Asymptotic Spectra of Trapping Sets in Regular and Irregular LDPC Code Ensembles *

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Abstract

We address the problem of evaluating the asymptotic, normalized distributions of a class of combinatorial configurations in random, regular and irregular binary low-density parity-check (LDPC) code ensembles. Among the configurations considered are trapping and stopping sets[1]; these sets represent induced subgraphs in the Tanner graph of a code that, for certain classes of channels, exhibit a strong influence on the height and point of onset of the error-floor. The techniques used in the derivations are based on large deviation theory and statistical methods for enumerating random-like matrices. These techniques can also be applied in a setting that involves more general structural entities such as subcodes and/or minimal codewords, which are known to characterize other important properties of soft-decision decoders of linear block codes.

1 Introduction

The performance and complexity of many well-known classes of decoding algorithms are influenced to a large extent by combinatorial properties of the underlying linear block codes. An example includes bounded-distance decoding, the performance of which is completely determined by the weight distribution of the code and the transition parameter of the Binary Symmetric Channel (BSC). Similarly, the performance and complexity of certain soft-decision decoding algorithms for codes used over the Additive White Gaussian Noise (AWGN) channel can be characterized in terms of their *minimal codewords* (i.e. codewords with supports that do not contain the support of any other codeword) [1, 5] and the minimal codewords of their dual codes. Furthermore, for the Binary Erasure Channel (BEC), iterative decoding techniques fail if the errors are confined to subsets of variable nodes forming a *stopping set* [4], while for the AWGN channel a similar phenomena can be observed with respect to *trapping sets* [6], [15].

The problem of enumerating and classifying all the aforementioned and other combinatorial entities for a given code is usually intractable. It is therefore of interest to find probabilistic methods that would allow for computing the *average* of a chosen set of parameters for a given code ensemble, which can then be used to describe the average performance of the ensemble as well as the performance of a randomly chosen code from the given category.

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[1] The problem of finding the asymptotic stopping set distribution was also addressed in [14], but for a different class of code ensembles

In this paper, we will be concerned with random LDPC code ensembles and with combinatorial structures that bear an influence on the *error-floor* characteristics of these codes. LDPC codes are known for their excellent performance at low to medium signal-to-noise ratio (SNR), but in certain cases these codes tend to exhibit saturation levels in the Bit Error Rate (henceforth, BER) curve for sufficiently high SNRs. This saturation in the BER curve is known as the *error-floor* of the code. For a large class of codes, including those based on Cayley graphs [17] or certain types of designs, error-floors exist for relatively low BERs. This prohibits the use of such codes in storage and deep-space communication systems, which require extremely reliable signalling techniques with BERs below 10^{-15}. For many other codes, error-floors cannot be observed directly, because they are out of reach of standard simulation techniques [20]. In order to assess the properties of such codes, one has to utilize error-floor approximation techniques based on *stopping* and *trapping-set* classification. Stopping sets were introduced in [4], while the more general notion of a trapping sets (sometimes referred to as a *near-codeword*) was first described in the context of error-floor analysis in [11]. The latter concept was further developed in the seminal paper [15], in which trapping set configurations were shown to influence the point of onset as well as the slope of the error-floor curve of LDPC codes. The method described in [15] relies on a computer-search strategy and may not lead to the identification of all possible trapping sets.

This paper describes a novel and conceptually simple method for analyzing the asymptotic behavior of the distribution of both trapping and stopping sets, based on enumeration techniques described in [9] and large deviation theory [3], [2]. The latter problem was previously analyzed in [14], by using involved combinatorial methods. The techniques described in this work also provide a simple framework for evaluating the distributions of other, more complicated, combinatorial entities. The main contributions of the paper consist in identifying a universal method for analyzing configurations in LDPC code ensembles and in deriving novel results concerning trapping set distributions in random, regular and irregular LDPC code ensembles. The obtained results indicate a strong dependence of the asymptotic behavior of the number of such sets on the column and row weights of the ensemble, and the rates of the underlying code ensembles.

The paper is organized as follows. The terminology and definitions needed for subsequent derivations are given in Section 2. Section 3 contains the main results: the asymptotic distribution of trapping sets of constant size and size linear in the length of the codes for regular and irregular LDPC code ensembles. Section 4 provides a simplified proof for a set of results in [14] related to stopping sets. Section 5 contains numerical results complementing the analytical formulas derived in Section 4, as well as a description of the expected error-floor properties of the investigated code ensembles.

2 Definitions and Terminology

2.1 Stopping and Trapping Sets

Let H be the parity check matrix of an $[n,k,d]$ LDPC code, and let $G(H)$ denote its corresponding bipartite Tanner graph. The columns of H are indexed by variable (left-hand side) nodes $\mathcal{V}$ of $G(H)$, while the rows of H are indexed by check (right-hand side) nodes of $G(H)$. We are concerned with the following subsets of $\mathcal{V}$ that play an important role in decoding LDPC codes over the Binary Erasure Channel (BEC) and Additive White Gaussian Noise (AWGN) Channel.

Definition 2.1 A *stopping set* $\mathcal{S}_a(n)$ is a subset of a nodes in $\mathcal{V}$, for which the induced subgraph in $G(H)$ contains no check nodes of degree one.

Definition 2.2 A *general* $((a,b))$ *trapping set* $\mathcal{T}_{a,b}$ is a configuration of a nodes from $\mathcal{V}$, for which the induced subgraph in $G(H)$ contains $b>0$ odd-degree check nodes. An *elementary* $((a,b))$ *trapping set* is a trapping set for which all check nodes in the induced subgraph have either degree one or two, and there are exactly b degree-one check nodes. The sets of all $((a,b))$ general and elementary trapping sets in a Tanner graph of a code with parity check matrix H will be denoted by $\Omega_{a,b}(H)$ and $\Phi_{a,b}(H)$, respectively. Of special interest are trapping sets for which both a and b are relatively small.

Two examples of the incidence structures corresponding to elementary $((5,1))$ and $((3,9))$ trapping sets of a $(3,*)$ regular LDPC code of length $n \geq 30$ are shown below. The variables v_i and c_i represent the indices of the variable and check nodes in the induced subgraph.

The role of stopping sets in iterative decoding of codes used over the BEC channel is well-understood, and a detailed description of the connection between stopping sets and both the bit and frame error-probability can be found in [4]. For more general classes of binary-input, symmetric output channels, the influence of trapping sets on the onset of error-floors in LDPC codes is associated with the following phenomena arising both due to the properties of the code graph and decoding algorithm, as well as to the realization of certain special channel-noise configurations.

	v_1	v_2	v_3	v_4	v_5
c_1	1	0	0	0	0
c_2	**1**	**1**	0	0	0
c_3	**1**	0	**1**	0	0
c_4	0	**1**	0	**1**	0
c_5	0	1	0	0	1
c_6	0	0	**1**	**1**	0
c_7	0	0	1	0	1
c_8	0	0	0	1	1

	v_1	v_2	v_3
c_1	1	0	0
c_2	1	0	0
c_3	1	0	0
c_4	0	1	0
c_5	0	1	0
c_6	0	1	0
c_7	0	0	1
c_8	0	0	1
c_9	0	0	1

At the first several iterations of belief-propagation, due to the presence of special low-probability noise samples, variable nodes internal to one particular trapping set experience a large increase in the reliability estimates for incorrect bit values. This information gets "propagated" to other variable nodes in the trapping set, some of which already have very low-reliability channel values themselves. After this initial increase in the reliability, external variables usually start adjusting the incorrect estimates towards the correct values. But by that time, the variable nodes in a trapping set may already have significantly biased their decisions towards the incorrect values. Since there are very few check nodes capable of detecting errors within the trapping set, this unreliable information remains "trapped" within this set of variables until the end of the decoding process.

The exact relationship between the frame error probability and the distribution of trapping sets is still not well understood. It is conjectured [15] that the frame error rate (FER) of a code can be represented as

$$FER \approx \sum_{a,b} \sum_{\mathcal{T}_{a,b}} P\{\xi_{\mathcal{T}_{a,b}}\}, \tag{1}$$

where $\xi_{\mathcal{T}_{a,b}}$ denotes the set of decoder inputs that lead to a decoding failure on the trapping set $\mathcal{T}_{a,b}$; clearly, this set varies with the channel parameters. Although, at this point, the probabilities $P\{\xi_{\mathcal{T}_{a,b}}\}$ cannot be characterized analytically, there exist numerical methods for estimating their values [15]. Therefore, conditioned on the knowledge of these probabilities, the frame error rate is completely determined by the distribution of the trapping sets $\mathcal{T}_{a,b}$, for

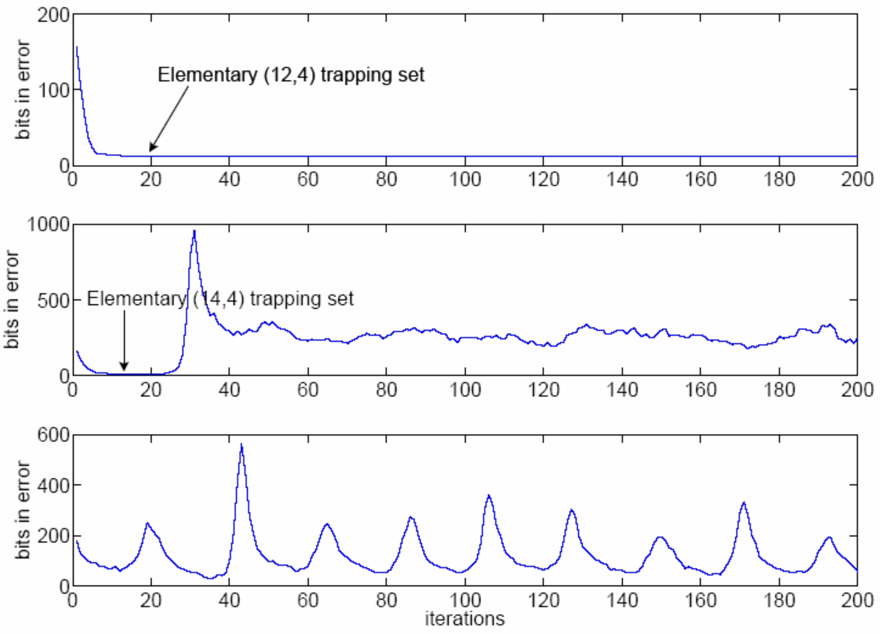

Figure 1: Frames with trapping set errors, standard belief propagation.

all *relevant* values of the parameters a and b. Usually, the smaller the trapping set (i.e. the smaller the values of a and b, with $a > b$), the stronger the influence of the set on the frame error rate. Nevertheless, there exist examples of trapping sets which bear a strong influence on the error-floor, for which a is relatively large, and b exceeds the value a [8]. Also, in some instances of belief propagation decoding, an $((a,b))$ trapping set will provide a significant contribution to the frame error rate, while an $((a,b))$ trapping set with $b' < b$ will have almost no impact on the code's performance. As an example, three different types of trapping sets observed during belief propagation decoding of a Margulis-type LDPC code [8] are shown in Figure 1. The number of erroneous variables is plotted with respect to the number of iterations of product-sum decoding. Here, the decoding algorithm gets "trapped" in a $((12,4))$ elementary trapping set (Figure 1 (a)), or in a $((14,4))$ trapping set (Figure 2(b)). In the later case, errors spread from the initial trapping set to a large number of variable nodes within 10-15 iterations.

The findings described above motivate the study of the most general class of $((a,b))$ trapping sets without any restrictions imposed on the relationship between the values of a and b. The distribution of trapping sets with more restrictive properties can be deduced from the general results.

2.2 Code Ensembles

Throughout the paper, we use $\Lambda_{m,n}$ and $\Lambda_{m,n}^{\mathbf{c},\mathbf{r}}$ to denote the ensembles of binary parity-check matrices H of dimension $m \times n$, for which c and r can take any positive integer value and for which the column and row weight distributions are given by the ordered vectors $\mathbf{c} = (c_1, \dots, c_n)$ and $\mathbf{r} = (r_1, \dots, r_m)$, respectively. The probability distributions over these random ensembles are uniform. If $c_i = c$, $\forall\, i$, and $r_i = r$, $\forall\, i$, the underlying ensemble $\Lambda_{m,n}^{c,r}$ is termed a *regular ensemble*. Otherwise, the ensemble is referred to as an *irregular ensemble*. In the former case, we use the following notation

$$0 < \zeta = \frac{c}{r} = \frac{m}{n} < 1,$$

and tacitly assume that $r \geq 2$. In our analysis we also consider a class of irregular matrix ensembles, denoted by $\mathcal{H}_{m,n}^{\mathbf{c},\mathbf{r}} \subset \Lambda_{m,n}^{\mathbf{c},\mathbf{r}}$ with the following set of properties. First, the rows and columns of any given matrix in $\mathcal{H}_{m,n}^{\mathbf{c},\mathbf{r}}$ are *ordered according to their type*: this means that the set of columns (rows) of a matrix in the ensemble are partitioned according to their weight, and where in every subset of the partition the vectors have the same weight. An example of a parity check matrix from an ensemble $\mathcal{H}_{m,n}^{\mathbf{c},\mathbf{r}}$ with $m = 5, n = 8$, $\mathbf{c} = \{3,3,3,2,2,1,1,1\}$ and $\mathbf{r} = \{4,4,4,3,1\}$ is shown below

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{H}_{m,n}^{\mathbf{c},\mathbf{r}}. \tag{2}$$

More formally, we define two sets of rational numbers in the interval $(0,1]$, namely $\kappa_1, \ldots, \kappa_h$ and $\chi_1, \ldots, \chi_g$, with the following set of properties. The first is the normalization property,

$$\sum_{i=1}^{h} \kappa_i = \sum_{i=1}^{g} \chi_i = 1,$$

where $g = \max\ r_i$, and $h = \max\ c_i$. The numbers κ_i and χ_i describe the proportion of columns (rows) of weight i in a matrix from the ensemble $\mathcal{H}_{m,n}^{\mathbf{c},\mathbf{r}}$. The second property is the enumerative property, which implies that $m\kappa_i$ and $n\chi_i$ are integers determining the number of different columns and rows of weight i in a matrix. Furthermore, we use the sets of integers s_i and t_i to denote the column and row weights, respectively, of the i-th strip of $\mathcal{H}_{m,n}^{\mathbf{c},\mathbf{r}}$. The principle of conservations of ones implies that $m \sum_i \kappa_i s_i = n \sum_j \chi_j t_j = N$, where N denotes the total number of ones in a parity-check matrix. Note that both classes of ensemble $\Lambda_{m,n}^{\mathbf{c},\mathbf{r}}$ and $\mathcal{H}_{m,n}^{\mathbf{c},\mathbf{r}}$ are identical to the ones considered in [9] and [10] for the purpose of evaluating the underlying codes average weight enumerators.

For the example in (2), $h = 3, g = 4$, while $\kappa_3 = 3/8, \kappa_2 = 1/4, \kappa_1 = 3/8$ and $\chi_4 = 3/5, \chi_3 = 1/5, \chi_2 = 0, \chi_1 = 1/5$.

Define next

$$\begin{aligned} E_{\theta,\lambda}(n) &\doteq \frac{1}{n} \log A_{a,b}(n), \\ T_{\theta,\lambda}(n) &\doteq \frac{1}{n} \log B_{a,b}(n), \end{aligned} \tag{3}$$

where

$$\begin{aligned} A_{a,b}(n) &= \frac{1}{|\Gamma_{m,n}|} \sum_{H \in \Gamma_{m,n}} |\ \mathbf{t} : \mathbf{t} \in \Omega_{a,b}(H)\ |, \\ B_{a,b}(n) &= \frac{1}{|\Gamma_{m,n}|} \sum_{H \in \Gamma_{m,n}} |\ \mathbf{t} : \mathbf{t} \in \Phi_{a,b}(H)\ |\ . \end{aligned} \tag{4}$$

In the expressions above, a and b stand for parameters of trapping sets that grow linearly with n, while $\Gamma_{m,n}$ denotes either the ensemble $\Lambda_{m,n}^{\mathbf{c},\mathbf{r}}$ or the ensemble $\mathcal{H}_{m,n}^{\mathbf{c},\mathbf{r}}$. Which one of the two ensembles is considered will be apparent from the context.

The central problem of this paper is to find asymptotic expressions for $E_{\theta,\lambda}(n)$ and $T_{\theta,\lambda}(n)$,

$$
\begin{aligned}
e(\theta,\lambda) &= \lim_{n\to\infty} E_{\theta,\lambda}(n), \\
t(\theta,\lambda) &= \lim_{n\to\infty} T_{\theta,\lambda}(n),
\end{aligned}
\tag{5}
$$

which we will alternatively refer to as the asymptotic *distribution* or *spectrum* of elementary and general trapping sets, respectively.

2.3 Large Deviation Theory and Random Matrix Enumeration

The task of computing the trapping set spectra is greatly simplified by using techniques borrowed from the field of large deviation theory. For a detailed treatment of the subject of large deviation theory, the interested reader is referred to [3]. For reasons of clarity, we will describe next several important results from this branch of probability, including *Sanov's theorem* [18], [2, p.292] and its extensions, as well as Varadhan's lemma [3]. Before stating these two important results, we need to define the following probabilistic notions. These notions are assembled from the exposition in [2], Chapter 12.

Definition 2.3 The type (i.e. empirical distribution) $P_{\mathbf{x}}$ of a sequence $\mathbf{x}$ over an alphabet $\mathcal{Y}$ is the set of relative frequencies of symbols in $\mathcal{Y}$ contained in the sequence. The relative frequencies are rational number, and $\mathcal{P}_N$ denotes the set of types for which the denominator of all the frequencies is N.

Definition 2.4 The type class of an empirical distribution is defined as

$$\Theta(P) = \{\mathbf{x} \in \mathcal{Y}^N : P_{\mathbf{x}} = P\}. \tag{6}$$

For a given subset Δ of the set of probability mass functions, $Q^N(\Delta)$ is used to denote $\sum_{\mathbf{x}:\, P_{\mathbf{x}} \in \Delta\cap\mathcal{P}_N} Q^N(\mathbf{x})$, where $Q^N(\mathbf{x}) = \prod_{i=1}^{N} Q(x_i)$.

Theorem 2.1 (Sanov's theorem) Let $\{X_1,\ldots,X_n\}$ be i.i.d random variables with probability mass function $Q(x)$ over a bounded alphabet of K elements. Let $\mathcal{F} \subset \mathcal{P}$ be a set of probability distributions. Then

$$Q^n(\mathcal{F}) = Q^n(\mathcal{F}\cap\mathcal{P}) \leq (n+1)^K\, 2^{-nD(P^*||Q)}, \tag{7}$$

where

$$P^* = \arg\min_{P\in\mathcal{F}} D(P||Q),$$

and

$$D(P||Q) = \sum_i p_i \log(p_i/q_i)$$

denotes Shannon's relative entropy. Furthermore, if $\mathcal{F}$ is the closure of its interior, then

$$\frac{1}{n}\log Q^n(\mathcal{F}) \to -D(P^*||Q).$$

A straightforward corollary of the theorem is a set of rules that allows one to find the asymptotic behavior of the normalized logarithm of the probability of a union of events of the form $\{\frac{1}{n}\sum_{i=1}^{n} g_j(X_i) \geq \alpha_j, j = 1,2,...,t\}$, for a given set of functions g_j of i.i.d random variables X_i, $i = 1,\ldots,n$. This can be accomplished by defining a set of probability distributions

$$Z = \{P : \sum_{f\in\mathcal{F}} P(f)g_j(f) \geq \alpha_j,\ j = 1,\ldots,t\}, \tag{8}$$

and then finding the distribution in Z minimizing the relative entropy $D(P||Q)$. The minimum of the relative entropy function represents, according to Sanov's theorem, the growth exponent of the probability under investigation. This growth exponent is usually referred to as the *rate function* of the given probability, and is denoted by I.

In the derivations pertaining to the average trapping set spectra of irregular codes, one encounters certain summations for which the asymptotic expressions can be found by invoking *Varadhan's lemma* [3]. This important result in large deviation theory can be stated as follows.

Theorem 2.2 (Varadhan's lemma) Let $P_n(x) = P\{S_n > x\}$ denote a sequence of probability distributions satisfying the large deviation principle with rate function $I(x)$, that is, a function I such that

- $P_n(x) \simeq e^{-nI(x)}$ for $x > m$ or $x < m$, where m denotes the mean of the variable S_n;
- The rate function $I(x)$ is lower semi-continuous;
- For each real number a, the set $\{x \in \mathcal{R} : I(x) \leq a\}$ is compact;
- For each closed subset F of $\mathcal{R}$,

$$\lim_n \sup \frac{1}{n} \ln P\{S_n \in F\} \leq - \inf_{x \in F} I(x).$$

- For each open subset G of $\mathcal{R}$,

$$\lim_n \inf_n \frac{1}{n} \ln P\{S_n \in G\} \geq - \inf_{x \in G} I(x).$$

Then, provided that $g(x)$ is continuous and bounded,

$$\lim_{n \to \infty} \frac{1}{n} \ln E[e^{ng(x)}] = \max_x \left[g(x) - I(x)\right]. \tag{9}$$

Another important component of the analysis of trapping sets is related to the structure of the random code ensemble. With respect to this issue, one is concerned with asymptotic enumeration techniques for matrices with prescribed row and column weight profiles. The following result on the number of zero-one matrices with such weight distribution properties will be used in several subsequent derivations [6]. It represents a strengthening of the theorems in [7] and [13] that allows the claims to hold true for a significantly larger range of parameters.

Theorem 2.3 Let $\mathcal{N}_{\mathbf{c},\mathbf{r}}$ denote the number of square zero-one matrices of dimension n, with row and column weight distributions $\mathbf{r}$=$(r_1,\ldots,r_m)$ and $\mathbf{c}$=$(c_1,\ldots,c_m)$, respectively. Furthermore, let g and h denote the largest row and column weight, respectively, and let $S = \sum_i r_i = \sum_i c_i$. If $n \to \infty$ and $1 \leq D = \max(g,h) = O(n^{1/4})$, then

$$\mathcal{N}_{\mathbf{c},\mathbf{r}} = \frac{S!}{\prod_{i=1}^m r_i! \, \prod_{j=1}^n c_j!} \exp\left(-\frac{1}{2S^2} \sum_{i=1}^m r_i(r_i-1) \sum_{i=1}^m c_i(c_i-1) + O(D/S)\right).$$

Observe that the corresponding theorem in [13] used for finding the asymptotic weight enumerators of LDPC codes pertains to a maximum column/row weight bounded from above by $n^{1/4}$, rather than $\log n$. Furthermore, it can be easily shown that the same type of enumeration result holds for matrices that are not necessarily square, but of dimension $m \times n$, where m/n is some constant.

As a final remark, throughout the paper, all logarithms will are taken with respect to the natural base e.

3 Asymptotic Distribution of Trapping Sets in Regular Code Ensembles

Here we find the asymptotic distribution of elementary and general trapping sets in the ensemble of matrices $\Lambda^{c,r}_{m,n}$. Although the described approach represents a special instance of the derivations concerning irregular ensembles, a separate proof is provided in order to highlight the different ways in which large deviation theory is utilized. The corresponding results are presented in theorems 3.1 and 3.4, respectively. The main components of the proof can be summarized as follows. First, a set of combinatorial conditions is imposed on the structure of random-like matrices containing a trapping set with parameters $((a,b))$. Matrices of this form are enumerated using theorem 2.3. The result of this enumeration process is recognized as a constrained sum which represents a probability function. This finding allows for a simple deduction of the asymptotic behavior of the sum in terms of large deviation theory. The use of Sanov's theorem mitigates the need for demonstrating the uniqueness of the underlying solutions and for proving other relevant properties of the sum, due to the fact that the minimization task is performed over a relative entropy function.

3.1 Elementary Trapping Sets

The next theorem and corollary establish the asymptotic behavior of elementary trapping sets in regular LDPC code ensembles.

Theorem 3.1 Let $\zeta < 1$, and let $0 < \theta, \lambda < 1$. The asymptotic average ensemble distribution $e(\theta,\lambda)$ of $((a = \theta n, b = \lambda n))$ elementary trapping sets in a code from the ensemble $\Lambda^{c,r}_{m,n}$ with constant values c, r is given by

$$H\left(1-\zeta,\lambda,\frac{c\theta-\lambda}{2},\frac{2\zeta-c\theta-\lambda}{2}\right)-H(\zeta,1-\zeta)-(c-1)H(\theta,1-\theta)+\frac{c\theta-\lambda}{2}\log\binom{r}{2}+\lambda\log\binom{r}{1},$$

where $H(p_1,\ldots,p_N) = -\sum p_i \log p_i$, with $\sum_i p_i = 1$, denotes Shannon's (natural) entropy.

Proof. The proof represents a generalization of the arguments used in [9] for evaluating average weight distributions of code ensembles. For reasons of simplicity, we will work with transposes of parity-check matrices, H^T.

Let us write

$$H^T = \begin{bmatrix} M_u \\ \hline M_l \end{bmatrix}, \tag{10}$$

where M_u is an $a \times m$ and M_l is an $(n-a) \times m$ zero-one matrix. Let the column weights of M_u be $w_1, \ldots, w_m$, and let α_j denote the number of columns of weight j in the matrix M_u:

$$\alpha_j = |\{i : w_i = j, i = 1, \ldots, a\}|. \tag{11}$$

If $w_i \in \{0,1,2\}$, i.e., if all columns of M_u have weight zero, one, or two, then the rows of M_u correspond to variables of an elementary $((a, b = \alpha_1))$ trapping set.

Define

$$\tilde{\Lambda}^{\zeta,r}_{a,b,m} \doteq \left\{ M_u \in \Lambda_{a,m} : \sum_j M_u(i,j) = c,\, \alpha_3 = \alpha_4 = \ldots = \alpha_a = 0 \right\},$$

$$\hat{\Lambda}^{\zeta,r}_{a,b,m} \doteq \left\{ M_l \in \Lambda_{n-a,m} : \sum_j M_l(i,j) = c,\, \sum_i M_l(i,j) = \beta_j \right\},$$

where $\beta_j \in \{r, r-1, r-2\}$, and exactly α_0 values of β_j are equal to r, exactly α_1 values are equal to $r-1$ and the remaining values are equal to $r-2$. Furthermore, let

$$\mathcal{L}^{\zeta,r}_{a,b,m} \doteq \left\{ H \in \Lambda^{c,r}_{m,n} : M_u \in \tilde{\Lambda}^{\zeta,r}_{a,b,m}, M_l \in \hat{\Lambda}^{\zeta,r}_{a,b,m} \right\}. \tag{12}$$

Note that by going over all column orderings,

$$| \mathcal{L}^{\zeta,r}_{a,b,m} | = \sum \binom{\zeta n}{\alpha_0 \ \alpha_1 \ \alpha_2} | \tilde{\Lambda}^{\zeta,r}_{a,b,m} || \hat{\Lambda}^{\zeta,r}_{a,b,m} |, \tag{13}$$

where the sum is taken over all triples $(\alpha_0, \alpha_1, \alpha_2)$ such that

$$\alpha_0 + \alpha_1 + \alpha_2 = m = \zeta n, \quad \alpha_1 + 2\alpha_2 = ac = \theta cn. \tag{14}$$

The probability $P^{\zeta,r}_{a,b,n}$ that the transpose of a parity-check matrix H takes the form (10) with condition (12) is given by

$$P^{\zeta,r}_{a,b,n} = \frac{| \mathcal{L}^{\zeta,r}_{a,b,m} |}{| \Lambda^{c,r}_{m,n} |}. \tag{15}$$

From theorem 2.3 it follows that

$$| \tilde{\Lambda}^{\zeta,r}_{a,b,m} | = \frac{(ac)!}{c!^a \, 2!^{\alpha_2}} \exp\left(-\frac{(c-1)\alpha_2}{ac} \right) \left(1 + o\left((\zeta n)^{-1+\delta_1} \right) \right),$$

for some $\delta_1 > 0$. Similarly, one can also show that

$$| \hat{\Lambda}^{\zeta,r}_{a,b,m} | = \frac{((n-a)c)! \exp\left(-\frac{(c-1)}{(n-a)c} \left[\binom{r}{2} \zeta n - (r-1)ac + \alpha_2 \right] \right)}{(c!)^{n-a} \, r!^{\alpha_0} \, (r-1)!^{\alpha_1} \, (r-2)!^{\alpha_2}} \left(1 + o\left((\zeta n)^{-1+\delta_2} \right) \right),$$

for some $\delta_2 > 0$. The last expression is obtained by incorporating the constraining equations (14) into the sum

$$\alpha_0 r(r-1) + \alpha_1 (r-1)(r-2) + \alpha_2 (r-2)(r-3).$$

Because of the relations (14), for a given value of α_1, the sum in (13) contains only one term. This term is

$$M \times \frac{\exp\left(-\frac{(c-1)(ac-\alpha_1)}{2ac} - \frac{(c-1)}{(n-a)c} \left[\binom{r}{2} \zeta n - (r - \frac{3}{2})ac - \frac{1}{2}\alpha_1 \right] \right)}{\left(\frac{2\zeta n - ac - \alpha_1}{2} \right)! \, \alpha_1! \left(\frac{ac-\alpha_1}{2} \right)! \, 2!^{(ac-\alpha_1)/2} \, r!^{(2\zeta n - ac - \alpha_1)/2} \, (r-1)!^{\alpha_1} \, (r-2)!^{(ac-\alpha_1)/2}} \left(1 + o\left((\zeta n)^{-1+\delta} \right) \right),$$

where $\delta > 0$ and

$$M = \frac{(\zeta n)! \, (ac)! \, ((n-a)c)!}{c!^n},$$
$$\alpha_2 = \frac{ac - \alpha_1}{2}, \ \alpha_0 = \frac{2\zeta n - ac - \alpha_1}{2}.$$

Consequently, the probability of interest can be written as

$$\frac{(\zeta n)! \binom{r}{2}^{(ac-\alpha_1)/2} \binom{r}{1}^{\alpha_1}}{\binom{nc}{ac} \left(\frac{2\zeta n - ac - \alpha_1}{2} \right)! \, \alpha_1! \left(\frac{ac-\alpha_1}{2} \right)!} \exp\left(\frac{\zeta r(c-1)(r-1)n}{2nc} \right) \times$$
$$\exp\left(-\frac{(c-1)(ac-\alpha_1)}{2ac} - \frac{(c-1)}{(n-a)c} \left[\binom{r}{2} \zeta n - (r - \frac{3}{2})ac - \frac{1}{2}\alpha_1 \right] \right) \left(1 + o\left((\zeta n)^{-1+\delta} \right) \right).$$

The asymptotic behavior of $\log\left(P_{a,b,n}^{\zeta,r}\right)$ can be determined in two steps. First, it is straightforward to show that

$$\frac{1}{n}\log\left(P_{a,b,n}^{\zeta,r}\right) \simeq \frac{1}{n}\log\left(\frac{(\zeta n)!\,\binom{r}{2}^{(ac-\alpha_1)/2}\binom{r}{1}^{\alpha_1}}{\binom{nc}{ac}\left(\frac{2\zeta n-ac-\alpha_1}{2}\right)!\,\alpha_1!\,\left(\frac{ac-\alpha_1}{2}\right)!}\right). \tag{16}$$

Second, the asymptotic expression for the probability above can be obtained by invoking Stirling's approximation formula, $\log(n!) \simeq n\log n - n$, and the binomial approximation formula $\log\binom{N}{\theta N} \simeq NH(\theta,1-\theta)$, where $H(\theta,1-\theta)$ denotes Shannon's natural entropy.

Assume next that $a=\theta n$, $\alpha_1=\lambda n$, $0<\theta,\lambda<1$. Based on the previous results, one can obtain the following asymptotic formula for $\frac{1}{n}\log\left(P_{a,b,n}^{\zeta,r}\right)$:

$$\zeta\log\zeta + \frac{c\theta-\lambda}{2}\log\binom{r}{2} + \lambda\log\binom{r}{1} - cH(\theta,1-\theta)$$
$$-\lambda\log\lambda - \frac{c\theta-\lambda}{2}\log\left(\frac{c\theta-\lambda}{2}\right) - \frac{2\zeta-c\theta-\lambda}{2}\log\left(\frac{2\zeta-c\theta-\lambda}{2}\right),$$

where both $c\theta>\lambda$ and $2\zeta-c\theta>\lambda$. The last expression can be re-written as

$$H\left(1-\zeta,\lambda,\frac{c\theta-\lambda}{2},\frac{2\zeta-c\theta-\lambda}{2}\right) - H(\zeta,1-\zeta) - cH(\theta,1-\theta) + \frac{c\theta-\lambda}{2}\log\binom{r}{2} + \lambda\log\binom{r}{1}.$$

By adding the asymptotic formula for $\frac{1}{n}\log\binom{n}{\theta n}$ to the terms listed above, accounting for the possible choices of the $a=\theta n$ rows of H^T describing the trapping set, one obtains the claimed result. □

The same expression for the distribution of elementary trapping sets are valid for the case that either c or r, or both, grow sublinearly with n. As a result of this observation and theorem 2.3, analogue distribution results hold for *dense* matrix (code) ensembles, where the maximum row and/or column weights are $O(n^{1/4})$.

The result of theorem 3.1 allows for evaluating the most general class of configurations involving induced check degrees of value zero, one or two only. With respect to the problem of trapping set evaluation, one is usually interested in the case $\lambda << \theta$. By introducing $\varepsilon=\lambda/\theta$, one obtains the following result.

Corollary 3.1 For $\varepsilon << 1$ and $c,r\geq 3$ one has

$$e(\theta,\lambda) = H\left(1-\zeta,\frac{c\theta}{2},\frac{2\zeta-c\theta}{2}\right) - H(\zeta,1-\zeta) - (c-1)H(\theta,1-\theta) + \frac{c\theta}{2}\log\binom{r}{2},$$

which is independent of ε, and corresponds to the average ensemble distribution of codewords with induced check degrees equal to zero or two.

Based on the derivations presented in the previous theorem, one can also establish the asymptotic behavior of $P_{a,b,n}^{\zeta,r}$ for the case of constant parameters a and $b=\alpha_1$ (independent of n).

Corollary 3.2 For a constant value a for the size of an elementary trapping set with a constant number of b degree-one check nodes, one has

$$P_{a,b,n}^{\zeta,r} \simeq C\,n^{-(ac-b)/2}, \tag{17}$$

for some constant C independent on n.

Proof. The result follows from the observation that the only term in (16) contributing to the negative exponent of n is $(\zeta n)!/[\binom{nc}{ac}((2\zeta n-ac-\alpha_1)/2)!]$. A straightforward application of Stirling's formula establishes the desired result. □

3.2 General Trapping Sets

The next theorem and corollary establish the asymptotic behavior of the number of general trapping sets in regular LDPC code ensembles.

Theorem 3.4 Let $\zeta < 1$, and let $0 < \theta, \lambda < 1$. The average ensemble distribution $t(\theta,\lambda)$ of $((a = \theta n, b = \lambda n))$ trapping sets in a code from the ensemble $\Lambda^{c,r}_{m,n}$ with constant values c, r is given by

$$-(c-1)H(\theta,1-\theta) - \zeta \log \rho^{\theta r} - \zeta \left(1-\frac{\lambda}{\zeta}\right) \log\left(\frac{2(1-\lambda/\zeta)}{(1+\rho)^r+(1-\rho)^r}\right) - \zeta\frac{\lambda}{\zeta} \log\left(\frac{2\lambda/\zeta}{(1+\rho)^r-(1-\rho)^r}\right).$$

where ρ is the (unique) positive solution of the equation

$$\left(1-\frac{\lambda}{\zeta}\right) \frac{(1+\rho)^{r-1}+(1-\rho)^{r-1}}{(1+\rho)^r+(1-\rho)^r} + \frac{\lambda}{\zeta}\, \frac{(1+\rho)^{r-1}-(1-\rho)^{r-1}}{(1+\rho)^r-(1-\rho)^r} = 1-\theta.$$

Proof. In this case, the proof begins in a similar manner as the proof of theorem 3.1. The partition of the matrix H^T remains of the same form, except for the requirements that

$$\tilde{\Gamma}^{\zeta,r}_{a,b,m} \doteq \left\{ M_u \in \Lambda_{a,m} : \sum_j M_u(i,j) = c, \sum_i \alpha_{2i+1} = \lambda n \right\},$$

$$\hat{\Gamma}^{\zeta,r}_{a,b,m} \doteq \left\{ M_l \in \Lambda_{n-a,m} : \sum_j M_l(i,j) = c, \sum_i M_l(i,j) = \beta_j \right\},$$

where $\beta_j \in \{1,\ldots,r\}$ and exactly α_0 values of β_i are r, exactly α_1 values of β_i are $r-1$, and so on, exactly values of $\alpha_{\zeta n}$ are equal to 0. Furthermore, let

$$\mathcal{G}^{\zeta,r}_{a,b,m} \doteq \left\{ H \in \Lambda^{c,r}_{m,n} : M_u \in \tilde{\Gamma}^{\zeta,r}_{a,b,m}, M_l \in \hat{\Gamma}^{\zeta,r}_{a,b,m} \right\}.$$

The probability of interest takes the form

$$\tilde{P}^{\zeta,r}_{a,b,n} = \frac{\mid \mathcal{G}^{\zeta,r}_{a,b,m} \mid}{\mid \Lambda^{c,r}_{m,n} \mid},$$

and by using the same type of argument as described in the previous theorem, one arrives at the following formula for $\tilde{P}^{\zeta,r}_{a,b,n}$:

$$\tilde{P}^{\zeta,r}_{a,b,n} \overset{\log}{\simeq} \frac{1}{\binom{nc}{ac}} \sum \binom{\zeta n}{\alpha_0,\ldots,\alpha_r} \binom{r}{0}^{\alpha_0} \binom{r}{1}^{\alpha_1} \cdots \binom{r}{r}^{\alpha_r} \tag{18}$$

where

$$A \overset{\log}{\simeq} B$$

stands for $\log A \simeq \log B$, and where the sum is taken over all $\alpha_0,\ldots,\alpha_r$ such that

$$\sum_i \alpha_i = \zeta n, \;\; \sum_i i\alpha_i = \theta c n, \;\; \sum_i \alpha_{2i+1} = \lambda n. \tag{19}$$

We introduce next a simple method for evaluating the sum in (18) based on a set of results from large deviation theory [2]. Let

$$p_j \doteq 2^{-r}\binom{r}{j} \tag{20}$$

[2]This type of analysis can also be used to significantly simplify the proofs in [9] and [14] related to weight and stopping set enumerators

so that $\{p_j\}_{j=0}^{r}$ represents a probability vector. Multiplying both the numerator and denominator of the expression in (18) by $2^{r\zeta n}$, one obtains the following asymptotic formula for $\tilde{P}_{a,b,n}^{\zeta,r}$

$$\frac{2^{r\zeta n}}{\binom{nc}{ac}} \sum \binom{\zeta n}{\alpha_0, \ldots, \alpha_r} p_0^{\alpha_0} p_1^{\alpha_1} \ldots p_r^{\alpha_r}, \tag{21}$$

where the constraining equations (19) hold.

According to Sanov's theorem, the problem of estimating the probability in (21) reduces to finding a probability mass function q_i that minimizes $\sum q_i \log(q_i/p_i)$, such that

$$\sum_i q_i = 1, \ \sum_i i\, q_i = \theta r, \ \sum_i q_{2i+1} = \frac{\lambda}{\zeta}, \tag{22}$$

with probabilities p_i defined according to (20). By using the Lagrangian multiplier method, with multiplier function

$$\sum_i q_i \log \frac{q_i}{p_i} + \mu_1 \sum_i q_i + \mu_2 \sum_i i\, q_i + \mu_3 \sum_i q_{2i+1}, \tag{23}$$

one can show that the unique [3]optimizing distribution for both r even and r odd is of the form

$$\begin{aligned} q_i^* &= \frac{2\binom{r}{i}(1-\lambda/\zeta)\,\rho^i}{(1+\rho)^r + (1-\rho)^r}, \quad \text{for even values of } i, \\ q_i^* &= \frac{2\binom{r}{i}\,(\lambda/\zeta)\,\rho^i}{(1+\rho)^r - (1-\rho)^r}, \quad \text{for odd values of } i. \end{aligned} \tag{24}$$

Here, we used the following straightforward identities

$$(1+\rho)^r + (1-\rho)^r = 2\sum_i \binom{r}{2i}\rho^{2i}, \qquad \rho\left[r(1+\rho)^{r-1} - r(1-\rho)^{r-1}\right] = 2\sum_i 2i\binom{r}{2i}\rho^{2i},$$

$$(1+\rho)^r - (1-\rho)^r = 2\sum_i \binom{r}{2i+1}\rho^{2i+1}, \quad \rho\left[r(1+\rho)^{r-1} + r(1-\rho)^{r-1}\right] = 2\sum_i (2i+1)\binom{r}{2i+1}\rho^{2i+1}.$$

In all the above expression, ρ represents the (only) positive root of the equation

$$\left(1-\frac{\lambda}{\zeta}\right)\frac{(1+\rho)^{r-1} + (1-\rho)^{r-1}}{(1+\rho)^r + (1-\rho)^r} + \frac{\lambda}{\zeta}\,\frac{(1+\rho)^{r-1} - (1-\rho)^{r-1}}{(1+\rho)^r - (1-\rho)^r} = 1-\theta. \tag{25}$$

This shows that

$$\lim_{n\to\infty} \frac{1}{n} \log P_{a,b,n}^{\zeta,r} = r\zeta \log 2 - cH(\theta, 1-\theta) - \zeta D(q^*||p), \tag{26}$$

where

$$D(q^*||p) = \log \rho^{\theta r} + \left(1-\frac{\lambda}{\zeta}\right) \log\left(\frac{2^{r+1}(1-\lambda/\zeta)}{(1+\rho)^r + (1-\rho)^r}\right) + \frac{\lambda}{\zeta} \log\left(\frac{2^{r+1}\lambda/\zeta}{(1+\rho)^r - (1-\rho)^r}\right). \tag{27}$$

Again, by adding the asymptotic value of $\frac{1}{n}\log\binom{n}{\theta n}$ to the expression, accounting for the possible choices of the $a = \theta n$ variable nodes of the trapping set, one arrives at the claimed result. □

Corollary 3.3 For constant values $((a,b))$ of a trapping set, one has

$$C_1\, n^{-\frac{(r-1)}{r}(ac-b)} \leq P_{a,b,n}^{\zeta,r} \leq C_2\, n^{-(ac-b)/2}, \tag{28}$$

for some constants C_1 and C_2 independent on n.

[3]Uniqueness follows from the strict convexity of the relative entropy function with respect to the first argument, q

Proof. Without loss of generality, assume that r is even. In order to prove the claimed result, we observe that the constraining relations (19) imply that

$$\begin{aligned}
&\sum_i 2i\,\alpha_{2i} + \sum_i (2i+1)\alpha_{2i+1} = ca, \\
&\sum_i 2i\,\alpha_{2i} = ca - \sum_i (2i+1)\alpha_{2i+1} \leq c\,a - b, \text{ i.e} \\
&\sum_{i\neq 0} \alpha_{2i} \leq \frac{ca-b}{2}, \text{ so that } \sum_{i\neq 0} \alpha_i \leq \frac{ca+b}{2}.
\end{aligned} \tag{29}$$

Similarly, one can show that $\sum_{i\neq 0} \alpha_i \geq [ca + b(r-1)]/r$. Based on the previous expressions, it is straightforward to see that

$$\frac{\binom{r}{1}^{\alpha_1}}{\alpha_1\,!}\,\frac{\binom{r}{2}^{\alpha_2}}{\alpha_2\,!}\cdots\frac{\binom{r}{r}^{\alpha_r}}{\alpha_r\,!} \tag{30}$$

is bounded from above by a constant independent from n. Therefore,

$$P^{\zeta,r}_{a,b,n} \overset{\log}{\simeq} \frac{1}{\binom{nc}{a\,c}} \sum \binom{\zeta n}{\alpha_1 + \alpha_2 + \ldots + \alpha_r}, \tag{31}$$

where the sum contains only a constant number of terms. Inserting the upper and lower bounds yields the claimed result. □

As for the case of elementary trapping sets, the same distribution results hold for *dense* matrix (code) ensembles, where the maximum row and/or column weights are $O(n^{1/4})$.

4 Asymptotic Distribution of Trapping Sets in Irregular Code Ensembles

Here we find the asymptotic distribution of elementary trapping sets, as well as trapping sets in general, in the ensemble of matrices $\mathcal{H}^{\mathbf{c},\mathbf{r}}_{m,n}$. The corresponding results are presented in Theorems 4.1 and 4.2, respectively. The main components of the proof consist in establishing asymptotic expressions for two "nested" sums. For the first sum, a straightforward application of Sanov's theorem is needed. For the second sum, Varadhan's lemma provides an expression which has to be evaluated numerically.

4.1 Elementary Trapping Sets

The next theorem and corollary establish the asymptotic behavior of elementary trapping sets in regular LDPC code ensembles.

Theorem 4.1 Let $\zeta < 1$, and let $0 < \theta, \lambda < 1$. The average ensemble distribution $e(\theta,\lambda)$ of $((a = \theta n, b = \lambda n))$ elementary trapping sets in a code from the ensemble $\mathcal{H}^{\mathbf{c},\mathbf{r}}_{m,n}$ with column and row weights described by the distribution vectors $\kappa_1, \ldots, \kappa_h$ and $\chi_1, \ldots, \chi_g$ is given by the supremum of

$$\sum_j \kappa_j\, H\left(\frac{\theta_j}{\kappa_j}, 1 - \frac{\theta_j}{\kappa_j}\right) - \zeta I(\lambda, \tilde{\theta}) - \left(\sum_j \kappa_j s_j\right) H\left(\tilde{\theta}/\sum_j \kappa_j\, s_j, 1 - \tilde{\theta}/\sum_j \kappa_j\, s_j\right),$$

where

$$
\begin{aligned}
&I(\lambda,\tilde{\theta}) = \frac{\tilde{\theta}}{\zeta}\log\rho + \frac{\lambda}{\zeta}\log A - \sum_i \chi_i \,\log\left(1 + A\rho t_i + \begin{pmatrix} t_i \\ 2 \end{pmatrix}\rho^2\right), \\
&\theta_j = \frac{\kappa_j\, e^{y s_j + z}}{1 + e^{y s_j + z}},\; j = 1,\ldots,h, \\
&\sum_j \theta_j = \theta, \\
&\sum_j s_j\,\theta_j = \tilde{\theta},
\end{aligned}
$$

and were A, ρ, $\tilde{\theta}$, y and z represent the parameters over which the supremum is evaluated. These parameters satisfy the following set of equations

$$
\begin{aligned}
y &= \log\left(\frac{\rho\,\left(\sum_j \kappa_j s_j - \tilde{\theta}\right)}{\tilde{\theta}}\right), \\
A\rho\sum_i \frac{\chi_i t_i}{1 + A\rho t_i + \binom{t_i}{2}\rho^2} &= \frac{\lambda}{\zeta}, \\
\sum_i \frac{\chi_i\left(A t_i\,\rho + t_i\,(t_i-1)\rho^2\right)}{1 + A\rho t_i + \binom{t_i}{2}\rho^2} &= \frac{\tilde{\theta}}{\zeta}.
\end{aligned} \tag{32}
$$

Proof. The proof represents a generalization of the arguments used in [10] for evaluating average weight distributions of code ensembles. Again, we will work with transposes of parity-check matrices, H^T.

For

$$
H^T = \left[\frac{M_u}{M_l}\right], \tag{33}
$$

where M_u is an $a \times m$ and M_l is an $(n-a)\times m$ zero-one matrix, with $a = \theta n, 0 < \theta < 1$. Furthermore, let α_j be defined as in 3.1.

Define $\tilde{\Lambda}^{\zeta,r}_{a,b,m}$ and $\hat{\Lambda}^{\zeta,r}_{a,b,m}$ in the same manner as in theorem 3.1, except for an adequate redefinition of the ensembles involved and for changing $\Lambda^{c,r}_{m,n}$ into $\mathcal{H}^{\mathbf{c},\mathbf{r}}_{m,n}$. In this setting, we seek an expression for the probability

$$
P^{\mathbf{c},\mathbf{r}}_{m,n} = \frac{1}{\mid \mathcal{H}^{\mathbf{c},\mathbf{r}}_{m,n} \mid}\sum\prod_{i=1}^{g}\begin{pmatrix} m\chi_i \\ \alpha_0^{(i)}\;\alpha_1^{(i)}\;\alpha_2^{(i)} \end{pmatrix} \mid \tilde{\Lambda}^{\zeta,r}_{a,b,m} \mid\mid \hat{\Lambda}^{\zeta,r}_{a,b,m} \mid, \tag{34}
$$

where the sum is taken over all triples $(\alpha_0^{(i)}, \alpha_1^{(i)}, \alpha_2^{(i)})$ such that

$$
\begin{aligned}
&\sum_{l=0,1,2} \alpha_l^{(i)} = \chi_i\, m,\; i = 1,\ldots,g, \\
&\sum_{i=1}^{g}\left(\alpha_1^{(i)} + 2\,\alpha_2^{(i)}\right) = n\,\sigma_\theta, \\
&\sum_{i=1}^{g}\alpha_1^{(i)} = \lambda n.
\end{aligned} \tag{35}
$$

Here, we used σ_θ to denote

$$
\left(s_1\,\kappa_1 + \ldots + s_{q-1}\kappa_{q-1} + s_q(\theta - \kappa_1 - \kappa_2 - \ldots - \kappa_{q-1})\right),
$$

and q to denote the unique integer satisfying $\theta \in \left[\sum_{l=1}^{q-1} \kappa_l, \sum_{l=1}^{q} \kappa_l\right]$. Let $N = m\sum_{i=1}^{g} t_i \chi_i = n\sum_{j=1}^{h} s_j \kappa_j$ be the number of non-zero elements in H, since we are working with the first $n\theta$ rows. Then

$$\begin{aligned} &\mid \mathcal{H}_{m,n}^{\mathbf{c},\mathbf{r}} \mid \simeq \frac{N!}{s_1!^{n\kappa_1} \dots s_h!^{n\kappa_h}\, t_1!^{m\chi_1} \dots t_g!^{m\chi_g}}, \\ &\mid \tilde{\Lambda}_{a,b,m}^{\zeta,r} \mid \simeq \frac{(n\sigma_\theta)!}{s_1!^{n\kappa_1} \dots s_q!^{n(\theta-\kappa_1-\dots-\kappa_{q-1})} \prod_i 2!^{\alpha_2^{(i)}}}, \\ &\mid \hat{\Lambda}_{a,b,m}^{\zeta,r} \mid \simeq \frac{(N-n\sigma_\theta)!}{s_q!^{n(\kappa_1+\dots+\kappa_q-\theta)} \dots s_h!^{n\kappa_h} \prod_i t_i!^{\alpha_0^{(i)}} (t_i-1)!^{\alpha_1^{(i)}} (t_i-2)!^{\alpha_2^{(i)}}}. \end{aligned} \tag{36}$$

By combining the above expressions, one obtains

$$\log P_{m,n}^{\mathbf{c},\mathbf{r}} \simeq \log \binom{N}{n\sigma_\theta}^{-1} + \log \sum \prod_{i=1}^{g} \binom{m\chi_i}{\alpha_0^{(i)}\ \alpha_1^{(i)}\ \alpha_2^{(i)}} \binom{t_i}{0}^{\alpha_0^{(i)}} \binom{t_i}{1}^{\alpha_1^{(i)}} \binom{t_i}{2}^{\alpha_2^{(i)}}. \tag{37}$$

where the sum ranges over the same set of values of parameters $\alpha_l^{(i)}$ as given in equation (35). Write

$$G_i = \binom{t_i}{0} + \binom{t_i}{1} + \binom{t_i}{2}. \tag{38}$$

The last expression in (37) reduces to

$$\log \left(\prod_{i=1}^{g} G_i^{m\chi_i}\ P\{\sum_{i=1}^{g} \sum_{l=0,1,2} (\alpha_l^{(i)}/n) = \sigma_\theta\} \right), \tag{39}$$

where the variables $\alpha_l^{(i)}/n$ obey the binomial distribution given by

$$p_l^{(i)} = \frac{1}{G_i} \binom{t_i}{l},\ l = 0,1,2. \tag{40}$$

The last two formulas imply that the asymptotic behavior of the logarithm of the sum considered can be obtained by using Sanov's theorem. The problem reduces to minimizing

$$\sum_i \chi_i D(\underline{q}^{(i)} || \underline{p}^{(i)}), \tag{41}$$

subject to the constraints

$$\begin{aligned} &q_0^{(i)} + q_1^{(i)} + q_2^{(i)} = 1, \\ &\sum_i \chi_i \left(q_1^{(i)} + 2q_2^{(i)} \right) = \frac{\sigma_\theta}{\zeta}, \\ &\sum_{i=1}^{g} \chi_i q_1^{(i)} = \lambda/\zeta. \end{aligned}$$

More generally if we take θ_j rows with weight s_j such that $\sum_j \theta_j = \theta$ then σ_θ is replaced with $\tilde{\theta} = \sum_j \theta_j s_j$. By using Lagrange multipliers one obtains the following expressions for the probability mass function minimizing the relative entropy function

$$\tilde{q}_l^{(i)} = C_i \binom{t_i}{l} \rho^l A(l), \tag{42}$$

where $A(1) = A$, and $A(l) = 1$ otherwise. Furthermore ρ and A satisfy the following set of fixed point equations which have a unique solution,

$$\begin{aligned}(C_i)^{-1} = 1 + A\rho\, t_i + \binom{t_i}{2}\rho^2,\ \ i &= 1,\ldots,g,\\ A\rho \sum_i \frac{\chi_i t_i}{1 + A\rho t_i + \binom{t_i}{2}\rho^2} &= \frac{\lambda}{\zeta},\\ \sum_i \frac{\chi_i \left(A\rho\, t_i + t_i(t_i-1)\rho^2\right)}{1 + A\rho t_i + \binom{t_i}{2}\rho^2} &= \frac{\tilde{\theta}}{\zeta},\end{aligned}$$

with $\tilde{\theta}$ a constant independent on i. Therefore, it follows that

$$\sum_i \chi_i D\left(\tilde{\mathbf{q}}^{(i)} \,||\, \mathbf{p}^{(i)}\right) = I(\lambda, \tilde{\theta}) + \sum_i \chi_i \log G_i$$

where

$$I(\lambda, \tilde{\theta}) = \frac{\tilde{\theta}}{\zeta}\log\rho + \frac{\lambda}{\zeta}\log A - \sum_i \chi_i \log\left(1 + A\rho t_i + \binom{t_i}{2}\rho^2\right),$$

regarding λ as fixed.

The asymptotic behavior for $P^{\mathbf{c},\mathbf{r}}_{m,m}$ can be obtained by combining the above expression with the asymptotic formula $\log\binom{N}{n\tilde{\theta}} = \log\binom{N}{N\,(n/N)\,\tilde{\theta}} \simeq NH\left(\tilde{\theta}/\sum_{i=1}^{h}\kappa_i s_i\right)$. To complete the proof, one needs to estimate the spectrum by using the previously evaluated probability: for the case of regular ensembles, one simply had $\binom{n}{n\theta}$ ways of selecting the columns in the support set of a codeword. Here, we set $\underline{\theta} = (\theta_1, \ldots, \theta_h)$ so that θ_i denotes the fraction of columns one chooses from a strip of column weight i so that $\sum_j \theta_j = \theta$. We are then interested in the sum

$$P_n = \sum_{\underline{\theta}} \binom{N}{n\tilde{\theta}}^{-1} \prod_j \binom{n\kappa_j}{n\theta_j} \Psi_n(\lambda, \tilde{\theta}) \tag{43}$$

where

$$\Psi_n(\lambda, \tilde{\theta}) = \exp\left(-mI(\lambda, \tilde{\theta})\right). \tag{44}$$

By observing that

$$\prod_j \binom{n\kappa_j}{n\theta_j} 2^{-n}$$

is a probability, we may apply Varadhan's integral lemma to show that

$$-\lim_{n\to\infty}\frac{1}{n}\log P_n = \sup_{\underline{\theta}}\left[\sum_j \kappa_j H\left(\frac{\theta_j}{\kappa_j}, 1 - \frac{\theta_j}{\kappa_j}\right) - \zeta I(\lambda, \tilde{\theta}) - \left(\sum_j \kappa_j s_j\right) H\left(\tilde{\theta}/\sum_j \kappa_j s_j, 1 - \tilde{\theta}/\sum_j \kappa_j s_j\right)\right]. \tag{45}$$

For the case under consideration the supremum is actually a maximum that can be obtained by varying over the parameter $\tilde{\theta}$. Observe next that *only one* of the constraining equations found so far involves θ, namely $\sum_i \theta_i = \theta$. Hence, since the entropy terms in (45) involve θ only in a implicit way, one can perform the optimization of these function by first fixing the value of $\tilde{\theta}$. Therefore, one starts by considering

$$\max \sum_j \kappa_j H\left(\frac{\theta_j}{\kappa_j}, 1 - \frac{\theta_j}{\kappa_j}\right),\ \ 0 \le \theta_j \le \theta,\ j = 1, \cdots, h$$

subject to the constraints $\sum_j s_j\,\theta_j = \tilde{\theta}$ and $\sum_j \theta_j = \theta$. The Lagrangian multipliers for this optimization process are y and z, and the maximizing set of values for the parameters θ_j is given by

$$\theta_j = \frac{\kappa_j\, e^{y s_j + z}}{1 + e^{y s_j + z}},\ j = 1, \ldots, h.$$

Here, y and z are the unique real numbers chosen to satisfy

$$\sum_j s_j\, \frac{\kappa_j\, e^{y s_j + z}}{1 + e^{y s_j + z}} = \tilde{\theta}$$

and

$$\sum_j \frac{\kappa_j\, e^{y s_j + z}}{1 + e^{y s_j + z}} = \theta.$$

Now, one has to perform optimization over the previously fixed $\tilde{\theta}$ parameter. This can be achieved by differentiating the terms in (45) with respect to $\tilde{\theta}$ and by setting this derivative to zero. In fact, this derivative of the optimal objective function with respect to the constraint variable $\tilde{\theta}$ will equal to the Lagrange multiplier of the overall optimization problem. Assuming that the optimum value of $\tilde{\theta}$ is in the interior, it must be obtained at a point for which

$$y - \log\rho + \log\tilde{\theta} - \log\left(\sum_j \kappa_j s_j - \tilde{\theta}\right) = 0.$$

The first term in the expression arises from the original multiplier, the second term from the expressions in the rate function explicitly involving $\tilde{\theta}$, while the last two terms appear due to the entropy functions in (45). The equation above can be rewritten as

$$y = \log\left(\frac{\rho\left(\sum_j \kappa_j s_j - \tilde{\theta}\right)}{\tilde{\theta}}\right), \tag{46}$$

and this completes the proof of the claimed result. □

Remark 4.1: It is tedious, but straightforward to show that for regular LDPC ensembles, the parameters of theorem 4.1 reduce to

$$t_i = r,\ i = 1, \ldots, g,\ \ \rho = \sqrt{\frac{\theta c - \lambda}{\binom{r}{2}(2\zeta - \theta c - \lambda)}},$$

$$C_i = \frac{2\zeta - \theta c - \lambda}{2\zeta},\ \ \text{and}\ \ A = \lambda\sqrt{\frac{2(r-1)}{r(2\zeta - \theta c - \lambda)(\theta c - \lambda)}}.$$

By substituting these expressions into the formula for the average trapping set ensemble spectrum, one recovers the results for the regular code ensemble, given in theorem 3.1.

For many practical applications, one is interested in codes for which the sets of parameters κ_i and χ_j are optimized in terms of *density evolution*; optimization is, in this case, performed with respect to the code's gap to capacity [16]. Near-optimal degree distributions (for a given signal-to-noise ratio) usually exhibit a so-called "concentration property", which implies that the row-weights of the parity-check matrix are drawn from a small set of different values or that the row-weights are constant. In the latter case, corresponding to row-regular ensembles (with $t_i = r$) one can find closed-form expressions for A and ρ in terms of $\tilde{\theta}$ as

$$\begin{aligned} A &= \lambda \sqrt{\frac{2(r-1)}{r(2\zeta-\tilde{\theta}-\lambda)(\tilde{\theta}-\lambda)}}, \\ \rho &= \sqrt{\frac{\tilde{\theta}-\lambda}{\binom{r}{2}(2\zeta-\tilde{\theta}-\lambda)}}. \end{aligned} \tag{47}$$

Consequently, one only has to solve the system of equations

$$\begin{aligned} &\sum_j s_j \frac{\kappa_j e^{ys_j+z}}{1+e^{ys_j+z}} = \tilde{\theta}, \\ &\sum_j \frac{\kappa_j e^{ys_j+z}}{1+e^{ys_j+z}} = \theta, \\ &y = \log\left(\frac{\rho\left(\sum_j \kappa_j s_j - \tilde{\theta}\right)}{\tilde{\theta}}\right), \end{aligned} \tag{48}$$

for $\tilde{\theta}, z$ and y, since ρ can be expressed in terms of $\tilde{\theta}$ only.

4.2 General Trapping Sets

The next theorem and corollary establish the asymptotic behavior of the number of general trapping sets in irregular LDPC code ensembles.

Theorem 4.2 Let $0 < \zeta < 1$, and let $0 < \theta, \lambda < 1$. The average ensemble distribution $t(\theta,\lambda)$ of $((a = \theta n, b = \lambda n))$ general trapping sets in a code from the ensemble $\mathcal{H}^{\mathbf{c},\mathbf{r}}_{m,n}$ with column and row weights described in terms of the vectors $\kappa_1,\ldots,\kappa_h$ and $\chi_1,\ldots,\chi_g$ is given by the supremum of

$$\sum_j \kappa_j H\left(\frac{\theta_j}{\kappa_j}, 1-\frac{\theta_j}{\kappa_j}\right) - \zeta I(\lambda,\tilde{\theta}) - \left(\sum_j \kappa_j s_j\right) H\left(\tilde{\theta}/\sum_j \kappa_j s_j, 1-\tilde{\theta}/\sum_j \kappa_j s_j\right),$$

where

$$\begin{aligned} &I(\lambda,\tilde{\theta}) = -\sum_i \chi_i \log\left(\frac{(1+\rho)^{t_i}-(1-\rho)^{t_i}}{2} + \tau\,\frac{(1+\rho)^{t_i}+(1-\rho)^{t_i}}{2}\right) + \frac{\tilde{\theta}}{\zeta}\log\rho + \frac{\lambda}{\zeta}\log\tau, \\ &\theta_j = \frac{\kappa_j e^{ys_j+z}}{1+e^{ys_j+z}},\ j = 1,\ldots,h, \\ &\sum_j \theta_j = \theta, \\ &\sum_j s_j \theta_j = \tilde{\theta}, \end{aligned}$$

and were τ, ρ, $\tilde{\theta}$, y and z represent the parameters over which the optimization is performed. These parameters satisfy the following system of equations:

$$\begin{aligned} &y = \log\left(\frac{\rho\left(\sum_j \kappa_j s_j - \tilde{\theta}\right)}{\tilde{\theta}}\right), \\ &\tilde{\theta}/\zeta = \sum_{i=1}^{g} \chi_i\,\rho\, t_i \left(\frac{(1+\rho)^{(t_i-1)}-(1-\rho)^{(t_i-1)}+\tau\left((1+\rho)^{(t_i-1)}+(1-\rho)^{(t_i-1)}\right)}{(1+\rho)^{t_i}-(1-\rho)^{t_i}+\tau\left((1+\rho)^{t_i}+(1-\rho)^{t_i}\right)}\right), \\ &\lambda/\zeta = \sum_{i=1}^{g} \chi_i\,\tau\,\frac{(1+\rho)^{t_i}-(1-\rho)^{t_i}}{(1+\rho)^{t_i}-(1-\rho)^{t_i}+\tau\left((1+\rho)^{t_i}+(1-\rho)^{t_i}\right)}. \end{aligned} \tag{49}$$

Proof. The derivations follow along the same line as the one presented for elementary trapping sets, up to the point where the following expression is encountered

$$\log P_{m,n}^{\mathbf{c},\mathbf{r}} = \log \binom{N}{n\sigma_\theta}^{-1} + \log \sum \prod_{i=1}^{g} \binom{m\chi_i}{\alpha_0^{(i)} \ldots \alpha_{t_i}^{(i)}} \binom{t_i}{0}^{\alpha_0^{(i)}} \cdots \binom{t_i}{t_i}^{\alpha_{t_i}^{(i)}}. \tag{50}$$

Here the sum ranges over the set of parameters $\alpha_l^{(i)}$ constrained by

$$\begin{aligned} &\sum_l \alpha_l^{(i)} = \chi_i\, m,\ i = 1, \ldots, g, \\ &\sum_{i=1}^{g} \sum_l l\, \alpha_l^{(i)} = n\sigma_\theta, \\ &\sum_{i=1}^{g} \sum_l \alpha_{2l+1}^{(i)} = \lambda n. \end{aligned} \tag{51}$$

Write

$$G_i \doteq \sum_j \binom{t_i}{j} = 2^{t_i} \tag{52}$$

The logarithm of the last expression in (50) can be expressed as

$$\log \left(\prod_{i=1}^{g} G_i^{m\chi_i}\, P\{\sum_{i=1}^{g} \sum_l (\alpha_l^{(i)}/n) = \sigma_\theta\} \right), \tag{53}$$

where the variables $\alpha_l^{(i)}/n$ follow the binomial distribution given by

$$p_l^{(i)} = \frac{1}{G_i} \binom{t_i}{l},\ l = 0, \ldots, t_i. \tag{54}$$

The last formulas imply that the asymptotic behavior of the logarithm of the sum considered can be obtained by using Sanov's theorem. The problem reduces to minimizing

$$\sum_i \chi_i D(\underline{q}^{(i)} || \underline{p}^{(i)}), \tag{55}$$

subject to

$$\begin{aligned} &\sum_{l=0}^{t_i} q_l^{(i)} = 1, \\ &\sum_{i=1}^{g} \chi_i \sum_{l=0}^{t_i} l\, q_l^{(i)} = \frac{1}{\zeta} \sigma_\theta, \\ &\sum_{i=1}^{g} \chi_i \sum_l q_{2l+1}^{(i)} = \lambda/\zeta. \end{aligned} \tag{56}$$

The analysis proceeds along the same lines as for the case of elementary trapping sets. We find that

$$\tilde{q}_l^{(i)} = \begin{cases} C_i \binom{t_i}{l} \rho^l, & l \ \text{even} \\ C_i \binom{t_i}{l} \tau\rho^l, & l \ \text{odd} \end{cases} \tag{57}$$

where $\tau > 0$ is determined via the final constraint in (56).

Going to the more general case of taking θ_j rows with weight s_j where $\sum_j \theta_j = \theta,\ \sum_j s_j\theta_j = \tilde{\theta}$, we thus obtain,

$$\begin{aligned}(C_i)^{-1} &= \sum_l \begin{pmatrix} t_i \\ 2l \end{pmatrix} \rho^{2l} + \tau \sum_l \begin{pmatrix} t_i \\ 2l+1 \end{pmatrix} \rho^{2l+1}, \text{ i.e} \\ (C_i)^{-1} &= \frac{(1+\rho)^{t_i} - (1-\rho)^{t_i}}{2} + \tau\, \frac{(1+\rho)^{t_i} + (1-\rho)^{t_i}}{2},\end{aligned} \tag{58}$$

and the resulting pair of fixed point equations takes the form

$$\begin{aligned}\tilde{\theta}/\zeta &= \sum_{i=1}^{g} \chi_i\, C_i\, \rho\, t_i \left(\frac{(1+\rho)^{(t_i-1)} - (1-\rho)^{(t_i-1)}}{2} + \tau\, \frac{(1+\rho)^{(t_i-1)} + (1-\rho)^{(t_i-1)}}{2} \right), \\ \lambda/\zeta &= \sum_{i=1}^{g} \chi_i\, C_i\, \tau\, \frac{(1+\rho)^{t_i} - (1-\rho)^{t_i}}{2},\end{aligned}$$

for C_i given as above. Consequently, the minimum exponent is $\sum_i \chi_i G_i + I(\lambda, \tilde{\theta})$ where,

$$I(\lambda, \tilde{\theta}) = -\sum_i \chi_i \log\left(\frac{(1+\rho)^{t_i} - (1-\rho)^{t_i}}{2} + \tau\, \frac{(1+\rho)^{t_i} + (1-\rho)^{t_i}}{2} \right) + \frac{\tilde{\theta}}{\zeta} \log\rho + \frac{\lambda}{\zeta} \log\tau.$$

The rate function for the probability of interest is determined based on similar techniques as demonstrated for elementary trapping sets, so that the details are omitted. □

5 Stopping Set Analysis

An analysis similar to the one performed for trapping sets can be conducted for stopping sets as well. Although this problem was already addressed in [14], there exist some interesting differences in the problem settings. The ensemble used in [14] corresponds to a *random permutation* model, which is easier to analyze, but less realistic in terms of its modelling of LDPC code properties. Our analysis handles the more involved case in a very simplistic manner, producing results of the same form [4].

Assume that the size of a stopping set $\mathcal{S}_a(n)$ is linear in n, i.e. $a = \theta n$. The probability of interest for stopping set analysis takes the form

$$\tilde{P}_{a,n}^{\zeta,r} = \frac{\mid \mathcal{A}_{a,n}^{\zeta,r} \mid}{\mid \Lambda_{m,n}^{c,r} \mid},$$

where

$$\tilde{\mathcal{A}}_{a,m}^{\zeta,r} \doteq \left\{ M_u \in \Lambda_{a,m} : \sum_j M_u(i,j) = c,\ \alpha_1 = 0 \right\},$$

$$\hat{\mathcal{A}}_{a,m}^{\zeta,r} \doteq \left\{ M_l \in \Lambda_{n-a,m} : \sum_j M_l(i,j) = c, \sum_i M_l(i,j) = \beta_j \right\},$$

with $\beta_i \in \{1, 2, ..., r-1, r\}$ so that exactly α_i values of β_i are equal to $r - i$. Furthermore, let

$$\mathcal{G}_{a,m}^{\zeta,r} \doteq \left\{ H \in \Lambda_{m,n}^{c,r} : M_u \in \tilde{\mathcal{A}}_{a,m}^{\zeta,r}, M_l \in \hat{\mathcal{A}}_{a,m}^{\zeta,r} \right\}.$$

[4] In [19], while deriving the bounds on the tails of the spectrum of regular LDPC codes, the authors also showed how this method can be used to simplify the derivations of the weight distribution of this class of codes as well

By using the same type of argument as described in the previous two proofs, one arrives at the following formula for $\tilde{P}_{a,n}^{\zeta,r}$:

$$\tilde{P}_{a,n}^{\zeta,r} \overset{\log}{\simeq} \frac{1}{\binom{nc}{ac}} \sum \binom{\zeta n}{\alpha_0, \alpha_2, \ldots, \alpha_r} \binom{r}{0}^{\alpha_0} \binom{r}{2}^{\alpha_2} \ldots \binom{r}{r}^{\alpha_r} \tag{59}$$

where the variables $\alpha_0, \ldots, \alpha_r$ satisfy the following constraints:

$$\sum_i \alpha_i = \zeta n, \ \ , \sum_i i\alpha_i = ac, \ \ \alpha_1 = 0. \tag{60}$$

These relations translate into the following expression for the distribution minimizing the relative entropy in Sanov's theorem

$$\sum_i q_i = 1, \ \ \sum_i i\, q_i = \theta r, \ \ q_1 = 0, \tag{61}$$

with the probabilities p_i given by (20). By using the Lagrangian multiplier method, with multiplier function

$$\sum_i q_i \log \frac{q_i}{p_i} + \mu_1 \sum_i q_i + \mu_2 \sum_i i\, q_i + \mu_3\, q_1, \tag{62}$$

one obtains that the entropy minimizing probability function is of the form

$$q_i^* = \frac{1}{(1+\rho)^r - r\rho} \binom{r}{i} \rho^i, \ i \neq 1, \tag{63}$$

and $q_1^* = 0$. The parameter ρ in the above equation represents the unique positive root of the equation

$$\theta = \frac{\rho\,(1+\rho)^{r-1} - \rho}{(1+\rho)^r - r\rho}. \tag{64}$$

For the given distribution, the relative entropy is of the form

$$\log \frac{(2^r - r)\rho^{r\theta}}{(1+\rho)^r - \rho r}. \tag{65}$$

Substituting for q_i in Sanov's bound, one obtains

$$\frac{1}{n} \log \tilde{P}_{a,n}^{\zeta,r} = -\zeta \log \frac{\rho^{r\theta}}{(1+\rho)^r - \rho r} - cH(\theta, 1-\theta), \tag{66}$$

and consequently, the "stopping exponent" is equal to

$$-\zeta \log \frac{\rho^{r\theta}}{(1+\rho)^r - \rho r} - (c-1)H(\theta, 1-\theta). \tag{67}$$

This result agrees with the corresponding expressions derived in [14] by using more elaborate combinatorial methods.

Similarly, for the irregular case one obtains

$$\log P_{m,n}^{\mathbf{c},\mathbf{r}} \simeq \log \binom{N}{n\sigma_\theta}^{-1} + \log \sum \prod_{i=1}^{g} \binom{m\chi_i}{\alpha_0^{(i)}\ \alpha_2^{(i)} \ldots \alpha_{t_i}^{(i)}} \binom{t_i}{0}^{\alpha_0^{(i)}} \binom{t_i}{2}^{\alpha_2^{(i)}} \ldots \binom{t_i}{t_i}^{\alpha_{t_i}^{(i)}}, \tag{68}$$

where the sum ranges over the set of parameters $\alpha_l^{(i)}$ given by:

$$\sum_{l\neq 1} \alpha_l^{(i)} = \chi_i\, m,\ i=1,\ldots,g,$$
$$\sum_{i=1}^{g}\sum_{l=2}^{t_i} l\,\alpha_l^{(i)} = n\sigma_\theta, \tag{69}$$
$$\alpha_1^{(i)} = 0,\ \ i=1,\ldots,g.$$

Consequently,

$$\tilde{q}_l^{(i)} = \begin{cases} C_i \begin{pmatrix} t_i \\ l \end{pmatrix} \rho^l, & l\neq 1 \\ 0, & l=1, \end{cases} \tag{70}$$

were based on the fixed point equations one has

$$(C_i)^{-1} = (1+\rho)^{t_i} - t_i\,\rho,$$
$$\rho \sum_{i=1}^{g} \chi_i\, t_i\, \frac{(1+\rho)^{t_i-1}-1}{(1+\rho)^{t_i}-t_i\,\rho} = \sigma_\theta. \tag{71}$$

Using Varadhan's lemma as outlined in Section 4, one recovers the same results as presented in [14].

6 Numerical Results

The expressions in Section 3 describing the asymptotic distribution of elementary and general trapping sets in regular LDPC code ensembles are most easily analyzed trough numerical simulations. Results of this analysis are summarized in Figure 2,3, and 4. Figure 2 provides a comparison of the spectra of three different ensembles with column weight three, and rates 0.4, 0.5 and 0.8. It is interesting to observe that for a fixed size of the a parameter, the number of trapping sets with parameter b grows much faster for codes of higher then for codes of lower rate. Let the smallest value of λ (recall that $b=\lambda n$) for which the number of $((a,b))$ elementary trapping sets is exponential in the length of the code be called the *elementary trapping coefficient*. For the code ensembles shown in Figure 2, these numbers are $3\,10^{-4}$, $2.4\,10^{-4}$ and $1.8\,10^{-5}$. Consequently, the trapping coefficient decreases with the increase of the row weight, for a fixed column weight.

A similar phenomena is observed with respect to the column weight of the regular code ensembles. In Figure 3, one can see that for a fixed code rate $1/2$, the smaller the column weight of a code, the smaller the elementary trapping coefficient (i.e. the larger the number of trapping sets with the given parameters). The trapping coefficients for column weight three, four and five codes are $3.2\,10^{-4}$, $9.1\,10^{-4}$ and $0.9\,10^{-2}$. This finding may imply that the rate $1/2$ code ensembles which are known to exhibit the best threshold characteristics, namely the $(3,6)$ ensemble, can have poor average error-floor properties. Finally, Figure 4 demonstrates the intuitively expected fact that the trapping coefficient depends on the value of θ, and increases as θ increases. Similar findings hold for generalized trapping sets. Figure 5 plots the generalized trapping set spectrum for three different ensembles and $\theta = 0.001$. The interesting observation is that for this case the spectra of general and elementary trappings sets are almost identical, i.e. *the asymptotic growth exponents of these two classes of sets are approximately equal*. Furthermore, the difference between the spectra of elementary and general trapping sets grows (although only slightly) with the rate of the code. Observe that these findings may explain some of the numerical results presented in [15], regarding the error floor of random-like codes in the regular $(3,15)$ ensemble. The error floor of these codes is very high, and can be decreased by eliminating short cycles from the graphs of such codes. The latter finding can be explained in terms of the easily observable fact that trapping sets contain short cycles. An illustrative example is the eight-cycle present in the $((5,1))$ trapping set in Section 2.1. The edges of this cycle are given by the boldfaced ones in the sub-block in Section 2.1.

The situation becomes quite different when one increases the value of θ to 0.1: a substantial difference between the distributions arises for the case of a $(3,15)$ ensemble, as illustrated in Figure 6.

The expressions in Section 4 describing the asymptotic distribution of elementary trapping sets in irregular LDPC code can be evaluated only through numerical methods. We will present only one example pertaining to the solutions of the system of equations in theorem 4.1. We consider a code ensemble of rate $1/2$ for which

$$\kappa_2 = 0.505, \kappa_3 = 0.3, \kappa_4 = \kappa_5 = \kappa_6 = 0, \kappa_7 = 0.195,$$
$$\chi_1 = \chi_2 = \chi_3 = \chi_4 = \chi_5 = 0, \chi_6 = 0.438, \chi_7 = 0.562;$$

For $\lambda = 0.02$ and $\theta = 0.1$ the *unique* optimizing solutions for the parameters of the elementary trapping set problem are

$$\tilde{\theta}_{\mathrm{opt}} = 2.415 \times 10^{-1}, \rho = 0.128, A = 0.0646, \text{ and } y = -0.476.$$

For other instances of the problem related to irregular ensembles, the roots of the fixed point equations cannot be found for all conceivable values of λ and $\tilde{\theta}$. This is especially apparent for the case when χ_i is non-zero for only two different values of the index i.

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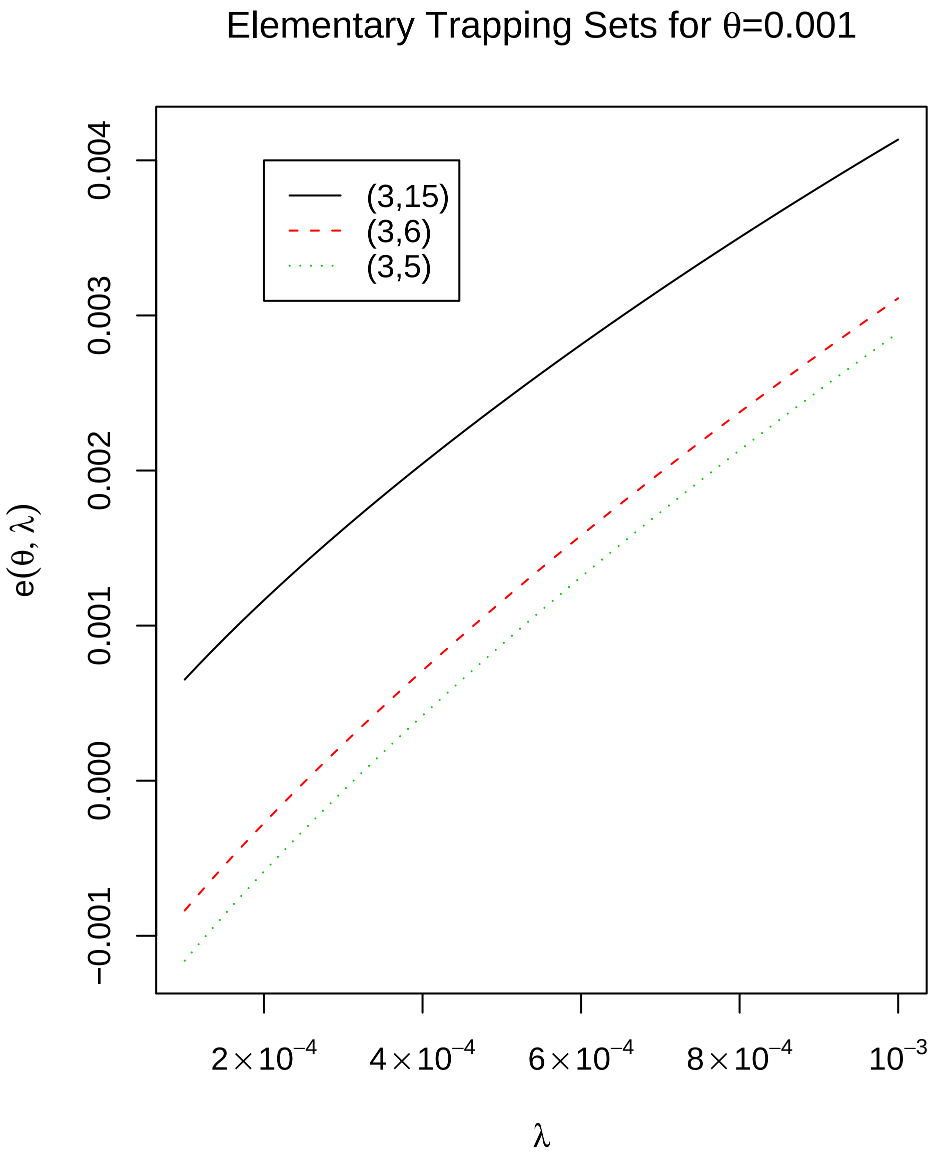

Figure 2: Trapping set spectrum for three regular ensembles, and $\theta = 0.001$

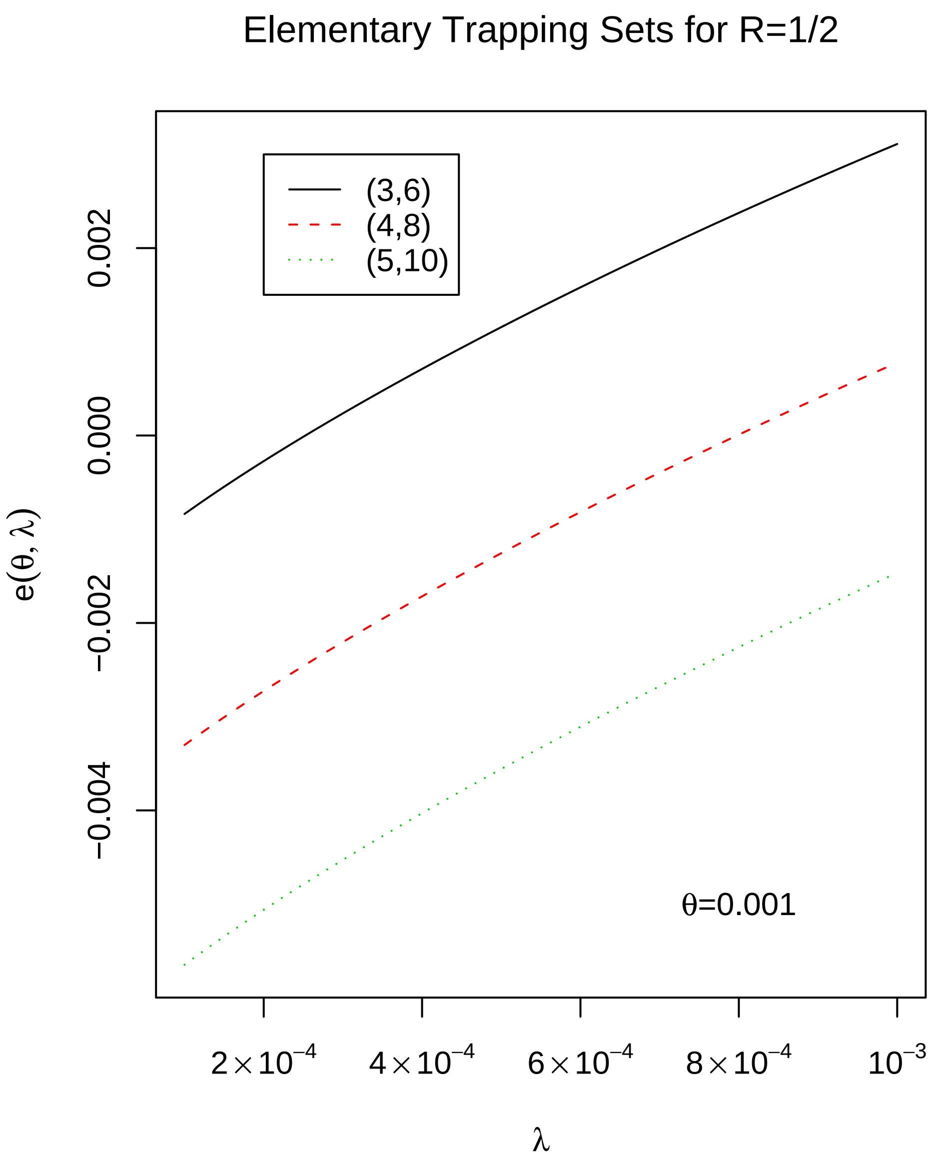

Figure 3: Trapping set spectrum for ensembles of fixed rate $R = 1/2$, and $\theta = 0.001$

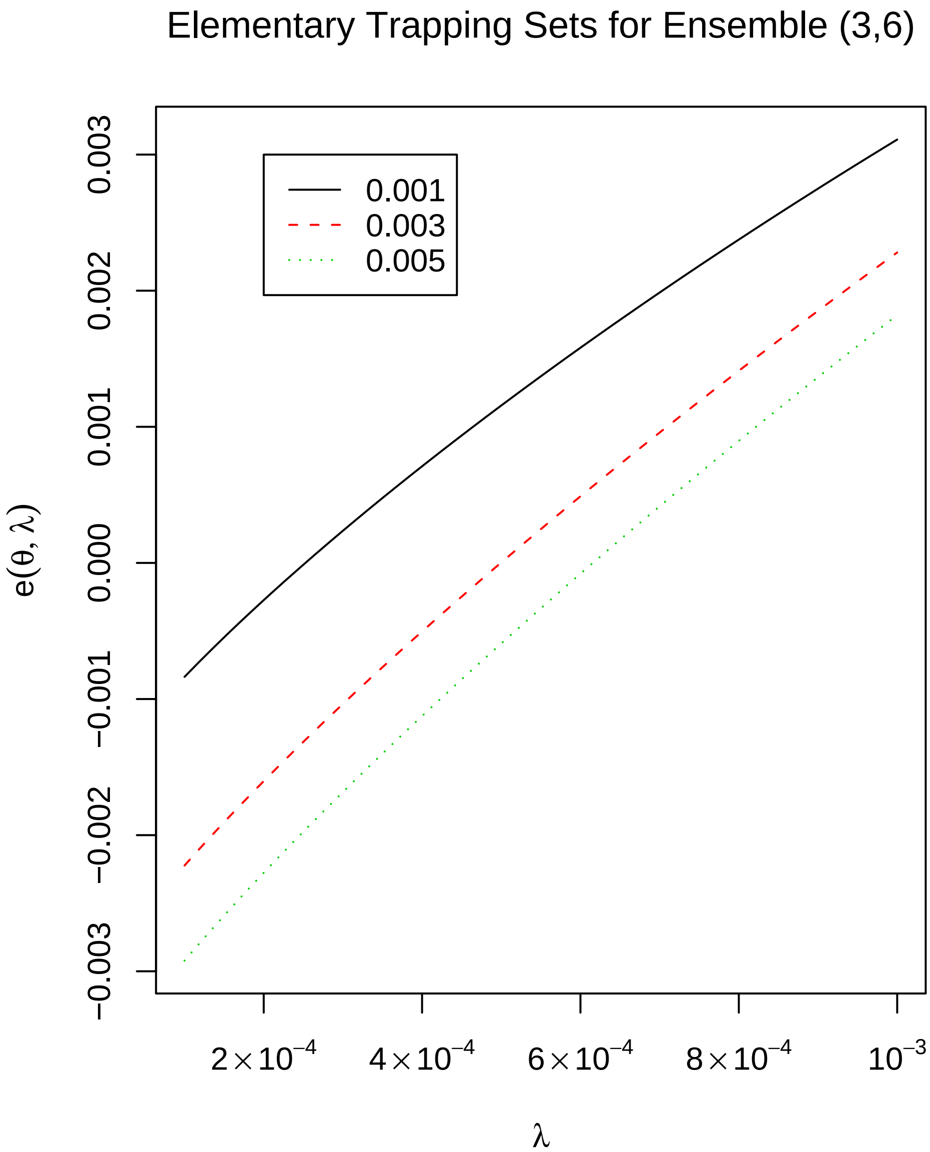

Figure 4: Trapping set spectrum for (3,6) regular ensemble and three different values of the parameter θ

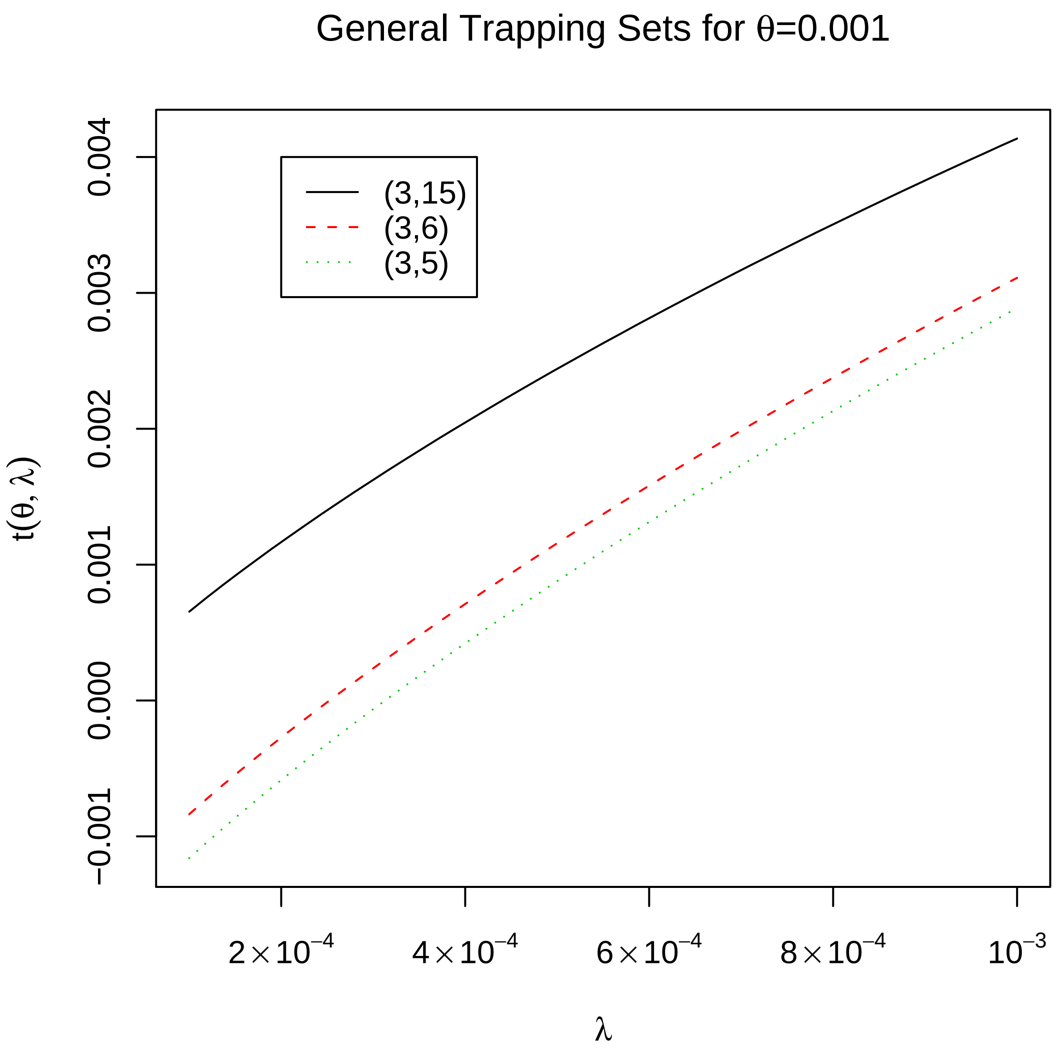

Figure 5: Trapping set spectrum for (3,6) regular ensemble and three different values of the parameter θ

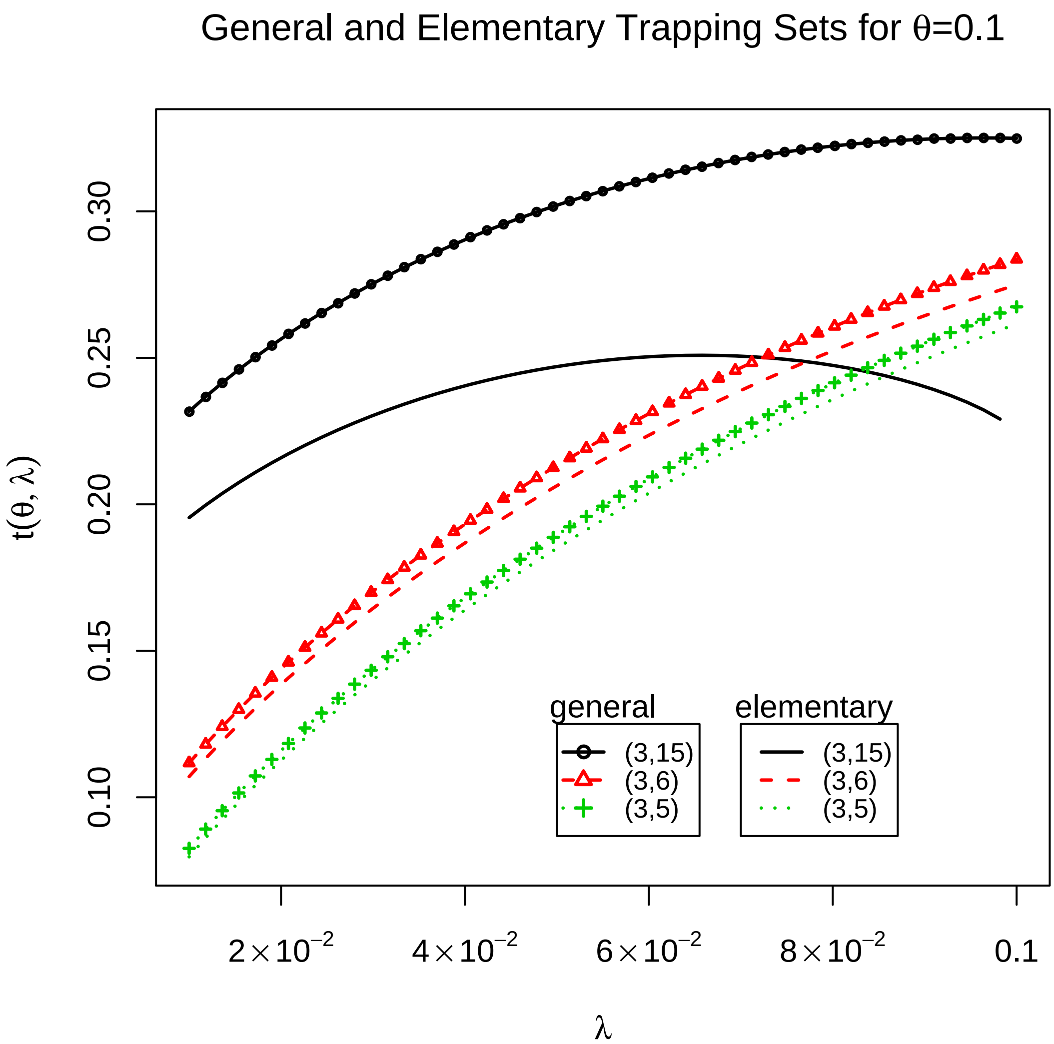

Figure 6: Comparison of trapping set spectra for $\theta = 0.1$ and regular $(3,5)$, $(3,6)$ and $(3,15)$ ensembles