} = \frac{1 - e^{-0.046 \times 25}}{0.046} = 14.86 \, [km]$$

The required gain is

$$G_R = P_S^{out} \, [dB] - P_S^{in} \, [dB] + \alpha \, [dB/km] \times l \, [km] = 20 \, [dB]$$

From which we can find the required pump power:

$$P_P = \frac{A_{eff} \ln(G_R)}{l_{eff} g_R} = \frac{5 \times 10^{-11} \ln(100)}{(14.86 \times 10^3)(6.45 \times 10^{-14})} = 240 \, [mW]$$
Stimulated Raman gain
Strong gain limit on signal

Consider a strong, narrow-band intensity introduced into the fiber. Spontaneous Raman scattering will generate a Stokes shifted wave. That spontaneously-generated wave will then be amplified by stimulated Raman scattering. Call the initial, strong wave the pump (even though it may be at the com band of 1.55) and the generated wave the signal (which will be at ~1.65 microns) to keep the same notation as before. In this case, a simplified solution to the strongly coupled equations reveals that the 1.55 micron “pump” will decrease and the 1.65 micron “signal” will decrease proportional to the same gain factor found before:

\[ e^{g_R I_P(0)l_{eff}} \]

Solving the complete solution for the case where the signal and pump have become equal in intensity at the end of the fiber leads to a threshold gain that depends on the fiber geometry and medium characteristics. For forward Raman Stokes generation,

\[ G_R^{th} = e^{g_R I_P(0)l_{eff}} = e^{16} \]

or

\[ P_P^{th} = A_{eff} I_P(0) = 16 \frac{A_{eff}}{g_R l_{eff}} \]

Continuing the previous example,

\[ P_P^{th} = 16 \frac{5 \times 10^{-11}}{(14.86 \times 10^3)(6.45 \times 10^{-14})} = 835 \text{[mW]} \]

If you approach this power, a significant fraction of your 1.55 micron wave is converted to the Stokes-shifted wave at 1.65 microns.