Abstract—This paper considers near time-optimal setpoint tracking of a second-order oscillator. First, the time-optimal setpoint tracking feedback law is derived by recasting the problem as a regulator problem in the error coordinates with shifted control limits. This result is used to construct a damped approximation of the optimal control, PTOS. Initially, the PTOS is derived for a regulator controller for a system with zero damping. This result is then extended to include setpoint tracking. Global stability is proven for the setpoint tracking controller. A numerical approach is considered to apply the same method for plants with non-zero damping. Both the damped and undamped controllers are validated experimentally.

Index Terms—Proximate time-optimal control, flexible structures.

I. INTRODUCTION

In this paper, we are concerned with the near minimum-time setpoint tracking of systems which can be described as a damped harmonic oscillator

\[ G(s) = \frac{b_0}{s^2 + 2\zeta \omega s + \omega^2} \]  

(1)

when the input is subject to saturation. Such systems arise in various applications such as DC-DC buck converters [1], and the x-y micro-positioners used in scanning probe microscope-based data storage systems [2]–[4]. Fast setpoint tracking is also of interest in certain Atomic Force Microscopy (AFM) imaging methods [5], [6]. The piezo-electric stage of some AFMs can also be adequately described by a damped harmonic oscillator [7, p.179].

For a linear time-invariant system, Pontryagin’s Minimum Principle leads to a time-optimal control that is bang-bang [8]. For low-order systems, the bang-bang control can be expressed by a feedback control, characterized by a switching surface. Excellent resources for synthesizing these feedback laws can be found in [9]–[11]. Unfortunately, the bang-bang feedback control is impractical. In any real control system, there will be process and measurement noise, uncertainty in the system parameters, and finite actuation bandwidth which will cause the control to chatter between its maximum and minimum values. It has furthermore been shown that for some plants, a bang-bang feedback control can lead to a limit cycle for arbitrarily small variations in plant parameters [12].

A large body of work exists which develops methods to combat these problems. Reference [13] suggests pre-computing the time-optimal trajectory which is then tracked with a stabilizing trajectory tracking control law. At the end of this reference trajectory, an end-game control law is implemented to eliminate any final error. Although such an approach is applicable to higher-order systems, a downside to this method is it requires pre-computation of individual trajectories for each initial condition and target state.

Reference [14] shows that within a certain subset of the state space, the non-linear switching surface can be replaced with a linear plane which intersects the switching surface at the switch points. They then show how to effectively approximate the optimal control with a high-gain linear feedback. The downside to this method is, again, that the intersecting plane changes for each initial condition and target point.

The Proximate Time-Optimal Servomechanism (PTOS) is one of the most popular techniques in robust time-optimal control. However, the theory has been developed largely with a focus on rigid-body systems. The PTOS was first developed by Workman [15]–[17] for the double integrator plant, and later extended to a triple integrator plant [18]. Numerous other extensions and improvements have been developed over the years. The damping properties of the PTOS have been strengthened by scheduling the gain as a function of position [19]. In [20], the linear region of PTOS is replaced with a non-linear function, yielding a faster settle time. The PTOS has been extended to include friction [21] and high frequency flexible modes [15]. Recently, a two degree of freedom PTOS controller has been proposed, the MPTOS, which allows additional flexibility in the design [22].

Two exceptions to the focus on rigid bodies are [2] and [3]. These works propose PTOS controllers for an x-y micro-scanner modeled as (1). However, both approximate the true oscillator switching curve with the switching curve for a rigid body plant. As they note, this approximation is only valid for a limited range of model parameters and a limited subset of the phase space.

To our knowledge, no PTOS-like controller has been developed for (1) which is derived from the true switching curve, aside from our preliminary work in [23] and [24]. In [23], we developed a near time-optimal controller, PTOS, for a system with purely imaginary eigenvalues and proved global stability. This controller is only applicable to regulation to the origin. In [24], we outlined how to extend the PTOS regulator to include setpoint tracking, though with only a heuristic stability analysis. This paper represents a complete exposition and extension of both works. We prove global stability of
the general setpoint tracking controller for an un-damped plant. We also demonstrate how to use these methods for systems with non-zero damping and validate both controllers experimentally.

This paper is organized as follows. Section II formulates the basic problem setting and reviews the synthesis of the time-optimal feedback control law for the general case of setpoint tracking. By considering the error coordinates, we recast setpoint tracking as a regulation problem with shifted control limits. We show that these shifted control limits imply an asymmetric switching curve, in contrast to both rigid-body time-optimal control as well as the more commonly derived minimum-time regulation problem for the harmonic oscillator. In Section III, we derive the special case of the PTOS regulator. With this insight, we then extend the PTOS regulator to include setpoint tracking in Section IV. We prove global stability in Section V. Design considerations are presented in Section VI. In Section VII, we show how to directly account for damping using numerical techniques and show how this can be efficiently accomplished using scaling. In Section VIII, we show experimental results for the setpoint tracking PTOS as well as for the version that accounts for damping. Finally we provide concluding remarks in Section IX.

II. Problem Formulation

The damped oscillator, (1), can be be represented in state space as

\[
\dot{x} = Ax + Bu \\
y = Cx,
\]

where

\[
A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\]

We assume throughout this paper that \(0 \leq \zeta < 1\) so that the imaginary part of the eigenvalues of \(A\) are non-zero. Equation (2) is solved by the variation of constants formula

\[
x = e^{A(t)}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau
\]

where \(x_0 = x(0)\) and the state transition matrix is given by

\[
e^{A\tau} = e^{-\sigma\tau} \begin{bmatrix} \cos(\omega_d\tau) + \frac{\zeta\sin(\omega_d\tau)}{\sqrt{1-\zeta^2}} & \frac{\sin(\omega_d\tau)}{\sqrt{1-\zeta^2}} \\ -\frac{\omega\sin(\omega_d\tau)}{\sqrt{1-\zeta^2}} & \cos(\omega_d\tau) - \frac{\zeta\sin(\omega_d\tau)}{\sqrt{1-\zeta^2}} \end{bmatrix}
\]

\(\sigma = \zeta\omega, \quad \omega_d = \omega\sqrt{1-\zeta^2}.\)

For the minimum-time problem to make sense, we must assume a bounded control input. In particular, we assume that the system has symmetric control limits and that \(u(t) \in [-c, +c]\). If the control authority saturates at a value other than \(\pm 1\), the problem is easily scaled by subsuming the actual saturation value, \(u_{\text{max}}\), into \(b_0\) by substituting \(b_0 \leftarrow u_{\text{max}}b_0\).

In this paper, we will draw a distinction between regulation and setpoint tracking. By regulation, we mean driving some initial condition to the origin of the state space and holding it there. By setpoint tracking, we mean driving some initial condition to a holdable equilibrium,

\[
H_{eq} = \{x : x = -A^{-1}Bu, \ u \in [-1, +1]\},
\]

and holding it there. Note that for the state space representation in (3), \(H_{eq}\) is a line segment in \(\mathbb{R}^2\) on the \(x_1\)-axis between \([-c, +c]\) where \(c = \frac{b_0}{\omega}\). Although regulation has certainly a subset of setpoint tracking, we draw this distinction since, as we will see, regulation has a convenient symmetry lacking in the setpoint tracking problem. More formally, we can state our objective as:

Problem 1: (Minimum-Time Regulation, \(\zeta = 0\))

Given the system (2)-(3) with \(\zeta = 0\), and any initial state \(x(0) = x_o\), transfer the system to the origin of the state space in minimum time \(t_f\) and hold it there \(\forall \ t \geq t_f\).

Problem 2: (Minimum-Time Setpoint Tracking, \(\zeta = 0\))

Given the system (2)-(3) with \(\zeta = 0\), and any initial state \(x(0) = x_o\), transfer the system to a setpoint, \(x_r \in H_{eq}\), in minimum time \(t_f\) and hold it there \(\forall \ t \geq t_f\).

A. Review of the Time-Optimal Feedback Control

The time-optimal solution to both problems can be derived from Pontryagin’s Minimum Principle. In standard texts, this is typically done for Problem 1, the regulator case [8]-[11]. The setpoint tracking case has also been derived previously [25], [26], which we briefly review here.

First, note that when \(x_e = 0\), we must apply the constant control

\[
\bar{u}_s(t) = \frac{x_{1r}}{c}, \quad \forall t \geq t_f
\]

if we are to hold the state at \(x_r\). Now, we recast setpoint tracking as a regulation problem by considering the error coordinates, \(x_e = x - x_r\). The error dynamics are described by

\[
\dot{x}_e = Ax_e + Ax_r + Bu
\]

\[
= Ax_e + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} \left( u - \frac{1}{c}x_{1r} \right).
\]

Thus, driving the system to \(x(t_f) = x_r\) is equivalent to driving the error state \(x_e\) to the origin if the error dynamics are driven by a control, \(\bar{u}(t)\), with asymmetric saturation, namely

\[
\bar{u}^+ = 1 - \frac{1}{c}x_{1r},
\]

\[
\bar{u}^- = -1 - \frac{1}{c}x_{1r}
\]

It will be convenient to refer to (10) and (9) together as \(\bar{u}^\pm\). Furthermore, since we assume that \(x_r \in H_{eq}\), we can parameterize \(\bar{u}^\pm\) as

\[
\bar{u}^+ = 1 - \gamma
\]

\[
\bar{u}^- = -1 - \gamma
\]

\[
x_{1r} = \gamma
\]

\[
\gamma \in (-1, 1).
\]
Thus the optimization problem to solve for Problem 2 is

$$\min \int_0^{t_f} 1 \cdot d\tau$$

s.t. $\dot{x}_e(t) = Ax_e(t) + B\bar{u}(t)$,

$$x_e(0) = x_o - x_r, \quad x_e(t_f) = 0$$

$\bar{u} \in \{\bar{u}^-, \bar{u}^+\}$.

The Hamiltonian for this problem is

$$H(x, u, p) = 1 + p(t)^T(Ax_e(t) + B\bar{u}(t)).$$

Solving the two-point boundary-value problem

$$\dot{x}_e(t) = H_p = Ax_e(t) + B\bar{u}(t)$$

$$-\dot{p}(t) = H_{x_e} = A^T p(t)$$

$$\bar{u}(t) = \arg \min_{\bar{u} \in [\bar{u}^-, \bar{u}^+]} H(x_e(t), u(t), p(t))$$

s.t. $x_e(0) = x_o - x_r, \quad x_e(t_f) = 0, \quad H(t_f) = 0$

yields the time-optimal control, and various methods exist to compute $u(t)$ as an open-loop control. For our purpose however, we would prefer the optimal control as a feedback law. Because $\bar{u}(t) \in [\bar{u}^-, \bar{u}^+]$ is bounded, the optimal control is

$$\bar{u} = -\text{sgn}_a(p(t)^T B) = -\text{sgn}_a(p_2(t) b_o)$$

where we define the asymmetric signum function as

$$\text{sgn}_a(\xi) := \begin{cases} 
\bar{u}^+, & \xi > 0 \\
0, & \xi = 0 \\
\bar{u}^-, & \xi < 0
\end{cases}$$

Thus, the control switches when the costate velocity, $p_2(t)$, vanishes. Observe that $p(t)$ will rotate in the costate space at a rate of $\pi/\omega$ rad per unit time. This implies that the control is never constant for longer than $\pi/\omega$ units of time. Hence, we can locate the final leg of any time-optimal trajectory by solving (8) backwards in time from the origin of the error-phase plane for half a period with $\bar{u}(t)$ alternately fixed at $\bar{u} = \bar{u}^+$ and $\bar{u} = \bar{u}^-$ which results in two curves, $S_{1}^\pm$ (see Figure 1). Because any time-optimal trajectory with an initial condition not on $S_{1}^\pm$ must have switched at some point on $S_{1}^\pm$, we can again integrate backwards for $\pi/\omega$ units of time from all points on $S_{1}^\pm$ with $\bar{u} = \bar{u}^+$ to locate a new curve, $S_{2}^\pm$. In other words, $S_{2}^\pm$ is obtained by rotating $S_{1}^\pm$ counter-clockwise by $\pi/\omega$ radians above the point $\bar{u}^+$. Similarly, $S_{3}^\pm$ is obtained by rotating $S_{1}^\pm$ counter-clockwise by $\pi/\omega$ radians about the point $\bar{u}^-$. Continuing this process generates the time-optimal switching curve, $S = \bigcup_{i=1}^\infty S_{i}^\pm$, which is characterized by an infinite series of semi-ellipses.

To obtain a closed-form expression for the first segment, $S_{1} = S_{1}^- \cup S_{1}^+$, of the switching curve, we perform the integration

$$x_e(0) = \bar{u}^+ \int_{t_f}^0 e^{A(0-\tau)} B d\tau$$

which yields

$$\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} \bar{u}^+ c (1 - \cos \omega(-t_f)) \\ \bar{u}^+ b_o \sin \omega(-t_f) \end{bmatrix}.$$
By eliminating time, we can solve for $x_{2e}$ as a function of $x_{1e}$, which we call $f_{to} = f_{to}^{-} \cup f_{to}^{+}$,

$$f_{to}^{-} = \omega \sqrt{2x_{1e}c\bar{u} - x_{1e}^2}, \quad 2c\bar{u}^2 \leq x_{1e} < 0 \quad (22)$$

$$f_{to}^{+} = -\omega \sqrt{2x_{1e}c\bar{u} + x_{1e}^2}, \quad 0 \leq x_{1e} \leq 2c\bar{u}^2. \quad (23)$$

We use the convention that the superscript "−" denotes switching curve segments of the left-half phase plane and "+" for the right-half phase plane through the rest of this paper. It is worth pointing out that the same result, less a translation of $x_{1r}$, will be obtained by generating the switching curve in the un-shifted coordinates by integrating (4) backwards from $x_r$ with $u(t) = \pm 1$.

Note that $f_{to}^{-}$ is an ellipse in the negative half-plane, located with a center at $c\bar{u}^2$ with a semi-minor axis length of $c|\bar{u}|$, while $f_{to}^{+}$ is an ellipse in the positive half-plane which has a center of $c\bar{u}$ and a semi-minor axis length of $c\bar{u}$ so that in general, each ellipse has a different size and a different center. Only in the regulator case, when $x_r = 0$ and $\bar{u}^- = -1$ and $\bar{u}^+ = +1$ will the switching curve exhibit symmetry.

Although it is possible to develop closed form expressions for the rest of the switching curve, for practical purposes, we can simplify the development. In Figure 2, we see that the first set of switching curves, $f_{to} = f_{to}^{-} \cup f_{to}^{+}$, extend twice as far as the maximum holdable equilibrium limits. Thus, we are primarily concerned with the region of the state space which can be described as

$$Q = \{x : |x_1| < 2c, \text{ and } x_2 \text{ s.t. } x \text{ can be driven to } S_1 \text{ in } t < \pi/\omega \text{ seconds} \} \quad (24)$$

This is depicted as the shaded region in Figure 2 for $x_r = 0$.

To make this physical, the interval $(-c, c)$ represents the maximum reachable displacement of say, an AFM piezoelectric stage. Here, we are assuming the stage stays within a reasonable set about the maximum expected displacement. However, if this fails to be the case, we would be likely to consider the control law to still be well defined. Thus, for all $x_{1e} \notin [2c\bar{u}^2, 2c\bar{u}^2]$, we let

$$f_{to}^{-} = 0, \quad x_{1e} > 2c\bar{u}^2 \quad (25)$$

$$f_{to}^{+} = 0, \quad x_{1e} < 2c\bar{u}^2 \quad (26)$$

i.e., we flatten the switching curve to a straight line for states that are beyond the first switching curve. This approximation far from the origin was suggested in [11]. Although this choice is sub-optimal, our interest here is in systems which move between setpoints and the choice allows considerable simplification while maintaining global stability if the system sustains an excessively large disturbance.

Finally, though the input to the error dynamics is $\bar{u} \in [-\bar{u}^-, \bar{u}^+]$, the input to the actual plant is still $u \in [-1, +1]$, which we obtain by adding back the required steady-state feedforward input, $u_{ss} = \frac{x_1}{c}$. This development can be implemented as a feedback control law as

$$u = \text{sgn}_a(\bar{u} - x_{2e} + f_{to}(x_{1e})) + \frac{x_{1e}}{c} \quad (27)$$

1Throughout this paper, we use “semi-minor axis” to refer to the axis of an ellipse aligned with the $x_1$-axis of the phase plane. In the ellipses we consider, this terminology is correct, provided $\omega > 1$. 

---

Fig. 3: Block diagram of time-optimal reference tracking controller.

where, in its entirety,

$$f_{to} = \begin{cases} 0, & x_{1e} < 2c\bar{u}^- \\ \omega \sqrt{2x_{1e}c\bar{u} - x_{1e}^2}, & 2c\bar{u}^- \leq x_{1e} < 0 \\ -\omega \sqrt{2x_{1e}c\bar{u} + x_{1e}^2}, & 0 \leq x_{1e} \leq 2c\bar{u}^+ \\ 0, & x_{1e} > 2c\bar{u}^+ \end{cases} \quad (28)$$

which we concatenate here for clarity. The control law is illustrated in the block diagram shown in Figure 3. The scheme has the desirable feature that it applies the necessary constant control to hold the state at $x(t) = x_r, \forall t \geq t_f$.

---

### B. Time-Optimal Issues

Although time-optimal, the controller given by (27) and (28) is impractical to implement. On real systems, there will be process noise, model uncertainties, and finite actuation bandwidth. All of these will contribute to the system chattering as it attempts to follow the final switching curve to the origin. This phenomenon is illustrated in Figure 4 for a plant with a 5% deviation in $b_t$. This problem has been successfully addressed for a large class of systems exhibiting a rigid-body mode with PTOS-like controllers. In this paper, we build on those methods to develop a similar controller, which we call PTOS$\omega$, for the oscillatory system described by (2)-(3). With foresight, Figure 4 also shows the trajectory in the phase plane subjected to the PTOS$\omega$ control law. We begin by deriving a special case of PTOS$\omega$, the regulator.

---

### III. PTOS$\omega$: REGULATOR

In this section, we develop the PTOS$\omega$ for (2)-(3) for the special regulator case when $\zeta = 0$. Observe that in this case, $x_{1r} = 0$, $\bar{u}^- = -1$, $\bar{u}^+ = +1$ and the switching curves are symmetric as seen in Figure 2. Moreover, the asymmetric signum function, $\text{sgn}_a(\cdot)$ is the standard signum function, $\text{sgn}(\cdot)$.

To motivate the development, note that after the last switch, the time-optimal control law, (27), can be seen as tracking an optimal velocity profile with infinite gain and that the optimal velocity profile, $f_{to}^{-}$, has infinite slope at the origin. As with the rigid body PTOS [15], the PTOS$\omega$ seeks to approximate the optimal control with three key features: (i) the infinite gain of the $\text{sgn}(\cdot)$ is replaced with a large yet finite gain and a saturator, (ii) near the origin, the velocity profile is approximated with a constant, finite slope, yielding a linear feedback region, and (iii) the optimal velocity profile is discounted by a factor $\alpha \in (1/2, 1]$ to prevent saturation during deceleration.
\begin{align*}
\text{Typically, in rigid-body PTOS controllers } \alpha \in (0, 1) \text{ [15], [22]. Here, we restrict the discount factor to } \alpha \in (1/2, 1). \text{ If we discount the available acceleration by a factor } \alpha, \text{ then } f_{to}(x_1) \text{ scales according to } b_o \leftarrow \alpha b_o \text{ and } c \leftarrow \alpha c \text{ (this can be seen by integrating (20) backwards with } u(t) = \pm 1 \cdot \alpha). \text{ Thus, the ellipse described by the discounted velocity profile has an } x_1\text{-axis length of } 2\alpha c. \text{ Requiring } \alpha > 1/2 \text{ has the following effect: when subjected to } u = +1 \text{ (resp., } u = -1), \text{ the system rotates in an ellipse about the rotation center } (+c, 0) \text{ (resp., } (-c, 0)). \text{ By fixing } \alpha > 1/2, \text{ we ensure that this rotation center is always inside the endpoints of the discounted velocity profile. As we show in Section V, this permits a globally stable controller.}
\end{align*}

\subsection{A. Development of } \mathcal{F}_\ell(x_1)

\begin{align*}
\text{Consider the controller given by,}
\begin{cases}
   u_p(t) = 
   \begin{cases}
      \text{sat}(k_2(-x_2 + f_p(x_1))), & |x_1| < 2\alpha c \\
      \text{sgn}(-x_2 + f_p(x_1)), & \text{otherwise}
   \end{cases} \\
\end{cases}
\end{align*}

\begin{align*}
\text{where}
\begin{align*}
\text{sat}(\xi) = \begin{cases}
   1, & \xi \geq 1 \\
   \frac{\xi}{|\xi|}, & \xi < 1 \\
   -1, & \xi \leq -1
\end{cases}
\end{align*}
\end{align*}

\begin{align*}
\text{When } |x_1| < 2\alpha c, f_p(\cdot) \text{ is an approximation to the optimal velocity profile, } f_{to}(\cdot). \text{ The derivation of } f_p(\cdot) \text{ is the main focus of this section. First, note that a saturator and a gain gives a finite slope approximation to the sgn(\cdot) function. By making } f_p(x_1) \text{ approximate } f_{to}(x_1) \text{ and making } k_2 \text{ large, then in (29) we have an approximation to the optimal control, (27). Furthermore, we can employ a linear feedback controller near the origin (i.e., for } |x_1| \leq x_\ell \text{) by defining } f_p \text{ as the piecewise function}
\begin{align*}
f_p(x_1) = \begin{cases}
   f_\ell(x_1), & |x_1| \leq x_\ell \\
   f_{to}(x_1), & x_\ell < |x_1| \leq 2\alpha c \\
   0, & |x_1| > 2\alpha c
\end{cases}
\end{align*}
\end{align*}

\begin{align*}
\text{where we require that}
\begin{align*}
f_\ell(x_1) &= f_{st}(x_1) \\
f'_\ell(x_1) &= f'_{st}(x_1).
\end{align*}
\end{align*}

\begin{align*}
\text{By making } f_\ell(x_1) \text{ a linear function of } x_1, \text{ then for } |x_1| \leq x_\ell, (29) \text{ describes the familiar equation for linear state feedback, with the sat(\cdot) function enforcing respect for the control limits. Specifically, define the linear portion of } f_p \text{ as}
\begin{align*}
f_\ell(x_1) := -\left(\frac{k_1}{k_2}\right)x_1, & \quad |x_1| < x_\ell.
\end{align*}
\end{align*}

\begin{align*}
\text{We construct the entire } f_p \text{ by connecting this linear } f_\ell \text{ to vertical translations of } f_{to} \text{ such that (32) and (33) are satisfied. Taking the Taylor approximation of (28) about } x_\ell \text{ yields}
\begin{align*}
f_{to}(x_1) \approx \frac{1}{k_2} - \frac{k_1}{k_2}x_1
\end{align*}
\end{align*}

\begin{align*}
\text{where}
\begin{align*}
k_1 &= \frac{\alpha c - x_\ell}{\alpha c \ell} \\
k_2 &= \frac{\sqrt{2\alpha c^2 - x_\ell^2}}{\omega \alpha c \ell}.
\end{align*}
\end{align*}

\begin{align*}
\text{The solid black curve in Figure 5 is (35). Since our new velocity profile must go through the origin, add the } x_2\text{-intercept, } \frac{1}{k_2}, \text{ to (35) to yield } f_\ell(x_1), \text{ the blue-dashed curve in Figure 5. We connect the parts of } f_{to} \text{ outside } [-x_\ell, x_\ell] \text{ to } f_\ell \text{ by shifting the right portion of } f_{to} \text{ up by } \frac{1}{k_2} \text{ and the left}
\end{align*}
portion down by \( \frac{1}{k_2} \). Together, this yields the concatenation of all the dashed curves in Figure 5, which is \( f_p(x_1) \). We thus obtain
\[
f_t(x_1) = -\frac{k_1}{k_2} x_1 - \omega \sqrt{2\alpha |x_1| - x_1^2} + \frac{1}{k_2},
\]
\[
f_{nl}(x_1) = \text{sgn}(x_1) \left[ -\omega \sqrt{2\alpha |x_1| - x_1^2} + \frac{1}{k_2} \right].
\]
Note that if \( x_t \to 0 \) and \( \alpha \to 1 \), then \( k_2 \to \infty \) and \( \frac{1}{k_2} \to 0 \) and we recover (27).

B. Regulator PTOS\( \omega \) Stability

The PTOS\( \omega \) regulator is a special case of the setpoint tracking PTOS\( \omega \) developed in Section IV. We prove stability for this more general case in Section V.

IV. PTOS\( \omega \): Reference Tracking

Inspired by the development of the regulator PTOS\( \omega \), we will now develop the setpoint tracking PTOS\( \omega \). Examining Figure 1, the first challenge immediately presents itself. If we are to maintain the continuous differentiability of \( f_p(x_{1e}) \), we cannot have \( x^-_t = x^+_t \), since at a single \( x_t \) the curves \( f^-_t \) and \( f^+_t \) have different slopes, i.e., \( f^-_t (-x_t) \neq f^+_t(x_t) \). Rather, we need to enforce
\[
M := \frac{k^-_1}{k_2} = \frac{k^+_1}{k_2}.
\]
Because the curves are geometrically similar, choosing \( x^-_t \) and \( x^+_t \) as a fraction of the distance to the center of each ellipse gives us what we need, i.e., for \( 0 < \lambda \leq 1 \), choose
\[
x^+_t = \lambda \alpha c u^+_t,
\]
\[
x^-_t = \lambda \alpha c u^+_t.
\]
We can easily calculate \( M \) and the \( x_2 \)-intercepts, (i.e., \( 1/k_2 \)) previously \( 1/k_2 \) from (36) and (37) by the substitutions \( c \leftarrow c |\bar{u}^\pm| \) and \( x_t \leftarrow |x^\pm_t| \) from (41) and (42). This yields
\[
k^\pm_2 = \frac{\sqrt{2\lambda - \lambda^2}}{\omega \alpha c |\bar{u}^\pm|},
\]
\[
k^\pm_1 = \frac{1 - \lambda}{\lambda \alpha c |\bar{u}^\pm|}.
\]
It follows that \( M = \frac{k^-_1}{k^-_2} = \frac{k^+_1}{k^+_2} \). However, in order for the control to be well defined, we need a single gain, \( k_2 \). Notice that \( k^\pm_2 \) and \( k^\pm_1 \) as well as \( k^+_2 \) and \( k^-_1 \) differ only by the factor \( \bar{u}^\pm \) and that this factor cancels in \( k^+_2/k^-_2 \). Thus, we define
\[
k_1 = \frac{1 - \lambda}{\alpha c \lambda},
\]
\[
k_2 = \frac{\sqrt{2\lambda - \lambda^2}}{\omega \alpha c \lambda},
\]
and we see that \( k_1/k_2 = M \). Although the fraction of \( k_1/k_2 \) yields the correct slope, we must be careful to shift the curve by the appropriate amount to maintain continuity. This yields \( f_p \) as
\[
f_p(x_{1e}) = \begin{cases} 
0, & x_{1e} < 2\alpha c u^- \\[\omega \sqrt{2\alpha c u^- x_{1e} - x_{1e}^2} - \frac{\bar{u}}{k_2} \end{cases},
\]
\[
2\alpha c u^- \leq x_{1e} \leq x^-_t,
\]
\[
x^-_t < x_{1e} < x^+_t, \quad -\omega \sqrt{2\alpha c u^+ x_{1e} - x_{1e}^2} + \frac{\bar{u}}{k_2},
\]
\[
x^+_t \leq x_{1e} \leq 2\alpha c u^+, \quad x_{1e} > 2\alpha c u^+.
\]
Thus the PTOS\( \omega \) control for the error dynamics is given by
\[
\begin{align*}
\bar{u}_p &= \begin{cases}
\text{sgn}_a(-x_{2e} + f_p(x_{1e})) & x_{1e} < 2\alpha c u^- \\
\text{sat}_a[k_2(-x_{2e} + f_p(x_{1e}))] & 2\alpha c u^- \leq x_{1e} \leq 2\alpha c u^+ \\
\text{sgn}_a(-x_{2e} + f_p(x_{1e})) & x_{1e} > 2\alpha c u^+,
\end{cases}
\end{align*}
\]
where the asymmetric saturator is given by
\[
\text{sat}_a(\xi) = \begin{cases}
\bar{u}^+, & \xi > \bar{u}^+ \\
\bar{u}^-, & \bar{u}^- \leq \xi \leq \bar{u}^+
\end{cases},
\]
\[
\text{sat}_a(\xi) = \begin{cases}
\bar{u}^+, & \xi > \bar{u}^+ \\
\bar{u}^-, & \bar{u}^- \leq \xi \leq \bar{u}^+
\end{cases}.
\]
Just as in the time-optimal case, we add back the required steady-state feedforward control which yields
\[
u_p = \bar{u}_p + \frac{x_{1e}}{c}.
\]
Finally, it will be useful in the upcoming sections to define the following divisions of the error-coordinate state space:
\[
\mathcal{P} = \{ x_e : \frac{\bar{u}^-}{k_2} \leq (-x_{2e} + f_p(x_{1e})) \leq \frac{\bar{u}^+}{k_2},
\]
\[
x_{1e} \in [2\alpha c u^-, 2\alpha c u^+]
\]
\[
\mathcal{T}^+ = \{ x_e : x_e \in \mathcal{P}, x_{1e} > \alpha c u^+, x_{2e} > 0 \}
\]
\[
\mathcal{T}^- = \{ x_e : x_e \in \mathcal{P}, x_{1e} < \alpha c u^-, x_{2e} < 0 \}
\]
\[
\mathcal{B} = \{ x_e : x_e \in \mathcal{P} \setminus (\mathcal{T}^- \cup \mathcal{T}^+) \}
\]
\[
\mathcal{L} = \{ x_e : x \in \mathcal{B}, x_1 \in [x^-_t, x^+_t] \}
\]
\[
\mathcal{U}^- = \{ x_e : x \notin \mathcal{P}, x_2 \geq f_p(x_{1e}) \}
\]
\[
\mathcal{U}^+ = \{ x_e : x \notin \mathcal{P}, x_2 \leq f_p(x_{1e}) \}.
\]
These regions are illustrated in Figure 7. Regions \( \mathcal{U}^+ \) and \( \mathcal{U}^- \) are the regions of the state space which result in a saturated control. The entire region \( \mathcal{P} \) results in an unsaturated control. The region \( \mathcal{B} \) is an invariant subset of \( \mathcal{P} \), though the “tails” of \( \mathcal{P} \), called, \( \mathcal{T}^\pm \) are not invariant. Together, \( \mathcal{T}^- \cup \mathcal{B} \cup \mathcal{T}^+ = \mathcal{P} \). Finally, we note that \( \mathcal{L} \subset \mathcal{B} \subset \mathcal{P} \), where \( \mathcal{L} \) defines the region of unsaturated linear feedback.

V. Stability of the Reference Tracking PTOS\( \omega \)

**Theorem 1**: If the system described by (8) under the control law given by (47)–(49) satisfies
\[ 1/2 < \alpha < 1, \]
\[ |\gamma| < 1 - \frac{2\lambda}{\sqrt{2\lambda - \lambda^2}}, \]
CIII) \( \left( \frac{1}{c_x} \right)^2 < \frac{\varepsilon^2 \omega^2}{4} R \), where
\[
R = \min \{ R^+, \ R^- \} \tag{51}
\]
\[
R^+ = (2\bar{u}^+ - (\alpha + 1)\bar{u}^-)^2 - (2\alpha \bar{u}^+ - \bar{u}^-)^2 \tag{52}
\]
\[
R^- = ((\alpha + 1)\bar{u}^+ - 2\bar{u}^-)^2 - (2\alpha \bar{u}^- - \bar{u}^+)^2 \tag{53}
\]
then the system is asymptotically stable about the origin of the \( x_e \) phase plane.

**Proof:** Lemma 1 establishes that all trajectories enter region \( \mathcal{P} \) in finite time. Theorem 2 establishes that region \( \mathcal{B} \) is invariant. Lemma 8 establishes that all trajectories in \( T \) will leave \( T \) and enter \( \mathcal{B} \) in finite time. Finally, Theorem 3 shows that region \( \mathcal{B} \) is asymptotically stable about the origin by developing a Lyapunov function.

**Lemma 1:** Given an initial condition \( x(0) = x_0 \notin \mathcal{P} \), the system described by (8) under the control law given by (48) will drive the system to region \( \mathcal{P} \) in finite time.

**Proof:** Because the state rotates clockwise at a rate of \( \omega \) in the phase space and \( f_p(x_{1e}) \) divides the phase space into two disjoint regions, then after \( t < 2\pi/\omega \) units of time any trajectory under a constant input must cross \( f_p(x_{1e}) \).

**Case 1:** If the initial condition is such that, in \( t < 2\pi/\omega \) units of time, the trajectory crosses \( f_p(x_{1e}) \) with \( 2\alpha \bar{u}^- \leq x_{1e} \leq 2\alpha \bar{u}^+ \), we are done.

**Case 2:** Otherwise, the trajectory crosses \( f_p(x_{1e}) \) where \( f_p(x_{1e}) = 0 \). Hence, it is sufficient to show that all initial conditions of the form
\[
x_e(0) = [x_{1e}^0, 0]^T, \quad x_{1e}^0 \notin [2\alpha \bar{u}^- , 2\alpha \bar{u}^+] \tag{54}
\]
enter \( \mathcal{P} \) in finite time.

When \( \bar{u}_p(t) \) is held constant at \( \bar{u}_p = \bar{u}^+ \) (resp., \( \bar{u}_p = \bar{u}^- \)), it is straightforward to show using (4) and (5) that the resulting trajectory describes a portion of an ellipse centered at \( +\bar{u}^+ \) (resp., \( -\bar{u}^- \)). It follows that between every switch (besides the first and last), the absolute distance of \( x_{1e} \) to the origin has decreased by \( 2\bar{u}^+ \) (resp., \( 2\bar{u}^- \)).

If the trajectory crosses the \( x_{1e} \)-axis with \( 2\bar{u}^- < x_{1e} < 2\alpha \bar{u}^- \) (resp., \( 2\alpha \bar{u}^- < x_{1e} < 2\bar{u}^- \)), then the restriction that \( 0.5 < \alpha < 1 \) ensures that the rotation center, \( \bar{u}^- \) (resp., \( \bar{u}^+ \)) is contained between the endpoints of region \( \mathcal{P} \), so that the trajectory must intersect the boundary of \( \mathcal{P} \). The significance of this can be seen in Figure 6. If we allowed \( \alpha \leq 1/2 \), then at the final switch before entering \( \mathcal{P} \), the ellipse described by the trajectory could (with the right initial condition) rotate about a point outside \( \mathcal{P} \) and the state would never enter \( \mathcal{P} \).

Finally, after \( 2\pi/\omega \) units of time, i.e., after every two switches, the absolute distance of \( x_{1e} \) from the origin decreases by \( 4\epsilon \) so that any state within a finite distance must enter \( \mathcal{P} \) in finite time.

**A. The Region \( \mathcal{B} \) Is Invariant**

**Theorem 2:** The region \( \mathcal{B} \) is an invariant set, i.e., once a trajectory enters \( \mathcal{B} \) the state is trapped there.

**Proof:** Considering Figure 7, we see that the long upper and lower boundaries of \( \mathcal{B} \) are given by the segments \( \mathcal{D}C \) and \( \mathcal{C}D \), which separate \( \mathcal{B} \) from \( \mathcal{U}^\pm \). The shorter segments, \( \mathcal{D}C \) and \( \mathcal{C}D \), separate \( B \) from \( T^\pm \). To show that \( B \) is invariant, we show that at every point on the boundary, the vector field points toward the interior of \( B \). Lemmas 3-6 below establish this fact for segments \( \mathcal{D}C \) and \( \mathcal{C}D \). Lemma 7 shows this also holds for the boundary between \( B \) and \( T^\pm \).

To show that trajectories move to the interior of \( B \) at the boundaries between \( B \) and \( \mathcal{U}^\pm \), we will consider the time derivative of \( \bar{u}_p(t) \) along trajectories of the system evaluated at the upper and lower boundaries. For the system to remain trapped, we must have that on the upper boundary (\( \mathcal{D}C \)), \( \dot{\bar{u}}(x_e(t)) > 0 \); and on the lower boundary (\( \mathcal{C}D \)), \( \dot{\bar{u}}(x_e(t)) < 0 \) (since otherwise the control would be increasing past its saturated value which implies that the trajectory is leaving \( B \). To start, we take the time derivative of \( \bar{u}(x_e(t)) \) along a trajectory. This yields
\[
\dot{\bar{u}}(t) = k_2 f_p'(x_{1e}) f_p(x_{1e}) - f_p'(x_{1e}) \bar{u}_p(t) + k_2 \omega^2 x_{1e} - k_2 b_o \bar{u}_p(t). \tag{55}
\]
If we consider this derivative evaluated on either the upper or lower boundary of \( B \), we have \( \dot{\bar{u}}(t) = \bar{u}^- \) and \( \dot{\bar{u}}(t) = \bar{u}^+ \) respectively. Let \( \bar{u}_u(t) \) and \( \bar{u}_l(t) \) denote the time derivatives of \( \bar{u}(t) \) along the upper and lower boundary of \( B \), respectively. Then for invariance, the following inequalities must hold
\[
\dot{\bar{u}}_u(t) = k_2 f_p'(x_{1e}) f_p(x_{1e}) - f_p'(x_{1e}) \bar{u}^- + k_2 \omega^2 x_{1e} - k_2 b_o \bar{u}^- > 0 \tag{56}
\]
\[
\dot{\bar{u}}_l(t) = k_2 f_p'(x_{1e}) f_p(x_{1e}) - f_p'(x_{1e}) \bar{u}^+ + k_2 \omega^2 x_{1e} - k_2 b_o \bar{u}^+ < 0. \tag{57}
\]
Fig. 7: The entire shaded region is $P$, where the control is unsaturated. The lightly shaded tails are $T^\pm$. The darkly shaded region together with the hatch-marked region is $B$, which is invariant. The hatch-marked region is $L \subset B$.

Before stating and proving Lemmas 3-7, we note for $x \in P$

$$f_p'(x_1) = \begin{cases} \frac{\omega(\alpha \bar{u} - x_1)}{2\alpha c \bar{u}^+ - x_1} & -2\alpha c < x_1 < x_\ell^- \\ -\frac{k_1}{k_2} & x_\ell^- \leq |x_1| \leq x_\ell^+ \\ \frac{\omega(\alpha \bar{u} + x_1)}{2\alpha c \bar{u}^+ + x_1} & x_\ell^+ < x_1 < 2\alpha c. \end{cases} \tag{58}$$

Another useful observation is

**Lemma 2:** For $x_{1e} > 0$, $f_p'(x_{1e})$ is monotonically increasing with $x_{1e}$; and for $x_{1e} < 0$, $f_p'(x_{1e})$ is monotonically decreasing with $x_{1e}$.

**Proof:** Note that $f_p''(x_{1e}) = 0$ for $x_{1e} \in L$. Thus, we need only show that $\frac{\omega}{2\alpha c \bar{u}^+ - x_{1e}} f_p'(x_{1e}) > 0$ over the entire interval $x_{1e} \in [x_{1e}^-, 2\alpha c \bar{u}^+]$. Calculating the derivative, we see that

$$f_p''(x_{1e}) = \frac{\omega}{2\alpha c \bar{u}^+ x_{1e} - x_{1e}^2} + \frac{(\alpha \bar{u}^+ - x_{1e})^2}{(2\alpha c \bar{u}^+ x_{1e} - x_{1e}^2)^{3/2}}. \tag{59}$$

Both terms are always positive and real when $0 < x_{1e}^- \leq x_{1e} \leq 2\alpha c \bar{u}^+$, as desired. The case for negative $x_{1e}$ follows similarly.

Throughout the rest of the section, we will reference Figure 7, which shows the boundaries of region $B$ divided into different sections which we consider separately. However, even though the curves are not strictly symmetric, there is an equivalence between, for example, segment $BC$ and $BC$. These equivalent segments are considered concurrently in the ensuing proofs.

**Lemma 3:** Inequality (57) holds for segment $DE$ (the lower boundary in the non-linear region for positive $x_{1e}$) and (56) holds for segment $DE$ (the upper boundary in the non-linear region for negative $x_{1e}$).

**Proof:** Using (47) and (58) in (55), we obtain

$$\hat{u}_p = k_2 b_0 \alpha \bar{u}^\pm - k_2 b_0 \bar{u}^\pm$$

$$= k_2 b_0 \alpha \bar{u}^\pm (\alpha - 1) \tag{60}$$

Since $\alpha \in (1/2, 1)$ and $\bar{u}^- < 0$ and $\bar{u}^+ > 0$, we see that both (56) and (57) are satisfied.

**Lemma 4:** In the linear region, inequality (57) holds for $AE$ (lower boundary in $L^+$) and inequality (56) holds for $EA$ (upper boundary in $L^-$).

**Proof:** First consider the linear region on the lower boundary for $x_{1e} > 0$ (segment $AE$). Using (58) and (47) in (57), we obtain

$$k_1 k_2 x_{1e} + \frac{k_1}{k_2} \bar{u}^+ + k_2 \omega^2 x_{1e} - k_2 b_0 \bar{u}^- < 0 \tag{61}$$

We can maximize the LHS by letting $x_{1e} = x_\ell = \alpha c \bar{u}^+ \lambda$. Furthermore, using the expressions for $k_1$ and $k_2$ from (45) and (46), we obtain

$$\omega^2 (1 - \lambda)^2 \alpha \bar{u}^+ + \omega^2 (1 - \lambda) \alpha c \lambda \bar{u}^+ + \omega^2 \alpha \bar{u}^+ - b_0 \bar{u}^- < 0 \tag{62}$$

Now, using the fact that $c = b_0/\omega^2$ and canceling $\bar{u}^+$, we obtain, after further algebra

$$(2 - \lambda)(\alpha - 1) < 0.$$ 

Because $\lambda < 1$, the first factor is positive while the second factor is always negative. Therefore, the inequality holds.

The result for segment $EA$ follows similarly by noting the flip in inequalities that would occur when $\bar{u}^-$ is canceled in (62). Also, note that one would substitute $x_{1e} = \alpha c \lambda \bar{u}^-$ which is negative.

**Lemma 5:** Inequality (56) holds for segment $AB$ (the upper boundary in $L^-$) and inequality (57) holds for segment $AB$ (the lower boundary in $L^+$).

**Proof:** We begin with the upper boundary for positive $x_{1e}$ (segment $AB$). For this segment, (56) becomes

$$k_2 \left(\frac{k_1}{k_2}\right)^2 x_{1e} + \frac{k_1}{k_2} \bar{u}^- + k_2 \omega^2 x_{1e} - k_2 b_0 \bar{u}^- > 0. \tag{63}$$

We can minimize the LHS by letting $x_{1e} = 0$. Then (63) can be re-written as

$$\frac{k_1}{k_2} ( -|\bar{u}^-| - b_0 ( -|\bar{u}^-| ) > 0. \tag{64}$$
Now after canceling \(|\ddot{u}^-|\) and using \(k_1\) and \(k_2\) from (45) and (46) we obtain
\[
\lambda(1-\alpha) + 2 - \alpha > 0 \tag{65}
\]
which clearly holds. The case for segment \(\overline{AB}\) follows nearly identically.

**Lemma 6:** In the nonlinear region, inequality (56) holds for segment \(BC\) (the upper boundary for positive \(x_{1e}\)) and inequality (57) holds for segment \(\overline{BC}\) (the lower boundary for negative \(x_{1e}\)).

**Proof:** Consider segment \(BC\). Using (47) and (58), (56) becomes
\[
\dot{u}_e = k_2\alpha b_o \ddot{u}^+ + f'_p(x_{1e})\left(\ddot{u}^- - \ddot{u}^+\right) - k_2 b_o \ddot{u}^- > 0. \tag{66}
\]
Using the parameterization of \(\ddot{u}^+\) and \(\ddot{u}^-\) in (11) and (12), we have
\[
k_2\alpha b_o (1-\gamma) + k_2 b_o (1+\gamma) + 2f'_p(x_{1e}) > 0. \tag{67}
\]
Now, by Lemma 2, \(f'_p(x_{1e})\) is monotonically increasing with respect to \(x_{1e}\), so that we can minimize the right-hand side by letting \(x_{1e} = x^+_e = \lambda \alpha \ddot{u}^+\). This gives
\[
(\alpha(1-\gamma) + (1+\gamma) - 2(1-\lambda) \alpha > 0 \tag{68}
\]
After some algebra, this becomes
\[
(2-\lambda)(1+\gamma - \alpha \gamma) + \alpha \lambda > 0. \tag{69}
\]
Since each term is positive, the inequality holds. The case for \(\overline{BC}\) follows similarly.

**Lemma 7:** Suppose that Condition CI and CII are satisfied. Then (i) the boundary between \(B\) and \(T^+\) (resp., \(\overline{B}\) and \(T^-\)) is a line-segment, \(x_2 = 0\), between points \(CD\) (resp., \(\overline{C}\overline{D}\)), and (ii) at this boundary, the vector field points toward the interior of \(B\).

**Proof:** To show (i), we must ensure that at \(x_{1e} = \alpha \ddot{u}^+\), the upper boundary is less than zero and at \(x_{1e} = \alpha \ddot{u}^-\) the lower boundary is greater than zero.

The upper boundary at \(x_{1e} = \alpha \ddot{u}^+\) is given by
\[
x_{2e}^{upper} = -\frac{\ddot{u}^-}{k_2} + f_p(x_{1e}) \bigg|_{x_{1e} = \alpha \ddot{u}^+} < 0 \tag{70}
\]
\[
= -\ddot{u}^- + \frac{\ddot{u}^+}{k_2} - \omega \alpha \ddot{u}^+ < 0.
\]
Using our parameterization of \(\ddot{u}^+\) and \(\ddot{u}^-\) and (46), this becomes
\[
\frac{2\lambda}{\sqrt{2\lambda^2 - \lambda^2}} - \ddot{u}^+ < 0 \tag{71}
\]
where we have used (46). Performing a similar analysis for \(x_{1e} = \alpha \ddot{u}^-\) and requiring that the lower boundary be positive at this point, we find that
\[
|\gamma| < 1 - \frac{2\lambda}{\sqrt{2\lambda^2 - \lambda^2}} \tag{72}
\]
which is Condition CII.

Now, we will show (ii). Along the segments \(CD\) and \(\overline{CD}\) we have \(x_{2e} = \dot{x}_{1e} = 0\), so we must ensure that \(\dot{x}_{2e} < 0\) for positive \(x_{1e}\) and \(\dot{x}_{2e} > 0\) for negative \(x_{1e}\). Consider the case for \(x_{1e} > 0\). Then we have, noting that \(x_{2e} = 0\) on \(CD\),
\[
\frac{\dot{x}_{2e}}{b_o} = -\frac{1}{c} x_{1e} + k_2 f_p(x_{1e}) \tag{73}
\]
Furthermore, note that along \(CD\),
\[
\frac{1}{b_o} \frac{\partial^2}{\partial x_{1e}^2} \dot{x}_{2e} = k_2 f''_p(x_{1e}). \tag{74}
\]
We showed in Lemma 2 that \(f''_p(x_{1e}) > 0\) if \(x^+_e \leq x_{1e} \leq 2\alpha \ddot{u}^+\). This implies any stationary point of (73) is a minimum, so that the maximum of (73) must occur at the boundary. Hence, it is sufficient to check that (73) is negative when \(x_{1e} = 2\alpha \ddot{u}^+\) (it is clearly negative at point \(C\) because at point \(C\), \(f_p(x_{1e}) < 0\)). Indeed, we find that
\[
\frac{\dot{x}_{2e}}{b_o} \bigg|_{x_{1e} = 2\alpha \ddot{u}^+} = -2\alpha \ddot{u}^+ + \ddot{u}^+ = \ddot{u}^+ (1 - 2\alpha) < 0 \text{ for } \alpha > 1/2 \tag{75}
\]
which holds since we have restricted \(\alpha > 1/2\).

**B. All Trajectories Enter \(B\) in Finite Time**

We have shown that \(\mathcal{P}\) is a globally attractive set and that \(\mathcal{B}\) is an invariant set. We now show that \(\mathcal{B}\) is also globally attractive by showing that all initial conditions which begin in \(T^\pm\) enter \(\mathcal{B}\) in finite time.

**Lemma 8:** Suppose that Condition CIII holds and the conditions of Lemma 7 are satisfied. Then (i) the state exits \(T^\pm\) in finite time and (ii) all states which exit \(T^+\) (resp., \(T^-\)) along \(DG\) (resp., \(\overline{D}\overline{G}\)) enter \(\mathcal{B}\) by the end of one additional switch.

**Proof:** To show (i), we consider \(T^+\). By assumption, the conditions of Lemma 7 are satisfied so (75) gives an upper, strictly negative bound on the acceleration, \(\dot{x}_{2e}\), in \(T^+\) (when \(x_{2e} \neq 0\) but still in \(T^+\), the acceleration becomes more negative). Call this bound \(-\mu_1 = b_o \ddot{u}^+ (1 - 2\alpha), \mu_1 > 0\). Then for any \(x_{2e}(0) \in T^+\),
\[
x_{2e}(t) - x_{2e}(0) = \int_0^t \dot{x}_{2e}(\tau) d\tau \\
\leq -\mu_1 t.
\]
It follows, that after finite time, the state exits \(T^+\), which proves (i). A similar argument holds for \(T^-\). When the state exits \(T^+\) (resp., \(T^-\)), this only occurs when either the state enters \(\mathcal{B}\) via \(CD\) (resp., \(\overline{C}\overline{D}\)) or the state exits via \(GD\) (resp., \(\overline{G}\overline{D}\)). Part (ii) of the theorem deals with the second scenario.

To show (ii), suppose the state exits \(T^+\) along \(DG\) so that it re-enters \(U^-\). Then the next switch occurs when the state crosses the \(x_{1e}\)-axis. We can calculate this point by solving (4) for the time, \(t_s\), when \(x_{2e} = 0\) and using this in the \(x_{1e}\) component of (4). This gives,
\[
x_{1e}(t_s) = \sqrt{\left(\frac{x_{2e}(0)}{\omega}\right)^2 + \left(x_{1e}^0 - c\ddot{u}^-\right)^2 + \ddot{u}^-}. \tag{76}
\]
To prevent a limit cycle\footnote{We wish to eliminate the possibility that the state can, e.g., exit $T^+$, then enter $T^-$.}, we require that by the end of the next switch (i.e., in $\pi/\omega$ more units of time), the state enters $B$. Depending on our choices of $\alpha$, $\lambda$, and $\gamma$, the state could enter $B$ while $x_{1e}$ is still positive, in which case our job is done. However, if this does not happen, then the system will be under the constant control input $\hat{u}_p(t) = \hat{u}^+$ for at least half a rotation. One implication of \textit{CII} is that the $x_{1e}$ component of point $C$ is less than $\alpha c\bar{u}^-$. Thus if

$$-x_{1e}(t_0) + 2c\bar{u}^+ > \alpha c\bar{u}^- \quad (77)$$

holds, then we are guaranteed that the state enters $B$. The left-hand side is found by reflecting the point $x_{1e}(t_0)$ about the origin and decreasing its magnitude by $2c\bar{u}^+$. The worst case to consider is when the state exits $T^+$ with $x_{1e}^0 = 2\alpha c\bar{u}^+$, $x_{2e}^0 = 2/k_b$. Using these initial conditions with (76), it is straightforward to solve the inequality in (77) to give (52). Repeating a similar argument but starting at $x(0) = 2\alpha c\bar{u}^+ - 2/k_b$ yields (53).

It is noteworthy that simulation results indicate that satisfying \textit{CII} implies satisfaction of \textit{CII}, though this fact has so far eluded proof. We also point out that \textit{CII} is essentially required by convenience. If the condition is not satisfied, then, for example, the lower boundary of $P$ at $x_{1e} = \alpha c\bar{u}^-$ could sink below the $x_{1e}$-axis. If we allowed this, defining region $T^-$ becomes problematic. Ideally, we would integrate backwards from point $D$ until the trajectory crosses the lower boundary. All points in $P$ and to the left of this curve would be $T^-$ and points to the right would be $B$. Unfortunately, the non-linear feedback makes solving such a trajectory in closed-form unfeasible. Thus, we are left with Condition \textit{CII}, which, though overly restrictive, permits a workable definition of region $T^-$ and $T^+$.

\textbf{C. B Is Asymptotically Stable In The Sense of Lyapunov}

We have shown that all states will become trapped within $B$. Now, we show that for $x_e \in B$, $x_e$ tends asymptotically to the origin. We do this by determining a Lyapunov function $V(x_e)$ for the Region $B$. References [27] and [28] are excellent resources on Lyapunov Stability.

\textit{Theorem 3:} Define:

$$V(x_e) = \frac{1}{2}x_e^2 + \int_0^{x_{1e}} q(s)ds, \quad (78)$$

$$q(x_{1e}) = \omega^2 x_{1e} - b_k2f_p(x_{1e}).$$

Then (i) $V(x_e)$ is positive definite for $x_e \in B$ and (ii) $\dot{V}(x_e) \leq 0$ but does not vanish identically along any trajectory. Note that by LaSalle’s invariance principle [27], (i) and (ii) are sufficient to imply that $B$ is asymptotically stable.

\textit{Proof:} First we show (i). Clearly, $\frac{1}{2}x_e^2 \geq 0$. Thus it is sufficient to show that $q(x_{1e})$ is positive for positive $x_{1e}$ and negative for negative $x_{1e}$ since then the integral will always produce a positive area. This condition is readily checked for $x_{1e} \in \mathcal{L}$. For clarity, we consider the case for $x_{1e} > 0$. By construction, $f_p(x_{1e}) < 0$ until $x_{1e}$ is between the line segment defined by $CD$. For that region, $-q(x_{1e})/b_o$ is the same as (73) in Lemma 7 which we showed is strictly negative. Thus, the integral in (78) is always positive. Furthermore, since we have just shown that both terms of $V(x_e)$ individually are positive, it is a trivial observation (since $f_p(0) = 0$) that $V(x_e) = 0 \iff x = 0$. This proves (i).

Now we show (ii). Differentiating (78) yields

$$\dot{V}(x_e) = -b_k2x_{2e}^2 \leq 0, \quad \forall x_e \in B. \quad (79)$$

Clearly then, (79) is negative semi-definite for all $x \in B$.

Now, suppose $\dot{V}(x_e) = 0$. Then it must be that $x_{2e} = 0$. Thus when $\dot{V}(0) = 0$, trajectories of (8) must have the form $[x_{1e}, 0]^T$. Hence,

$$x_{2e} = 0, \quad \dot{x}_{2e} = -\omega^2 x_{1e} + b_k2f_p(x_{1e}) \quad (80)$$

which is just $-q(x_{1e})$ which, by the same argumentation as in (i), is strictly negative (resp., positive) for $x_{1e} > 0$ (resp., $x_{1e} < 0$) so that (80) equal to zero implies $x_{1e} = 0$. Furthermore, noting the linear region about the origin, $x_{1e} = 0$ implies that (80) is zero so that $\dot{V}(x_e)$ does not vanish identically along any trajectory.

\textbf{VI. Design Considerations}

When the system is in the linear region, $\mathcal{L}$, the transient response behaves according to

$$\dot{x}_e = (A - B \begin{bmatrix} k_1 & k_2 \end{bmatrix}) x_e \quad (81)$$

which means the closed-loop damping $\zeta$ and natural frequency $\omega$ are

$$\omega = \sqrt{1 + \frac{1}{\alpha \lambda}}, \quad \zeta = \frac{\sqrt{2 - \lambda}}{\alpha(1 + \lambda(\alpha - 1))}. \quad (82)$$

Moreover, (\textit{CII}) can be written as

$$\lambda < \frac{2(1 - |\gamma|)^2}{(1 - |\gamma|)^2 + 4}. \quad (83)$$

Bearing in mind that $\gamma$ parameterizes the holdable setpoints, (83) says that the closer to the boundaries of $\mathcal{H}_{eq}$ we wish to visit, the smaller the linear region must be. A plot of the maximum allowable $\lambda$ vs $\gamma$ is shown in Figure 8. Additionally, we have plotted the minimum increase in closed-loop bandwidth vs $|\gamma|$ for $\alpha = 0.95$. For example, if we wish to visit setpoints, $x_{1r} \in (-0.8c, 0.8c)$, then we must have $\lambda < 0.02$ which, for $\alpha = 0.95$, implies that the closed-loop bandwidth has increased by approximately seven times. This has two immediate implications. First, if the control law is to be implemented digitally and we follow the rule of thumb that the sample frequency should be 30 times the closed-loop bandwidth [29], then the sample frequency would need to be about 210 times the system resonance. Second, the nearest mode that destroys our second-order assumption should be almost a decade away from the main second-order resonance. As we noted at the end of Section V-B, \textit{CII} appears to be more restrictive than is strictly necessary for stability. Hence, if \textit{CII} could be relaxed or replaced with a necessary and sufficient condition, these design considerations would be less limiting.
From (82), the closed-loop damping depends on $\lambda$. However, if $\lambda << 1$, then $2 - \lambda \approx 2$ and $\lambda(\alpha - 1) \approx 0$ so $\hat{\zeta} \approx \sqrt{2\xi}$. Thus, in general we expect the closed-loop response in the linear region to be well damped and that this damping increases with decreasing $\alpha$.

**VII. PTOS$\omega\zeta$: The Case With Damping**

We now consider the case for non-zero damping, $0 < \zeta < 1$. Note that the definition of the error coordinates, (8) as well as $H_{eq}$ and the asymmetric saturation associated with the error dynamics, (10)-(9) are unchanged. Just as we did before in Section II, integrating (20) locates the first segment of the time-optimal switching curve which yields

$$
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= \bar{u}^\pm_\alpha 
\begin{bmatrix}
  \alpha \sigma T & -c \\
  \frac{\sin \omega d \ell}{\cos \omega d \ell} & 0
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
+ \begin{bmatrix}
  c \\
  0
\end{bmatrix}. 
$$

(84)

Though it is still possible to eliminate time from the equations as we did with (21), the presence of the exponential makes it impossible to isolate $x_1$ or $x_2$ on one side. However, if $x_1$ is monotonic, then to each $(x_1(t_i))$ there corresponds a unique $x_2(t_i)$ so that the tuple $(x_1, x_2)$ represents a function

$$
x_1(t_i) \mapsto F(x_1(t_i)) = x_2(t_i)
$$

(85)

which is the switching curve. From a practical point of view, $F(x_1)$ is easily represented as a lookup-table whose values are calculated via numeric integration. Just as in the case without damping, the switching curve will change for each new setpoint.

Of course, it is impractical to store a different switching curve every possible setpoint. However, this can be solved through careful scaling. Let $F_{reg}$ denote the regulator switching curve (i.e., $x_r = 0$, $\bar{u}^- = -1$, $\bar{u}^+ = +1$) and let

$$
\frac{v_r(s)}{V_{in}(s)} = \frac{1/\ell}{s^2 + \frac{R}{\ell} s + \frac{1}{C\ell^2}}.
$$

(92)

For both experiments, we programmed the control laws into a National Instruments Compact RIO (NI cRIO-9082) FPGA, using a sample frequency of $F_s = 100$ kHz. Because our control law utilizes both states of the system, we implemented a digital prediction observer on the FPGA to provide an estimate of the state, denoted $\dot{x} = [\dot{x}_1 \dot{x}_2]^T$. For both
cases, we chose \( \lambda \) as small as possible while keeping the closed-loop bandwidth in the linear region reasonably close to 100kHz/30 \( \approx 3.3 \)kHz. All plant and controller parameter values are summarized in Table I.

In both cases, we compare the experimental results to two simulations. First, we compare to a simulation of the PTOS\( \omega(\zeta) \) controller implemented in continuous time with direct measurements of both states. To show the level of sub-optimality, we also show the ideal time-optimal solution.

A. No Damping PTOS\( \omega \)

If we could construct the circuit with \( R = 0 \), we would have exactly the plant described in (1) with \( \zeta = 0 \). Of course, this is impossible but we can get close. We chose a capacitor and inductor which have nominal values \( C = 0.235 \) \( \mu \)F and \( L = 100 \) mH. The inductor has an internal resistance of \( R = 82 \) \( \Omega \). In trying to construct a passive circuit with extremely high \( Q \), note that choosing a larger inductor tends to increase its internal resistance. Of course, we could try to increase \( \omega = \sqrt{1/LC} \) by choosing smaller capacitance values and could thus theoretically get arbitrarily low damping. However, since the control law is implemented digitally on an FPGA, this approach is limited by the achievable sample rate. The values chosen represent a compromise between these competing concerns.

We performed a white noise system identification which yields

\[
G_1(s) = \frac{4.47806e7}{s^2 + 843.519s + 4.44851e7}.
\]

(93)

Thus, \( \zeta = 0.06 \). We induced an initial condition by issuing a step command of \( V_{in} = 1 \) volt to the system. After a settling period, we turned on the PTOS\( \omega \) controller with a reference value of \( x_{1_r} = -0.5c \). We set \( \lambda = 0.1 \) and \( \alpha = 0.85 \). In the linear region, this gives a closed-loop bandwidth of approximately 3.61 kHz which is about 27.7 times less than the sample rate. Since our control law was designed for continuous time, this represents essentially the lower bound on \( \lambda \) given the sample rate.

The results of this experiment are displayed in Figures 10 and 11. Also plotted are the results of the two simulations. First, we simulate with the PTOS\( \omega \) controller designed as though the plant has zero damping, but with the simulation plant as (93) and with both states directly available. For comparison, we also simulate the time-optimal controller.

B. PTOS\( \omega \zeta \) with damping

We also experimentally verified the method outlined in Section VII. Now, we add a 1 k\( \Omega \) resistor in series with the inductor. The system identification yields

\[
G_2(s) = \frac{4.07385e7}{s^2 + 9844.84s + 4.0891e7}
\]

(94)

which has a damping of \( \zeta = 0.77 \). As outlined in Section VII, we generated the regulator \( F_{p,reg} \) as a lookup table. To ease implementation details, we defined an evenly spaced grid of 1024 points for \( x_1 \) in the interval \( (-2c, 2c) \). We then computed \( F_{p,reg} \) at these grid points offline. This data is stored in the memory of the FPGA and a routine was programmed to perform linear interpolation between the grid points. Here, we use \( \alpha = 0.85 \) and \( \lambda = 0.15 \) which gives a closed-loop bandwidth of 3.63kHz. For this experiment, we use the same initial conditions and setpoint as above. The results of this experiment are plotted in Figure 12 along with the simulated PTOS\( \omega \zeta \) and time-optimal responses.

C. Discussion

For the listed control parameters, both experiments performed quite well and match with the simulated results nicely. In theory, decreasing \( \lambda \) below the values used here should yield even faster responses that approach the time-optimal response. However, in other trials not shown, this resulted in a deterioration in performance, likely due to the discretization of
the control law. Although the simulated time-optimal results are significantly faster, these results represent a theoretical lower bound on the settle time which are not achievable in practice. As noted in Section II-B, this is due to a combination of finite actuation bandwidth, model uncertainty and process noise.

IX. CONCLUSIONS AND FUTURE WORK

In this paper, we reviewed the derivation of the time-optimal setpoint tracking controller for the harmonic oscillator, which we noted can be viewed as a time-optimal regulator with asymmetric saturation limits in the error coordinates. Using this development, we derived the PTOSω controller, which has finite bandwidth and smoothly blends a non-linear controller into a standard linear feedback law. We then proved stability of the PTOSω provided certain conditions are satisfied. Moreover, we showed how the method can be extended to damped harmonic oscillators by using a single lookup-table. Finally, we demonstrated that both controllers perform well in practice.

Future research should investigate digital implementation issues by determining the slowest sample rate for which stability can be guaranteed or by developing a fully discretized version. Replacing the sufficient conditions for stability, CII and CIII, with sufficient and necessary conditions should result in an increased range of permissible λ and γ. Due to the increased complexity over using the rigid-body switching curve as in [2], [3], further investigation could consider how much optimality is gained through our approach and for which parameter values their method fails.

REFERENCES


Roger A. Braker is a graduate student in the Electrical, Computer, and Energy Engineering Department at the University of Colorado Boulder. He earned the bachelor's degree in physics from the University of Oklahoma and is a student member of the IEEE Control Systems Society. His research focuses on the application of optimal control methods to Atomic Force Microscopy.

Lucy Y. Pao is currently a Professor in the Electrical, Computer, and Energy Engineering Department and a Professor (by courtesy) in the Aerospace Engineering Sciences Department at the University of Colorado Boulder. She earned B.S., M.S., and Ph.D. degrees in Electrical Engineering from Stanford University. Her research has primarily been in the control systems area, with applications ranging from atomic force microscopy to disk drives to digital tape drives to megawatt wind turbines and wind farms. Earlier major awards received include a National Science Foundation (NSF) Early Faculty CAREER Award, an Office of Naval Research (ONR) Young Investigator Award, and an International Federation of Automatic Control (IFAC) World Congress Young Author Prize. Selected recent honors include elevation to IEEE Fellow in 2012, the 2012 IEEE Control Systems Magazine Outstanding Paper Award (with K. Johnson), election to Fellow of IFAC in 2013, and the 2015 SIAM J. Control and Optimization Best Paper Prize (with J. Marden and H. P. Young). Selected recent and current professional society activities include being General Chair for the 2013 American Control Conference, an IEEE Control Systems Society (CSS) Distinguished Lecturer (2008-2014), a member of the IEEE CSS Board of Governors (2011-2013 (elected) and 2015 (appointed)), Fellow of the Renewable and Sustainable Energy Institute (2009-present), IEEE CSS Fellow Nominations Chair (2016–), member of the IFAC Fellow Selection Committee (2014-2017), and member of the International Program Committees for the 2016 Indian Control Conference, the 2016 IFAC Symposium on Mechatronic Systems, and the 2017 IFAC World Congress.