This exam has 23 questions, for a total of 100 points.

1. 3 points Find $s$, $t$, and $d$ such that $d$ is the greatest common denominator of 71 and 5 and $d = s71 + t5$.

   **Solution:**
   \[ d = 1, s = 1, t = -14 \]

2. 1 point What does it mean for $a$ to be congruent to $b$ modulo $m$ (that is, $a \equiv b \pmod{m}$)?

   **Solution:** It means that $m| (a - b)$.

3. 1 point What does it mean for $a$ to be the inverse of $b$ modulo $m$.

   **Solution:** It means that $ab \equiv 1 \pmod{m}$

4. 3 points
   
   1. What is the inverse of 71 modulo 5?
   
   2. What is the inverse of 5 modulo 71?

   **Solution:** 1 (mod 5) and 57 (mod 71) respectively.

5. 3 points Find a solution to the following system of congruences.

   \[
   x \equiv 3 \pmod{71} \\
   x \equiv 4 \pmod{5}
   \]

   **Solution:** $x \equiv 74 \pmod{355}$. Because $3 \cdot 5 \cdot 57 + 4 \cdot 71 \cdot 1 = 1139$ and $1139 \mod 355 = 74$. 
6. **4 points** For each of the following relations on $\mathbb{R}$, state whether it is reflexive, symmetric, antisymmetric, and which relations are functions?

\[ R_1 = \{(x, y) \mid x + y = 0\} \]
\[ R_2 = \{(x, y) \mid x = 1\} \]
\[ R_3 = \{(x, y) \mid x = y \lor x = -y\} \]

**Solution:**

- $R_1$ is not reflexive, it is symmetric, it is not antisymmetric, and it is a function.
- $R_2$ is not reflexive, it is not symmetric, it is antisymmetric, and it is not a function.
- $R_3$ is reflexive, it is symmetric, it is not antisymmetric, and it is not a function.

7. **2 points** Let $R$ be the relation on the set of all people such that $(a, b) \in R$ if person $a$ is a parent of person $b$. Give an intuitive characterization of when an ordered pair in the relation $R^3$.

**Solution:** The relation $R$ corresponds to the great grandparent relation.

8. **4 points** List the ordered pairs in the equivalence relation produced by the partition \{a, b\}, \{c, d\}, \{e, f, g\} of the set \{a, b, c, d, e, f, g\}.

**Solution:**

- $(a, a)$, $(b, b)$, $(c, c)$, $(d, d)$, $(e, e)$, $(f, f)$, $(g, g)$,
- $(a, b)$, $(b, a)$, $(c, d)$, $(d, c)$, $(e, f)$, $(f, g)$, $(e, g)$, $(f, e)$, $(g, f)$, $(g, e)$

9. **5 points** Write down a compatible total order for the partially ordered set specified by the following Hasse diagram.

```
  l
 / \
 j   k
 / \ / \ \
 d   h   g
 /   \   \ 
 a   b   f
```
10. Give the set of lower bounds for $\{j, k, m\}$ for the partially ordered set in Question 9.

Solution: $\{b, h\}$

11. What is the greatest lower bound of $\{j, m\}$, if there is one, for the partially ordered set in Question 9?

Solution: $h$ is the greatest lower bound of $\{j, m\}$.

12. The following recursive algorithm computes a compatible total order for a finite partially ordered set $S$ (topological sorting). Prove that this algorithm is correct, that is, prove that the output is a total ordering of $S$ compatible with $\leq$. (The notation $x\#L$ creates a list with $x$ at the front and $L$ as the rest. The notation [] is for the empty list.)

```
procedure topo_sort(S, $\leq$)
  if $S = \emptyset$ then
    return []
  else
    $x :=$ a minimal element of $S$ with respect to $\leq$.
    $L :=$ topo_sort($S - \{x\}$, $\leq$)
    return $x\#L$
```

Solution: The proof is by mathematical induction on the cardinality (size) of $S$.

Base case $|S| = 0$: So $S = \emptyset$ and [] is trivially a total order that is compatible with $\leq$.

Induction Step: The induction hypothesis is that for any set $S$ of size $k$, topo_sort($S, \leq$) returns a total ordering of $S$ compatible with $\leq$. We need to prove that for any set $S$ of size $k + 1$, topo_sort($S, \leq$) returns a total ordering of $S$ compatible with $\leq$. We fix a set $S$ of size $k + 1$. Let $x$ be a minimal element of $S$ with respect to $\leq$. Then the set $S - \{x\}$ has size $k$ and we can apply the induction hypothesis to infer that topo_sort($S - \{x\}, \leq$) returns a total ordering $L$ of $S - \{x\}$ that is compatible with $\leq$. Because $x$ is a minimal element of $S$, $x$ is no greater than any element of $S$. Thus, $x\#L$ is a total ordering of $S$ that is compatible with $\leq$.  

13. **4 points** Are the following two graphs isomorphic? If so, give the isomorphism.

![Graphs](image)

**Solution:** No, they are not isomorphic. The graph on the right has two vertices of degree two, but the graph on the left only has one vertex of degree two. Thus, because degree is a graph invariant, these two graphs cannot be isomorphic.

14. **2 points** Give the adjacency matrix representation of the following graph.

![Graph](image)

**Solution:**

\[
\begin{array}{cccc}
  a & b & c & d \\
  a & 0 & 1 & 1 & 1 \\
  b & 1 & 0 & 0 & 1 \\
  c & 1 & 0 & 0 & 1 \\
  d & 1 & 1 & 1 & 0 \\
\end{array}
\]

15. **7 points** How many paths of length 2 are there from vertex \( a \) to \( d \) in the graph in Question 14? How many paths of length 3?

**Solution:** There are 2 paths of length 2. There are 5 paths of length 3.

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{bmatrix},
A^2 = \begin{bmatrix}
3 & 1 & 1 & 2 \\
1 & 2 & 2 & 1 \\
1 & 2 & 2 & 1 \\
2 & 1 & 1 & 3 \\
\end{bmatrix},
A^3 = \begin{bmatrix}
4 & 5 & 5 & 5 \\
5 & 2 & 2 & 5 \\
5 & 2 & 2 & 5 \\
5 & 5 & 5 & 4 \\
\end{bmatrix}
\]
16. **7 points** Identify a shortest path from vertex $a$ to $z$. What is its length?

\[
\begin{array}{c}
\text{Solution:} \quad \text{The shortest path is } a \stackrel{1}{\rightarrow} e \stackrel{1}{\rightarrow} i \stackrel{2}{\rightarrow} j \stackrel{1}{\rightarrow} k \stackrel{1}{\rightarrow} o \stackrel{1}{\rightarrow} z, \text{ which has length 7.}
\end{array}
\]

17. **6 points** Apply breadth-first search to the following undirected graph, starting at vertex $h$, and draw the resulting search tree using arrows that point from parent to child.

\[
\begin{array}{c}
\text{Solution:} \quad \text{The following search tree is denoted by the double lines in the following graph. There are other solutions.}
\end{array}
\]
18. **6 points** Apply depth-first search to the graph below, starting at vertex $h$, and draw the resulting search tree using arrows that point from parent to child. When choosing which vertex to visit next, give precedence to vertices whose label is lower in the alphabet.

![Graph](image)

**Solution:**

![Solution Graph](image)

19. **7 points** Identify a minimum spanning tree for the following graph. What is its total weight?
Solution: 18  
(Here’s one minimum spanning tree with total weight 18:

![Minimum Spanning Tree](image)

20. **7 points** Given the following grammar

\[
E \rightarrow EpE \\
E \rightarrow nE \\
E \rightarrow 0 \\
E \rightarrow 1 \\
E \rightarrow 2
\]

1. Draw all of the derivation trees for \(n0p1p2\).
2. Draw all of the derivation trees for \(0p1np2\).

Solution: 1.

\[
\begin{array}{c|c|c}
E & E & E \\
\hline
/|\ & /\ & /|\
|/| & |/| & |/| \\
||| & ||| & ||| \\
| | & | | & | |
\end{array}
\]

\[
\begin{array}{c|c|c}
E & E & E \\
\hline
/|\ & /\ & /|\
|/| & |/| & |/| \\
||| & ||| & ||| \\
| | & | | & | |
\end{array}
\]

\[
\begin{array}{c|c|c}
E & E & E \\
\hline
/|\ & /\ & /|\
|/| & |/| & |/| \\
||| & ||| & ||| \\
| | & | | & | |
\end{array}
\]

\[
\begin{array}{c|c|c}
E & E & E \\
\hline
/|\ & /\ & /|\
|/| & |/| & |/| \\
||| & ||| & ||| \\
| | & | | & | |
\end{array}
\]

\[
\begin{array}{c|c|c}
E & E & E \\
\hline
/|\ & /\ & /|\
|/| & |/| & |/| \\
||| & ||| & ||| \\
| | & | | & | |
\end{array}
\]

\[
\begin{array}{c|c|c}
n0p1p2 & n0p1p2 & n0p1p2 \\
\end{array}
\]
21. 2 points Does the following automata accept or reject the string 010010110? List the sequence of states that were visited as the automata processed this string.

![Automata Diagram]

**Solution:** The string 010010110 is accepted. The sequence of states visited is abbcdbcaab.

22. 7 points Minimize the finite-state automata given in Question 21.

**Solution:** The distinguishing table is

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>×</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td></td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
</tbody>
</table>

So states a and c can be merged and so can b and d. The minimized automata is therefore
23. **7 points** Construct a finite-state automata over the input alphabet \( \{a, b, c\} \) that recognizes the language of the regular expression \((ab \cup c)^*\).

**Solution:**