Support weight enumerators and coset weight distributions of isodual codes *

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Abstract

In this paper various methods for computing the support weight enumerators of binary, linear, even, isodual codes are described. It is shown that there exist relationships between support weight enumerators and coset weight distributions of a code that can be used to compute partial information about one set of these code invariants from the other. The support weight enumerators and complete coset weight distributions of several even, isodual codes of length up to 22 are computed as well. It is observed that there exist inequivalent codes with the same support weight enumerators, inequivalent codes with the same complete coset weight distribution and inequivalent codes with the same support weight enumerators and complete coset weight distribution.

1 Introduction

In [30], Wei introduced the notion of a weight hierarchy of a code, motivated by applications in keyless cryptography. Wei defined the r-th generalized Hamming weight of a code as the minimum support weight of any of its r-dimensional subcodes. He also showed that the performance of a linear code on the type II wire-tap channel can be completely characterized by its weight hierarchy. On the other hand, Kasami *et al.* [15] showed that generalized Hamming weights are also of practical interest for decoding purposes, since there exists a strong relationship between the weights and the trellis complexity of a code. Generalized Hamming weights were investigated further by a number of authors, [9], [14], [28], leading to the derivation of upper and lower bounds on their values [28]. For some classes of codes exact values of some generalized Hamming weights were presented in [20], [24], [27], [29].

Besides their applications in coding theory, generalized Hamming weights are also of interest from a purely combinatorial point of view. They provide information about the structure of a code and represent parameters that may be used for classifying them. It is therefore natural to consider an extension of the notion of generalized weights – the support weight enumerators. The support weight enumerators were first introduced in [11], and they contain information about the number of subcodes of a code with specified dimension and support weight [26]. For certain classes of codes these enumerators are helpful for computing the number of uniformly efficient permutations [3].

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Support weight enumerators can also be used for gaining more information about the structure of codes, deciding about the existence of codes with a given set of parameters as well as for computing the complete coset weight distribution of codes (the last problem is addressed in this paper). Finding the support weight enumerators, even for very short codes, is a difficult task. Up to date, there exist only a few results concerning these weight enumerators [3], [5], [16].

The work presented in this paper is concerned with investigating the support weight enumerators of even isodual codes. Isodual codes have many practical applications and their mathematical structure provides useful information for computing their support weight enumerators. Our contributions to the problem of determining support weight enumerators of isodual codes are two-fold. First, we prove that there exist functional relationships between support weight enumerators and coset weight distributions of isodual codes. These relationships can be used to compute the coset weight distributions of isodual codes based on some standard coding arguments and knowledge of the support weight enumerators. In a companion paper [19], we describe how to use the coset weight distributions to compute the third support weight enumerator of some self–dual codes. Our second contribution is in classifying most of the extremal, isodual codes of length up to 22 according to their support weight enumerators. This classification reveals that inequivalent codes can share the same support weight enumerators as well as the same complete coset weight distribution.

The paper is organized as follows: Section II states the definitions used throughout the paper. Section III contains generalizations of invariant theory for linear combinations of support weight enumerators. The results in invariant theory allow us to determine how many unknowns there are in every complete set of support weight enumerators. Section IV presents a new result that connects a function of the complete set of coset weight distributions with the first, second and a generalization of the third support weight enumerator. The main results are given in Section V, where the support weight enumerators and complete coset weight distributions of a number of even, isodual codes are computed. Section VI contains the concluding remarks.

2 Definitions and Background

Throughout this paper we use standard notation and terminology of algebraic coding theory [18]. Additionally, all definitions related to generalized weights are taken from [9].

Let C be a binary, linear code of length n and dimension k. The words in C are called codewords, and the Hamming weight (henceforth, weight) of a codeword is the number of its non–zero coordinates. The minimum weight d of C is the smallest weight of any of its non–zero codewords. We reserve the special notation $\mathbf{1}$ for a codeword of weight n, provided that C contains such a codeword. The weight distribution of C is an ordered sequence of non–negative integers, $A_0, A_1, A_2, ..., A_n$, such that A_i represents the number of codewords of weight i in C. Codes for which $A_i = 0$ whenever i is odd are called even, while codes for which $A_i = 0$ whenever i is not a multiple of four are called doubly–even. The weight enumerator of a code is the polynomial $A(z) = \sum_{i=1}^{n} A_i\, z^i$. By introducing the change of variables $z^i \to x^{n-i}y^i$, the weight enumerator becomes a homogeneous polynomial of degree n in two variables, x and y.

The translate $\mathbf{f} + C = \{\mathbf{f} + c : \ c \in C\}$ of a code by any vector $\mathbf{f} \in GF(2)^n$ is called a coset of the code, while a vector of smallest weight within the coset is called a coset leader. The weight distribution and weight enumerator $A(\mathbf{f} + C, z)$ of a coset are defined in the same way as the weight distribution and weight enumerator of a code, respectively. The set of all coset weight distributions of a code is sometimes called the complete coset weight distribution.

Let $C^{\perp}$ be the dual code of C. The code $C^{\perp}$ consists of all vectors $u \in GF(2)^n$ orthogonal to every codeword of C. If each word of C is orthogonal to all other codewords in C (including itself), then the code is called self–orthogonal. If a self–orthogonal code has dimension $k = n/2$, it is called self–dual. A code with the same weight distribution as its dual is called formally self–dual. Two codes are permutation equivalent (henceforth, equivalent) if one of the codes can be obtained by permuting the coordinates of the other code. If a code is equivalent to its dual, we say that the code is isodual. A self–dual code is isodual and even, but the claims do not hold in the other direction. If an even (respectively, doubly–even) formally–self dual code has minimum distance d that satisfies $d = 2\lfloor n/8 \rfloor + 2$ (respectively, $d = 4\lfloor n/24 \rfloor + 4$) then it is called extremal [7], [8].

Let us identify the positions of bits in a codeword by the index set $I = \{1, 2, ..., n\}$. The support of a codeword is a subset of I that corresponds to the positions of its non–zero coordinates. The size of the support is called the support weight. For $J \subset I$, the subcode C_J of C is the largest set of codewords of C such that the union of their supports is the subset J. The projection P_J of C on a set J is the code obtained from C by setting all coordinates in $I - J$ to zero. If $J = \{1, 2, ..., m\}$, then C_J and C_{I-J} are denoted by C_{m^-} and C_{m^+} and are called the past and future subcode of C at position m. The corresponding projections P_{m^-} and P_{m^+} are known as the past and future punctured code, respectively.

Assume next that *dim* (C)=*dim* ($C^{\perp}$) and that the generator matrices of C and $C^{\perp}$ are of the form shown below:

$$G(C) = \begin{bmatrix} [\quad A \quad] & [\quad \mathbf{0} \quad] \\ [\quad \mathbf{0} \quad] & [\quad B \quad] \\ [\quad D \quad] & [\quad E \quad] \end{bmatrix}, \quad G(C^{\perp}) = \begin{bmatrix} [\quad F \quad] & [\quad \mathbf{0} \quad] \\ [\quad \mathbf{0} \quad] & [\quad J \quad] \\ [\quad L \quad] & [\quad M \quad] \end{bmatrix}.$$

Assume also that C and $C^{\perp}$ have length $n = n_1+n_2$ and that A (respectively, B) and F (respectively, J) generate the subcodes of C and $C^{\perp}$ of largest dimension k_A (respectively, k_b) and k_F (respectively, k_J) with support under the first n_1 (respectively, last n_2) coordinates. If $\{A\}$ denotes the code generated by the matrix A, and $A \cup B$ the vertical concatenation of the matrices A and B, then [8]

- $k_D = k_E = k_L = k_M$,
- $\{A\}^{\perp} = \{F \cup L\}$, $\{B\}^{\perp} = \{J \cup M\}$, $\{F\}^{\perp} = \{A \cup D\}$ $\{J\}^{\perp} = \{B \cup E\}$,
- $n_1 - 2k_A = n_2 - 2k_J$, $\; n_1 - 2k_F = n_2 - 2k_B$.

We will refer to the above result as the gluing principle. For self–dual codes the gluing principle is also known as the balancing principle [17]. It states that $k_B = n/2 - n_1 + k_A$, $A^{\perp} = \{A \cup D\}$ and $B^{\perp} = \{B \cup E\}$.

The r-th generalized Hamming weight of a linear code C, $d_r(C)$, is the minimum support weight of any subcode of C that has dimension r. The sequence $\mathbf{d}(C) = \{d_r(C), 1 \le r \le k\}$ is the weight hierarchy of C. The weight hierarchy of a code and its dual satisfy Wei's duality formula:

$$\{d_r(C), 1 \le r \le k\} = I - \{n + 1 - d_r(C^{\perp}), 1 \le r \le k\}.$$

The generalized weights also satisfy the generalized Griesmer bound,

$$d_r \geq \sum_{i=0}^{r-1} \lceil \frac{d}{2^i} \rceil,$$

as well as a strict monotonicity rule of the form $d_i > d_j$, $\forall\, i > j$.

Let $A_j^{(i)}$ be the number of i–dimensional subcodes of C with support weight j. The i-th support weight enumerator is the polynomial $A^{(i)}(z) = \sum_{j=0}^{n} A_j^{(i)}\, z^j$. By definition, $A^{(0)}(z) = 1$ and $A^{(1)}(z) = A(z) - 1$, where $A(z)$ is the weight enumerator of the code. A support weight enumerator can also be represented in the form of a homogeneous polynomial of degree n in two variables, x and y, by writing $x^{n-i}\, y^i$ for z^i. Throughout the paper, we will use both the homogenous and non–homogeneous form for the support weight enumerators, depending on the specific application.

3 Support weight enumerators and invariant theory

We start this section by stating a generalization of the well known MacWilliams transformation formula [18, p. 128] for the case of support weight enumerators [16], [26]. For isodual codes the support weight enumerators of the code and its dual are identical, and the MacWilliams–type identities for support weight enumerators are given by Theorem 1.

Theorem 1 [16]: For all $s \geq 0$ let

$$(s)_r = \prod_{i=0}^{r-1} (2^s - 2^i). \tag{1}$$

Then one has

$$\sum_{r=0}^{s} (s)_r\, A^{(r)}(z) = 2^{-sn/2}\; (1 + (2^s - 1)z)^n \left(\sum_{r=0}^{s} (s)_r\, A^{(r)} \left(\frac{1 - z}{1 + (2^s - 1)z} \right) \right). \tag{2}$$

Throughout the paper, we will refer to the equations in Theorem 1 as the Kløve identities.

Based on an observation made by the authors of [11] that actually motivated the introduction of support weight enumerators, it follows that a s-dimensional subcode $C^{(s)}$ of C can be viewed as a codeword c^s of a linear code over the field $GF(2^s)$. In this case, the weight of c^s over $GF(2^s)$ is the support weight of the subcode $C^{(s)}$ over $GF(2)$. Therefore, $\sum_{r=0}^{s} (s)_r\, A^{(r)}(z)$ represents the weight enumerator of a linear code over the field $GF(2^s)$. This allows us to conclude without proof that the results stated in Lemma 1, 2 and 3 given below hold.

Lemma 1: Let C be an isodual $[n, n/2, d]$ code with homogeneous support weight enumerators $A^{(r)}(x, y)$, $0 \leq r \leq n/2$, and let

$$v^{(s)}(x, y) = \sum_{r=0}^{s} (s)_r\, A^{(r)}(x, y). \tag{3}$$

Then

$$v^{(s)}(x, y) = v^{(s)} \left(\frac{x + (2^s - 1)y}{2^{s/2}}, \frac{x - y}{2^{s/2}} \right), \tag{4}$$

i.e. $v^{(s)}(x, y)$ is invariant under the linear transformation $x \rightarrow (x + (2^s - 1)y)/2^{s/2}$ and $y \rightarrow (x - y)/2^{s/2}$.

Remark: Notice that the invariance property of $v^{(s)}(x,y)$ also follows from the statement of Theorem 1 directly, by writing y/x for z and by performing some simple algebra.

Next, define $v_i^{(s)}$ as

$$v_i^{(s)} = [y^i\, x^{n-i}] \sum_{r=0}^{s} (s)_r\, A^{(r)}(x,y) = \sum_{r=0}^{s} (s)_r\, A_i^{(r)}, \tag{5}$$

where $[y^i\, x^{n-i}]P(x,y)$ denotes the coefficient in front of $y^i\, x^{n-i}$ in the polynomial $P(x,y)$.

Lemma 2: The binomial moments of the coefficients $\{v_i^{(s)}\}$ are of the form

$$\sum_{i=m}^{n} \binom{i}{m} v_i^{(s)} = 2^{s(n/2-m)} \sum_{i=0}^{m} (-1)^i\, (2^s-1)^{m-i} \binom{n-i}{m-i} v_i^{(s)}\,. \tag{6}$$

Lemma 3: The power (Pless) moments of order $m \geq 0$, of the coefficients $\{v_i^{(s)}\}$ are

$$\sum_{i=0}^{n} i^m\; v_i^{(s)} = \sum_{j=0}^{n} (-1)^j\, v_j^{(s)} \sum_{l=0}^{m} l! \binom{n-j}{n-l} 2^{s(n/2-l)} (2^s-1)^{l-j}\, S(m,l)\,, \tag{7}$$

where $S(m,l)$ stands for the Stirling numbers of the second kind. Notice that the first summation on the right hand side actually ranges from 0 to m, since $\binom{n-j}{n-l} = 0$ for $j > m$ and $l \leq m$.

Theorem 2: In order to compute the support weight enumerators of an isodual code of any order s, it is sufficient to know only the coefficients $A_i^{(r)},\ 0 \leq i \leq n/2,\ 0 \leq r \leq s$.

Proof: For a fixed i, $v_i^{(s)}$ is a linear function of $A_i^{(r)}$, $0 \leq r \leq s$, as given by (5). Therefore, for fixed $s \leq n/2$ the coefficient matrix of the $A_i^{(r)}$'s, $0 \leq r \leq s$, is lower–triangular with diagonal elements strictly greater than zero, i.e.

$$\begin{bmatrix} v_i^{(0)} \\ v_i^{(1)} \\ \vdots \\ \vdots \\ v_i^{(s)} \end{bmatrix} = \begin{bmatrix} (0)_0 & 0 & 0 & \dots & 0 \\ (1)_0 & (1)_1 & 0 & \dots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ (s)_0 & (s)_1 & \dots & (s)_{s-1} & (s)_s \end{bmatrix} \begin{bmatrix} A_i^{(0)} \\ A_i^{(1)} \\ \vdots \\ \vdots \\ A_i^{(s)} \end{bmatrix}.$$

Since the inverse of a lower–triangular matrix is also a lower–triangular matrix, it follows that for any s, the value of $A_i^{(s)}$ can be uniquely determined from the values of $v_i^{(r)}$, $0 \leq r \leq s$.

Assume that s is fixed and that the values of $A_i^{(r)},\ 0 \leq i \leq n/2,\ 0 \leq r \leq s$ are known. Then the values of $v_i^{(r)}$ for $0 \leq i \leq n/2$ and $0 \leq r \leq s$ are known as well. For a specified r, the first $n/2$ equations (with respect to m) of (7) have $n/2$ unknowns, and the coefficient matrix of the equations is Vandermonde, and hence invertible. This implies that there is a unique solution for the $n/2$ unknown values of the variables $v_i^{(r)}$, $n/2 < i \leq n$. Since the same procedure can be performed for all $0 \leq r \leq s$, it is possible to compute the complete set of coefficients of any support weight enumerator up to order s. This proves the claimed result. ∎

Corollary 1: Let C be a self–dual $[n, n/2, d]$ code. If $n \leq 3d_2 - 2$ then the second support weight enumerator is uniquely determined by the weight enumerator of C and Kløve's identities.

Proof: The supports of two codewords of a self–dual code can intersect only in an even number of positions, so that any two–dimensional subcode of a self–dual code has even support weight. This implies that for a given weight enumerator the values of $v_i^{(2)}$ for $i = 0$ to $i = d_2 - 1$ as well as for all odd $i > d_2$ are known. Therefore, there are only $1 + (n - d_2)/2$ unknowns in the second support weight enumerator. Based on the arguments given in Theorem 2 one can deduce that if $1 + (n - d_2)/2 \leq d_2$, the complete second support weight enumerator is known. This proves the claimed result. ■

4 Coset structure of isodual codes

When analyzing the structure of a code, it is usually a very difficult problem to determine its complete coset weight distribution. There exist only a few classes of codes for which these distributions are known or can be derived without much effort, for example, perfect and quasi-perfect codes, or codes that support higher order designs and have a small covering radius [18]. It is intuitively clear that the coset weight distributions are closely related to support weight enumerators, since both contain information about the intersections of supports of codewords. Hence, one is inclined to investigate connections between the support weight enumerators and coset weight distributions, determine the set of invariants that is easier to find and then use it to obtain some information about the other set.

We start this section by stating a result concerning the relationship between the complete coset weight distribution of an isodual code and its first and second support weight enumerator [6]. Then we proceed to extend this result for the case of the first, second and a generalization of the third support weight enumerator.

Theorem 3 [6]: For an isodual code C, the set of weight enumerators $A(f + C, z)$ of the cosets of the codes generated by the vectors $f \in GF(2)^n$ satisfy the following two equalities:

$$\sum_{f \in GF(2)^n} A(f + C, z)^2 = \sum_{i=0}^{n} A_i (1 + z)^{2n-2i} (1 - z)^{2i}\,, \tag{8}$$

$$\sum_{f \in GF(2)^n} A(f + C, z)^3 = 2^{-n/2}\,(1 + z)^{3n} \left(1 + 3A^{(1)}\left(\left(\frac{1-z}{1+z}\right)^2\right) + 6A^{(2)}\left(\left(\frac{1-z}{1+z}\right)^2\right)\right). \tag{9}$$

Remark: In Section 3, we described the polynomial $v^{(i)}(z)$ that for $i = 2$ has the form $1 + 3A^{(1)}(z) + 6A^{(2)}(z)$. The coefficients of this polynomial are equal to the coefficients of the linear combination on the right hand side of Equation (9), with z replaced by $((1 - z)/(1 + z))^2$.

The result we derive next is an extension of Theorem 3 for the sum involving fourth powers of the weight enumerators of cosets of a code. We will use an approach similar to the one used for proving Theorem 3.

Theorem 4: Let C be an isodual code. Then the following relationship exists between the first, second and a modification of the third support weight enumerator on one side and the complete coset weight distribution of the code on the other side

$$\begin{aligned} &2^{-3n}(1+z)^{4n} \sum_{f \in GF(2)^n} A\left(f + C, \frac{1-z}{1+z}\right)^4 \\ &= 1 + 6A^{(1)}(z^2) + 3\left(A^{(1)}(z^2)\right)^2 - 2A^{(1)}(z^4) + 24A^{(2)}(z^2) + 24\sum_{C^{(3)}} p(C^{(3)})\, z^{4\,supp(C^{(3)})}\,. \end{aligned} \tag{10}$$

The last summation on the right hand side of (10) ranges over all three–dimensional subcodes $C^{(3)}$ of C, $supp(C^{(3)})$ denotes the support weight of $C^{(3)}$, and $p(C^{(3)})$ is given by

$$p(C^{(3)}) = \sum_{(v_i,v_j)} z^{-2supp(v_i,v_j)} . \tag{11}$$

The summation in Equation (11) ranges over all pairs of codewords v_i, v_j in $C^{(3)}$ that generate inequivalent two–dimensional subcodes of $C^{(3)}$ (a total of $\binom{7}{2}/3 = 7$ pairs), and $supp(v_i, v_j)$ is the support weight of the two–dimensional subcode generated by v_i, v_j.

Proof: Consider the expression

$$\sum_{f\in GF(2)^n} \left(\sum_{u\in C} (-1)^{f\cdot u} \; z^{w(u)} \right)^4 ,$$

where $w(u)$ denotes the weight of the codeword $u \in C$. This expression can be rewritten as

$$\begin{aligned} &\sum_f \left(\sum_x \sum_y \sum_u \sum_v (-1)^{f\cdot(x+y+u+v)} \; z^{w(x)+w(y)+w(u)+w(v)} \right) \\ &= \sum_x \sum_y \sum_u \sum_v z^{w(x)+w(y)+w(u)+w(v)} \sum_f (-1)^{f\cdot(x+y+u+v)} \\ &= 2^n \sum_x \sum_y \sum_u z^{w(x)+w(y)+w(u)+w(x+y+u)} , \end{aligned} \tag{12}$$

where the last equality follows from the well known fact that $\sum_f (-1)^{f\cdot(x+y+u+v)}$ is equal to 2^n for $x + y + u + v = 0$, and is zero otherwise [18].
The next step of the proof is to describe $w(x) + w(y) + w(u) + w(x + y + u)$ in terms of the support weights of codes spanned by the not necessarily different codewords x, y, u. We distinguish seven different cases as given below.

- $x = y = u = 0$ (I)
- Exactly one of the codewords x, y, u is zero (II)
- Exactly two of the codewords x, y, u are zero (III)
- $x, y, u \neq 0$ and
  - Exactly two of the codewords x, y, u are equal (IV)
  - $x = y = u$ (V)
  - $x \neq y \neq u$ and $x + y = u$ (VI)
  - $x \neq y \neq u$ and $x + y \neq u$. (VII)

It is tedious, but straightforward, to verify that the first six cases contribute the following terms to (12)

$$2^n \left(1 + 6A^{(1)}(z^2) + 3 \left(A^{(1)}(z^2) \right)^2 - 2A^{(1)}(z^4) + 24A^{(2)}(z^2) \right) . \tag{13}$$

For (VII), the codewords x, y, u are linearly independent and it is easy to show that

$$2\,supp(x,y,u) = w(x) + w(y) + w(u) + w(x+y+z) - 2w(x*y*u)\,, \tag{14}$$

where $x*y*u$ is a vector with ones only at positions where all three codewords x, y, u have ones, and where $supp(x,y,u)$ is the support of the three–dimensional subcode $C^{(3)}$ generated by the codewords x, y, u. From (14) it follows that the seventh case contributes the following sum to (12)

$$2^n \sum_{x\neq y\neq u,\ x+y\neq u} z^{2\,supp(x,y,u)+2\,w(x*y*u)}, \tag{15}$$

and the last summation ranges over all triples (x, y, u) of linearly independent codewords. Since each ordered quadruple of codewords $x, y, u, x+y+u$ is counted exactly $4! = 24$ times, (15) takes the form

$$24\ \cdot 2^n \sum_{C^{(3)}\subset C} z^{2\,supp(C^{(3)})} \sum_{span(x,y,u)=C^{(3)}} z^{2\,w(x*y*u)}. \tag{16}$$

From the principle of inclusion and exclusion, it holds that

$$w(x*y*u) = w(x) + w(y) + w(u) - supp(x,y) - supp(x,u) - supp(u,y) + supp(x,y,u), \tag{17}$$

and this can be rewritten as

$$2\,w(x*y*u) = 2\,supp(C^{(3)}) - 2supp(x+y, x+u). \tag{18}$$

Therefore,

$$\sum_{span(x,y,u)=C^{(3)}} z^{2\,w(x*y*u)} = z^{2\,supp(C^{(3)})} \sum_{span(x,y,u)=C^{(3)}} z^{-2\,supp(x+y,x+u)}, \tag{19}$$

so that (15) becomes

$$24\ \cdot 2^n \sum_{C^{(3)}\subset C} z^{4\,supp(C^{(3)})} \sum_{span(x,y,u)=C^{(3)}} z^{-2\,supp(x+y,x+u)}\,. \tag{20}$$

We note next that [18, p.166]

$$A(f+C, z) = 2^{-n/2} \sum_{u\in C} (-1)^{f\cdot u} (1+z)^{n-w(u)} (1-z)^{w(u)},$$

so that

$$\begin{aligned} A\left(f+C, \frac{1-z}{1+z}\right) &= 2^{-n/2} \sum_{u\in C} (-1)^{f\cdot u} \left(\frac{2}{1+z}\right)^{n-w(u)} \left(\frac{2z}{1+z}\right)^{w(u)} \\ &= 2^{n/2}\,(1+z)^{-n} \sum_{u\in C} (-1)^{f\cdot u} z^{w(u)}. \end{aligned}$$

The last equation implies that

$$\sum_f A\left(f+C, \frac{1-z}{1+z}\right)^4 = 2^{2n}\,(1+z)^{-4n} \sum_f \left(\sum_{u\in C} (-1)^{f\cdot u} z^{w(u)}\right)^4 \tag{21}$$

hold. Substituting the expressions (13) and (20) into (21) completes the proof. ■

Theorem 4 shows that the sum of the fourth powers of the coset weight enumerators of an isodual code depends on its first, second and third support weight enumerator. The dependence on the the third weight enumerator does not enter the formula directly, but rather through the requirement that one has to know not only the number of three–dimensional subcodes of a given support weight, but also the number of two–dimensional subcodes within any of these three–dimensional subcodes. There exist many codes for which Theorem 4 produces a linear equation involving coefficients of the third support weight enumerator that is linearly independent from the Kløve identities. Some of them are described in Section 5. Theorem 4 can also be used for computing unknown parameters of the coset weight distributions from the support weight enumerators. One can start by using the annihilator polynomials [18, pp.168-172] to find all possible coset weight distributions of a code and then deduce from Theorem 3 and Theorem 4 the number of cosets with a given weight distribution. Even though this approach does not always produce a complete answer, it is nevertheless useful because it allows one to express the number of cosets with leaders of large weight in terms of the number of cosets with leaders of small weight.

5 The support weight enumerators of some isodual codes of length up to 22

5.1 The three even $[12, 6, 4]$ isodual codes

There exist three inequivalent, extremal, even, formally self–dual $[12, 6, 4]$ codes, one of which is self–dual, while the other two are isodual [2]. The weight enumerator of these codes is

$$A(z) = 1 + 15z^4 + 32z^6 + 15z^8 + z^{12}.$$

Let $H(C)$ be the hull of the code C, defined as $H(C) = C \cap C^{\perp}$, i.e. the set of all codewords that are both in the code and its dual. Based on the dimension of their hull, denote the three inequivalent $[12, 6, 4]$ codes by sd$[12, 6, 4]$, fsd12-4$[12, 6, 4]$ and fsd12-2$[12, 6, 4]$. The code sd$[12, 6, 4]$ is self–dual, while the dimensions of $H(fsd12-2)$ and $H(fsd12-4)$ are two and four, respectively. It was shown in [3] and [6] that the weight hierarchy of the sd$[12, 6, 4]$ code is $\mathbf{d} = \{4, 6, 8, 10, 11, 12\}$. We will prove next that that all three even $[12, 6, 4]$ codes have the same weight hierarchy.

Lemma 4: The weight hierarchy of all the even $[12, 6, 4]$ codes is $\mathbf{d}([12, 6, 4]) = \{4, 6, 8, 10, 11, 12\}$.

Proof: Let the past subcode of a formally self–dual code be a $[4, 1, 4]$ code. The future punctured code in this case is a $[8, 5, 2]$ code, since there is no $[8, 5, 3]$ code [13]. Therefore, $d_2 = 6$. Wei's duality relation implies that $d_3 > 7$. This result together with the strict monotonicity property of generalized weights provide sufficient information for computing the complete weight hierarchy. ■

Theorem 6: The support weight enumerators of the three formally self–dual $[12, 6, 4]$ codes are given in Table 1.

Proof: From Corollary 1, it follows that for the sd$[12, 6, 4]$ code there are no unknowns in the Kløve identities; the fact that sd$[12, 6, 4]$ is self–dual, and the Kløve identities show that $A_6^{(2)} = 20$, and this suffices to determine all the remaining unknowns in the support weight enumerators. There is only one unknown in the Kløve identities for the isodual codes, namely $A_6^{(2)}$, and it can be computed by using the value of the dimension of the hull and residual code arguments.

Let the past subcode of an isodual $[12,6,4]$ code be a $[4,1,4]$ subcode that is in the hull. Based on the gluing principle, the dimension of the future subcode of the dual equals three, and the code contains the codeword **1**. This implies that its weight enumerator is $A(z) = 1 + 6z^4 + z^8$. The dual of this code is the future punctured code of the $[12,6,4]$ code at position four and it has the weight enumerator $1+4z^2+22z^4+4z^6+z^8$. Therefore, every word in the hull generates exactly four $[6,2,4]$ subcodes. On the other hand, if the past subcode is a $[4,1,4]$ subcode which does not belong to the hull, the future subcode in the dual is a $[8,3,4]$ code that does not contain the codeword **1**. There are only two possibilities for the weight enumerator of such a code, namely $A_1(z) = 1 + 5z^4 + 2z^6$ or $A_2(z) = 1 + 7z^4$. The second enumerator must have support weight equal to seven, since every linear constant weight code is obtained by a direct sum (concatenation) of simplex codes [12]. But we already proved that $d_3 = 8$ and since the code is isodual, this can not be a subcode of a formally self–dual $[12,6,4]$ code. One can also establish the same result by examining the dual of the $[8,3,4]$ code with weight enumerator $A_2(z)$. The dual code contains a word of weight one, and this gives $d_2 = 5$, a violation of the Griesmer bound. Therefore, the future subcode in the dual has the weight enumerator $A_1(x)$. The dual of this subcode has two codewords of weight two, so that each codeword of weight four that is not in the dual generates exactly two $[6,2,4]$ subcodes. From the description of the isodual $[12,6,4]$ codes in [2], the number of codewords of weight four in the hull of fsd12-4 and fsd12-2 is 3 and 0, respectively. Since an even $[6,2,4]$ code has exactly $\binom{3}{2} = 3$ different pairs of basis codewords, it follows that

$$A_6^{(2)}(fsd12-4) = \frac{3 \cdot 4 + 12 \cdot 2}{3} = 12,$$
$$A_6^{(2)}(fsd12-2) = \frac{15 \cdot 2}{3} = 10.$$

All other entries in Table 1 are computed using the Kløve identities. ■

From Table 1 one can notice that only the second weight enumerators of the codes differ. Also, one has $A_{11}^{(5)} = \binom{12}{1} = 12$. Similar results exist for other isodual codes. For example, isodual codes that support a t-design have $A_{d_{n/2-u}}^{(n/2-u)} = \binom{n}{u}$, $\forall\, u \leq t$.

5.1.1 The coset weight distributions of the $[12,6,4]$ codes

The $[12,6,4]$ codes are isodual and hence have dual distance equal to four. Additionally, there are only four non–zero weights in the code. Therefore, from Theorem 20 of [18, p.169] one can compute the weight distributions of all possible cosets of any of the $[12,6,4]$ codes. The unknown parameters in the complete coset weight distribution are the *numbers* of cosets of each type (i.e. number of cosets with a certain weight distribution). To compute these numbers, we will try first to eliminate as many of these unknowns as possible by using standard coding arguments.

Let us start by observing that the codewords of any weight in an even $[12,6,4]$ code form a 1-design. Therefore, there exists a unique weight distribution for cosets with leaders of weight one, and the number of cosets with this weight distribution is equal to $n = 12$. From a general result given in [1], it can be deduced that the cosets of the $[12,6,4]$ codes with leaders of weight three and weight four (if any) have to have a unique weight distribution. It is straightforward to see that there are 20 cosets with leaders of weight three. The non–existence of some other cosets can be proved by using the following result which, to the best of our knowledge, was not stated before.

Proposition 1: Let $C\,[n, n/2, d]$ be an even code containing the codeword $\mathbf{1}$, and assume that d divides $2n$. If $2n/d$ is even, then C does not contain a coset with exactly $(2n/d) - 1$ vectors of weight $d/2$.

Proof: Assume on the contrary that such a coset exists. Then the supports of $(2n/d) - 2$ codewords of weight d contain the support of one common coset leader of weight $d/2$. The support of each of these codewords also contains the support of one more coset leader, and no two such coset leaders have overlapping support. Since $(2n/d) - 2$ is even, the described codewords of weight d add up to a codeword of weight $((2n/d)-2)(d/2) = n-d$. When added to the codeword $\mathbf{1}$, this codeword produces an additional codeword of weight d. The support of this codeword contains the support of the leader that the supports of all the $(2n/d) - 2$ codewords under consideration share. Hence, the coset under consideration contains one more coset leader of weight $d/2$, and this conclusion leads to a contradiction of the starting assumption. ■

Based on Proposition 1, one can eliminate cosets with five leaders of weight two from consideration. Table 2 gives the coset weight distributions with unknown values of $w_1, ..., w_4, v$ that we will compute by using the support weight enumerators.

Since all the $[12, 6, 4]$ codes have the same weight enumerator, equation (8) of Theorem 3 gives the unknowns w_1, w_2, w_3 in terms of the remaining variables as follows:

$$w_1 = 6 - 3v - w_4 - 6w_6,\;\; w_2 = 15 + 3v + 3w_4 + 15w_6,\;\; w_3 = 10 - v - 3w_4 - 10w_6.$$

Equation (9) of Theorem 3 and the second weight enumerator produce an equation for w_4 expressed in terms of the remaining variables as:

sd: w_4=10+v-10w_6 fsd12-4: w_4=2+v-10w_6 fsd12-2: w_4=v-10w_6.

The fourth equation for the sd$[12, 6, 4]$ code gives sufficient information to determine all the remaining unknowns. Since $w_i \geq 0\,,\, \forall\, i$, one must have

$$w_1=0,\; w_2=30,\; w_3=0,\; w_4=0,\; w_6=1,\; v=0.$$

On the other hand, there is not sufficient information to determine all the unknowns for the two formally self–dual codes; it is only clear that $w_6 = 0$, but otherwise, there can be two possible sets of values for $w_1, ..., v$. In order to calculate the value of one additional unknown, we will use the information provided by the third weight enumerator.

Let us examine the fsd12-2 code first. For this code, $A_8^{(3)} = 15$ and there are only two possible weight enumerators for even $[8, 3, 4]$ subcodes, namely

$$A_1(z) = 1 + 6z^4 + z^8, \quad \text{or} \quad A_2(z) = 1 + 5z^4 + 2z^6. \tag{22}$$

Let p_1 and p_2 be the number of codes with weight enumerator $A_1(z)$ and $A_2(z)$, respectively, that are contained in fsd12-2. Then $p_1 + p_2 = A_8^{(3)} = 15$. Consider Equation (10) in Theorem 4. The left hand side of (10) equals

$$C_1(z) = 1 + 90z^8 + 432z^{12} + 1440z^{14} + (4335 + 96v)\, z^{16} + ...\,,$$

and

$$C_2(z) = 1 + 90z^8 + 480z^{12} + 1152z^{14} + (5103 + 96v)z^{16} + ...\,,$$

for the fsd12-2 and fsd12-4 codes, respectively. Also, the first five terms on the right hand side of (10) give the following enumerators

$$E_1(z) = 1 + 90z^8 + 432z^{12} + 1440z^{14} + 3975z^{16} + \ldots,$$

and

$$E_2(z) = 1 + 90z^8 + 480z^{12} + 1152z^{14} + 4695z^{16} + \ldots,$$

for the fsd12-2 and fsd12-4 codes, respectively. Now,

$$C_1(z) - E_1(z) = (4335 + 96v - 3975)z^{16} + \ldots = (360 + 96v)z^{16} + \ldots,$$

and

$$C_2(z) - E_2(z) = (5103 + 96v - 4695)z^{16} + \ldots = (408 + 96v)z^{16} + \ldots$$

are equal to the polynomial p given by (11). It is straightforward to compute the coefficient in front of z^{16} of the last summation on the right hand side of (10) directly, since the even $[8,3,4]$ subcodes under consideration are unique, and the two-dimensional subcodes can be counted without difficulty. For the fsd12-2 code this gives $3p_1 + p_2 = 15 + 4v$, or equivalently $p_1 = 2v$. Since based on the previous equation v can only be 0 or 1, $p_1 = 0$ or $p_1 = 2$. Assume $p_1 = 2$. Let the past subcode of fsd12-2 be a $[8,3,4]$ code with weight enumerator $A_1(z)$. Then the dimension of the future subcode of length four in the dual is one. Since there are no codewords of weight four in the hull of this code, the past and future subcodes at position eight are a $[8,3,4]$ and a $[4,0,4]$ code. This leads to a contradiction, since the fsd2-12 code has to contain the codeword **1**, and the codeword of weight eight in the past subcode can not be used to obtain it. Therefore, $p_1 = 0$ and $v = 0$. This gives the following values of the unknown parameters for the coset weight distributions:

$$w_4 = v = 0,\ w_1 = 6,\ w_2 = 15,\ w_3 = 10$$

For the fsd12-4 code the third weight enumerator gives $p_1 = 1 + 2v$. Since v can only take the value 0 or 1, it follows that $p_1 = 1$ or $p_1 = 3$. The first case can be ruled out, showing that $v = 1$. This gives values for all the remaining unknowns:

$$v = 1,\ w_1 = 0,\ w_2 = 27,\ w_3 = 0,\ w_4 = 3.$$

5.2 The $[14,7,4]$ formally self–dual codes

Fields et al. [7] classified all extremal, even $[14,7,4]$ formally self–dual codes. They showed that there exist exactly ten even, formally self–dual codes with these parameters. One of the codes is the unique self–dual $[14,7,4]$ code, the sd$[14,7,4]$ code, while the remaining codes are isodual. The weight enumerator of all these codes is

$$A(z) = 1 + 14z^4 + 49z^6 + 49z^8 + 14z^{10} + z^{14}.$$

The weight hierarchy of the sd$[14,7,4]$ was first derived in [6]. We prove this result by using a different approach.

Lemma 5: The weight hierarchy of sd$[14,7,4]$ is $\mathbf{d} = \{4,6,7,10,12,13,14\}$.

Proof: From the balancing principle, it follows that if the past subcode of a $[14,7,4]$ code is a $[4,1,4]$ code, then the future subcode is a $[10,4,4]$ code that contains the codeword **1**. The future subcode has the weight enumerator $1 + 7z^4 + 7z^6 + z^{10}$. The dual of this code contains three codewords of weight two and therefore $d_2 = 6$. The Griesmer bound for d_3 gives $d_3 \geq 7$. That $d_3 = 7$ can be shown by examining the future subcode for a given past $[6, 2, 4]$ code whose existence we proved above. The future subcode in this case is a $[8, 3, 4]$ subcode which does not contain the codeword **1** and is self–orthogonal. There are two possible weight enumerator for this code, namely $A_1(z) = 1 + 5z^4 + 2z^6$ and $A_2(z) = 1 + 7z^4$. As shown before, the second code must have support weight equal to seven. It can also be shown that the first code can not be self-orthogonal. Assume on the contrary that it is self-orthogonal; then every two codewords overlap in an even number of positions. From a result by Dodunekov and Manev [4] there exist three linearly independent codewords of weight four that generate the code. These codewords overlap only in two or zero positions, and there can not be a word of weight 6 in the code. This proves that $d_3 = 7$. The remaining values for the weights follow from Wei's duality relation and the strict monotonicity property. Since for a past $[7, 3, 4]$ subcode of sd$[14, 7, 4]$ the future subcode is also a $[7, 3, 4]$ code, $A_7^{(3)}$ has to be an even number. The Kløve identities show that this number is ≤ 2 and it is therefore exactly equal to two. This implies that the self–dual $[14, 7, 4]$ code is obtained by gluing two simplex codes with the codeword **1**, a result first shown in [23]. ■

All the $[14, 7, 4]$ isodual codes have $d_2 = 6$, but there exist codes that have $d_3 = 7$ and codes that have $d_3 = 8$. We omit the proofs that establish certain values for the coefficients of the support weight enumerators, pointing out that for several codes we performed a computer search to find some of the missing parameters.

Table 3 gives the only unknown coefficients in the Kløve identities for all $[14, 7, 4]$ even codes, and the complete set of support weight enumerators can be computed based on these numbers. We observe that two of the inequivalent codes with dimension of the hull equal to one have *exactly the same* set of support weight enumerators. To the best of our knowledge, this is the first such example in coding theory literature.

5.2.1 Coset weight enumerators of the fsd$[14, 7, 4]$ codes

The coset weight distribution of the sd$[14, 7, 4]$ can be completely determined from the fact that this code is obtained by gluing two simplex codes with the **1** codeword. In the arguments that follow, we will use the coset weight distributions of the simplex $[7, 3, 4]$ and Hamming $[7, 4, 3]$ codes, which are well known [18] and therefore not presented in the paper.

We start by noticing that adding the codeword **1** to a codeword obtained by gluing a codeword from the past (respectively, future) simplex code with a zero codeword from the future (respectively, past) simplex code generates a codeword of the following form: the past (respectively, future) half of the codeword belongs to the Hamming code, while the future (respectively, past) half is the codeword **1** of length seven. Consider a vector of weight one. Since the weight distribution of a coset with a leader of weight one is unique for both the simplex and Hamming code, the sd$[14, 7, 4]$ code has a unique weight distribution for a coset with a leader of weight one as well. This result can also be deduced from design theory [18]. There are four vectors of weight three in that coset, since a coset leader of weight one creates four vectors of weight three in the simplex code. The vectors of weight three created in the Hamming code by a coset leader of weight one are not counted, since the other half of those vectors has weight seven. Hence, there are exactly 14 cosets with one leader of weight one and four vectors of weight three. Similarly, we can examine a vector of weight two; this vector can either have both ones in the first or both ones in the second half of a codeword, or one in each half. In the first case, the vector of weight two creates two more vectors of weight two in the simplex code, but it does not create vectors of weight two in the Hamming codeword glued with a **1** word of length seven. Hence, the sd$[14, 7, 4]$ code contains cosets with three leaders of weight two. The number of these cosets is 14 since there are two simplex codes in the sd$[14, 7, 4]$ code and each contains seven cosets with a leader of weight one. In the second case, a coset can contain only one leader of weight two, and the number of cosets of this type is 49. This follows since we can choose one position in the first half in seven ways and one position in the second half in seven ways as well. For cosets with leaders of weight three we can use similar arguments, except that now we have to take into consideration that there are leaders of weight three that are codewords of the Hamming code. These leaders contribute to the number of vectors of weight three in a coset by annulling a codeword of the past (or future) Hamming code, while combining it with a codeword of weight three in the future Hamming code. In this case a coset with 14 vectors of weight three is formed, and there is only one such coset. On the other hand, if a vector of weight three has ones in both the first and second half of the codeword, then it is a leader of a coset with $1 + 2 + 3 = 6$ vectors of weight three. This holds since the leader itself creates two vectors of weight three from the simplex codes, and three vectors of weight three from the concatenated codewords of the Hamming code. It is easy to check that the number of cosets obtained in this way is 49. There are no cosets of sd$[14, 7, 4]$ with leaders of weight higher than three, as already observed in [1].

For the isodual $[14, 7, 4]$ codes, a similar approach as for the $[12, 6, 4]$ codes can be used to try to determine their complete coset weight distributions. For several codes the first, second and third weight enumerators did not provide sufficient information to determine the complete set of coset weight distributions. For these codes the missing parameters were found by computer search. Table 4 and 5 list the coset weight distributions of the formally self–dual $[14, 7, 4]$ codes, together with the number of cosets of each type. We notice that two inequivalent codes, fsd14-1a and fsd14-1c, have the same coset weight distributions. To the best of our knowledge, this is the first known example of this kind.

5.3 The $[16, 8, 4]$ self–dual codes

For this choice of parameters, we will analyze only the three self–dual $[16, 8, 4]$ codes, which we name E16, F16, $H_8 \oplus H_8 = 2H_8$, following the notation in [21]. Since these self–dual codes are not extremal, they do not necessarily have the same weight enumerators, as can be seen below.

$$
\begin{aligned}
A_{2H_8}(z) &= 1 + 28z^4 + 198z^8 + 28z^{12} + z^{16}, \\
A_{F16}(z) &= 1 + 12z^4 + 64z^6 + 102z^8 + 64z^{10} + 12z^{12} + z^{16}, \\
A_{E16}(z) &= 1 + 28z^4 + 198z^8 + 28z^{12} + z^{16}.
\end{aligned}
$$

Consider the $2H_8$ code first. This code is obtained as a direct sum of two unique extended, self–dual $[8, 4, 4]$ codes. This allows us to determine the weight hierarchy of this code based on the weight hierarchy of the $[8, 4, 4]$ code. It is well known that the $[8, 4, 4]$ code has the weight hierarchy $\mathbf{d} = \{4, 6, 7, 8\}$ [3], so that for the $2H_8$ code $d_2 = 6$, $d_3 = 7$ and $d_4 = 8$. The support weight enumerators for the extended Hamming code are $A^{(2)}(z) = 28z^6 + 7z^8$, $A^{(3)}(z) = 8z^7 + 7z^8$, and $A^{(4)} = z^8$, so that $2H_8$ must contain exactly 16 simplex subcodes, 14 $[8, 3, 4]$ and exactly two $[8, 4, 4]$ subcodes. Based on Theorem 2 and Corollary 1, this provides sufficient information to determine the complete set of support weight enumerators. The complete second, third and fourth weight enumerator of the $2H_8$ code are given below:

$$A^{(2)}_{2H_8}(z) = 56z^6 + 210z^8 + 2352z^{10} + 5320z^{12} + 2520z^{14} + 337z^{16},$$
$$A^{(3)}_{2H_8}(z) = 16z^7 + 14z^8 + 784z^{10} + 1568z^{11} + 8624z^{12} + 18816z^{13} + 30800z^{14} + 23632z^{15} + 12901z^{16},$$
$$A^{(4)}_{2H_8}(z) = 2z^8 + 224z^{11} + 1400z^{12} + 9408z^{13} + 39200z^{14} + 75616z^{15} + 74937z^{16}.$$

The complete coset weight distribution can be calculated from the coset weight distributions of the $[8, 4, 4]$ code [18], and is listed in Table 6.

Let us analyze the E16 code next. Since for this code $d_2 = 6$ [3], based on Corollary 1 the second support weight enumerator is uniquely determined. The coefficients of the third support weight enumerator can be found in two steps. First, it can be shown that if the past subcode of E16 is a $[4, 1, 4]$ code, then the future subcode is a $[12, 5, 4]$ subcode with weight enumerator $1 + 15z^4 + 15z^8 + z^{12}$. This subcode is self-orthogonal. The future punctured subcode at position four contains six codewords of weight two. Since the $[12, 5, 4]$ code itself has a basis consisting of codewords of weight four only and these codewords are even cyclic shifts of each other, the six codewords of weight six in the dual must have disjoint support. Therefore, the E16 code can not contain a $[7, 3, 4]$ simplex code. The number of different $[8, 3, 4]$ subcodes of E16 is $28 \cdot \binom{6}{2}/6 = 70$, since for E16 $A_4 = 28$, there are $\binom{6}{2}$ pairs of codewords of weight two in the future punctured code, and each $[8, 3, 4]$ subcode is counted exactly six times. One can use the second weight enumerator to derive the same result; there are 56 subcodes of dimension two that can be used as past subcodes and five codewords of weight two in the future punctured subcode that can be used to form a $[8, 3, 4]$ subcode. Obviously, there are $56 \cdot 5/4 = 70$ ways to do this, since very $[8, 3, 4]$ subcode obtained in this way is counted exactly four times. A different argument showing the same result also appeared in [3]. Based on Theorem 2, the complete set of support weight enumerators can now be computed from the Kløve identities, and is listed in Table 7.

The coset weight distributions for E16 can be determined based on the knowledge of $A^{(1)}(z)$, $A^{(2)}(z)$ and $A^{(3)}(z)$. We first observe that the number of non–zero weights in E16 is four, so that the code supports a 1-design. Hence, a coset with leader of weight one has a unique enumerator, and there are $n = 16$ such cosets. Since the covering radius of a code is upper bounded by its dual distance, only cosets with leaders up to weight four have to be considered. Additionally, the cosets with leaders of weight three and four (if any) must have a unique weight distribution [1]. Using the annihilator polynomial, one can obtain the possible coset weight distributions of E16 as listed in Table 8.

Based on Equations (8), (9) and (10), the unknown numbers $w_1, w_2, w_3, w_4, w_5, w_6, w_8, u$ satisfy the following equalities:

$$u = 70 + w,\ w_6 = 0,\ w_5 = 35 - w - 35w_8,\ w_4 = -105 + 5w + 105w_8,$$
$$w_3 = 126 - 10w - 126w_8,\ w_2 = -14 + 10w + 70w_8,\ w_1 = 15 - 5w - 15w_8,$$

where $w \geq 0$. Four of the equations follow from (8), and one from each of the results in (9) and (10). Since the unknowns have to be non–negative, there is only one possible solution for this system of equations, namely $w = 0$, $w_8 = 1$. This gives $u = 70, w_8 = 1, w_6 = w_5 = w_4 = w_3 = w_1 = 0$ and $w_2 = 56$.

Using the same approach as for E16, it is possible to calculate the complete set of support weight enumerators for the F16 code, as listed in Table 9. It is straightforward to see that for this code $d_3 = 8$ and that $A_8^{(3)} = 2$. All other values follow from the Kløve identities. The complete coset weight distribution of F16 is listed in Table 10.

5.4 The $[18, 9, 4]$ self–dual codes

As for the case of the isodual codes of length 16, there exists a large number of even isodual codes with parameters $[18, 9, 4]$. We will therefore analyze only the two self–dual codes H18 and I18 [21]. The weight enumerators of these codes are

$$A_{H18}(z) = 1 + 9z^4 + 75z^6 + 171z^8 + 171z^{10} + 75z^{12} + 9z^{14} + z^{18},$$
$$A_{I18}(z) = 1 + 17z^4 + 51z^6 + 187z^8 + 187z^{10} + 51z^{12} + 17z^{14} + z^{18}.$$

The weight hierarchies of H18 and I18 are known to be [3]

$$\mathbf{d}(H18) = \{4, 6, 9, 11, 12, 14, 16, 17, 18\},$$
$$\mathbf{d}(I18) = \{4, 6, 7, 10, 11, 14, 16, 17, 18\}.$$

It is straightforward to see that for I18, $d_2 = 6$ and $d_3 = 7$. We will present next a new argument that shows that for this code $d_4 = 10$. Assume on the contrary that for I18 $d_4 = 8$, i.e. let the unique, self–dual $[8, 4, 4]$ code be a subcode of I18. If the past subcode of I18 is a $[8, 4, 4]$ code, then the future subcode is is a $[10, 5, 4]$ code. The $[10, 5, 4]$ code has to be self–orthogonal, and therefore self–dual. But there is no self–dual code of length $n = 10$ and minimum distance $d = 4$ [22]. Therefore, $d_4 \geq 9$. Next, assume that $d_4 = 9$. In this case, the past and future subcodes of I18 at position nine are $[9, 4, 4]$ codes. This implies that the number of $[9, 4, 4]$ subcodes of I18 is even. But the Kløve identities show that $A_9^{(4)} \leq 1$, and hence $A_9^{(4)} = 0$. The desired result now follows from the strict monotonicity property of higher weights and Wei's duality relation.

The support weight enumerators of H18 and I18 are given in Table 11 and Table 12. Equations (8), (9) and (10) did not provide sufficient information for computing all the unknown values in the complete coset weight distribution.

5.5 The unique $[18, 9, 6]$ QR code

The extended quadratic residue code QR$[18, 9, 6]$ [18] is a unique, even, isodual code with these parameters. The weight enumerator of this code is

$$A(z) = 1 + 102z^6 + 153z^8 + 153z^{10} + 102z^{12} + z^{18},$$

and its weight hierarchy is $\mathbf{d} = \{6, 9, 11, 12, 14, 15, 16, 17, 18\}$.

Lemma 6: The support weight enumerators of the QR$[18, 9, 6]$ code are listed in Table 13.

Proof: Let a codeword of weight six be the past subcode of QR$[18, 9, 6]$. The future punctured subcode at position six is then a $[12, 8, \geq 3]$ code. Since a $[12, 8, 4]$ code does not exist [13], the minimum distance of the punctured future code is exactly equal to three. The dual of this punctured future code is a $[12, 4, 6]$ code, and this code is unique [25]. Therefore, the punctured future code must have the weight enumerator $1 + 16z^3 + 39z^4 + \ldots$. This establishes $d_2 = 9$ and $A_9^{(2)} = 16A_6/3 = 544$. The value of $A_9^{(2)}$ provides sufficient information for determining all the support weight enumerators of the code. ■

Since the number of non–zero weights in QR$[18, 9, 6]$ is five and the code is isodual, its codewords of any fixed weight support a 2-design. This information gives rise to a coset structure as given in Table 14. Solving the system in Equation (8) one gets three equations for w_2, w_3, w_4:

$$w_2 = -3w_1 - 3w_6 - 6w, \;\; w_3 = 136 + 3w_1 + 8w_6 + 8w, \;\; w_4 = 102 - w_1 - 6w_6 - 3w. \qquad (23)$$

The non–negativity constraint on the w_i's implies that $w_1 = w_2 = w_6 = w = 0$, $w_3 = 136$ and $w_4 = 102$. We notice that this is an example of a code for which the coset distributions can be completely determined based only on the weight enumerator of the code and non–negativity conditions on the w_i's. The second weight enumerator, for example, gives two additional equations, $w_1 = 0$, $w_6 = w$, which confirm the previous result. The coset weight distributions of the QR$[18, 9, 6]$ code were first computed in [26], using other techniques.

5.6 The formally self–dual $[20, 10, 6]$ codes

There exist exactly seven formally self–dual, even, extremal $[20, 10, 6]$ codes. None of these codes is self–dual, but all of them are isodual [8]. The weight enumerator of these codes is

$$A(z) = 1 + 90z^6 + 255z^8 + 332z^{10} + 255z^{12} + 90z^{14} + z^{20}.$$

The $[20, 10, 6]$ codes represent an interesting example for two reasons. One reason is that they contain subcodes with certain properties that allow us to reduce the number of unknowns in the Kløve identities. The other is that they provide examples of inequivalent codes with the same support weight enumerators and complete coset weight distribution.

Let us start by observing that when applied to these codes, the generalized Griesmer bound gives $d_2 \geq 9$ and $d_3 \geq 11$. Hence, there are at most two unknowns in the support weight enumerators of these codes - namely $A_9^{(2)}$ and $A_{10}^{(2)}$ (since $A_{10}^{(3)} = 0$). Consider a $[6, 1, 6]$ past subcode of any $[20, 10, 6]$ code. Since none of the codes contains codewords of weight six in the hull [8], the future subcode in the dual is a $[14, 5, 6]$ code that does not contain the codeword $\mathbf{1}$, and whose dual distance is at least three. MacWilliams identities show that there exist at most two different weight enumerators for such codes. These weight enumerators are listed in Table 15.

If the past subcode is a $[6, 1, 6]$ code, the future punctured code is one of the codes $E_1^{\perp}$ and $E_2^{\perp}$ whose weight enumerators are shown in Table 15. Let the number of different E_i subcodes of a $[20, 10, 6]$ code be e_i, $i = 1, 2$. Then $A_6 = 90 = e_1 + e_2$ holds, where A_6 is the number of codewords of weight six in the $[20, 10, 6]$ codes. Also, based on the number of codewords of weight three in $E_1^{\perp}$ and $E_2^{\perp}$, it follows that

$$\begin{aligned} A_9^{(2)} &= \frac{1}{3}\left(8e_1 + 10e_2\right) = \frac{2}{3}\left(4e_1 + 5e_2\right) \\ &= \frac{2}{3}(3(e_1 + e_2) + e_1 + 2e_2) = \frac{2}{3}(3 \cdot A_6 + e_1 + 2e_2), \end{aligned}$$

since adjoining a word of weight three from a future punctured $[14, 9, \geq 3]$ code to the $[6, 1, 6]$ subcode produces a $[9, 2, 6]$ subcode, which is counted exactly three times. The last equation can be rewritten as

$$e_1 + 2e_2 = \frac{3}{2} A_9^{(2)} - 3A_6. \tag{24}$$

Based on the number of codewords of weight four in $E_1^{\perp}$ and $E_2^{\perp}$ one has

$$A_{10}^{(2)} = \frac{1}{2}(42e_1 + 34e_2) = 21e_1 + 17e_2\,. \tag{25}$$

The last equality in (25) can be further expanded as

$$\begin{aligned} 4(5e_1 + 4e_2) + (e_1 + e_2) &= 4(6\,A_6\, - (e_1 + 2e_2)) + A_6 \\ &= 25\,A_6 - 4\,(e_1 + 2e_2). \end{aligned} \tag{26}$$

Substituting (24) into the last equation gives

$$A_{10}^{(2)} = 25\,A_6 - 4\left(\frac{3}{2}\,A_9^{(2)} - 3\,A_6\right) = 37\,A_6 - 6\,A_9^{(2)} = 3330 - 6\,A_9^{(2)}. \tag{27}$$

Therefore, there is only one unknown parameter in the Kløve identities for these codes. A similar type of result exists for five of the $[14, 7, 4]$ codes that do not contain codewords of weight four in the hull, namely $A_6^{(2)} = 70 - 3\,A_7^{(2)}$. The coefficients of the third weight enumerator of the five doubly–even, self–dual $[32, 16, 8]$ codes also satisfy an equation of this type (see the companion paper [19]).

In Table 16 we list the only unknown parameters in the Kløve identities for these codes. Based on Table 16, one can notice that all six formally self–dual $[20, 10, 6]$ codes with dimension of the hull equal to two have identical support weight enumerators. Even more interesting is the fact that among the first six codes, three have the same set of coset weight distributions as well. Codes fsd20-2a,c,e have a coset structure as shown in Table 17. In [8] it was shown that the codes fsd20-2e,c share the same automorphism group order (equal to $2 \cdot 5 = 10$), the same distribution of the intersection table, and the same weight enumerator of the hull. On the other hand, only one of the fsd20-2e,c codes is pure double–circulant [10]. Hence, since there exist two non–isomorphic groups of order 10, namely the cyclic group C_{10} and dihedral group D_5, one of the fsd20-2e,c codes has C_{10}, and the other D_5 as its automorphism group.

5.7 The self–dual $[22, 11, 6]$ code

There are 25 inequivalent self–dual codes of length 22, and only one of these codes (the G_{22} code according to the notation in [21]) has minimum distance $d = 6$. In [3] it was shown that the G_{22} code has the following weight hierarchy: $\mathbf{d}(G_{22}) = (6, 10, 12, 14, 15, 16, 18, 19, 20, 21, 22)$. We will present next an alternative proof of this result.

If the past subcode of G_{22} is a $[6, 1, 6]$ code, the future subcode is a $[16, 6, 6]$ code that contains the codeword $\mathbf{1}$ and has minimum dual distance ≥ 3. Hence, there exists only one solution for the weight enumerator of this future subcode, namely $A(z) = 1 + 16z^6 + 30z^8 + 16z^{10} + z^{16}$. The dual of this code is even and has 60 codewords of weight four, so that this result gives $d_2 = 10$. Since the future punctured code has minimum distance equal to four, any two of its codewords can overlap in at most two positions, so that $d_3 \geq 12$. That $d_3 = 12$ can be deduced from the following argument. The maximum number of codewords of weight four in a constant weight code of length 16 and minimum distance $d = 6$ is 20 [13]. Since the future punctured code at position six has 60 codewords of weight four, it must contain at least two codewords of weight four that overlap in exactly two positions. When adjoined to the $[6, 1, 6]$ subcode, the two codewords of G22 containing these codewords of weight four in the future punctured code form a $[12, 3, 4]$ code. This proves the claimed result concerning d_3. Applying Wei's duality principle, one obtains the complete higher weight enumerator for the $[22, 11, 6]$ code as given above.

Based on the Kløve identities and Theorem 2, all support weight enumerators of G22 can be determined. They are listed in Table 18.

6 Conclusion

In this paper, we addressed the problem of determining the support weight enumerators of isodual codes. Using invariant theory, we showed that the support weight enumerators can be completely determined based only on the knowledge of coefficients of order up to half of the code length. We also demonstrated several methods that can be used to compute these unknown coefficients.

Since the support weight enumerators contain information about the intersections of supports of codewords, there exist relationships between them and the set of coset weight distributions. We derived a formula relating the coset weight distributions and the support weight enumerators up to order three and used it to compute partial information about the coset weight distributions for several codes. In the course of our analysis we observed that inequivalent codes can have the same support weight enumerators and the same complete coset weight distribution. These are, to the best of our knowledge, the first known examples of this kind.

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