Subtyping: motivation

- Recall that type systems are conservative predicates that filter out bad programs, but also some good programs.
- Much of the research on more advanced type systems tries to allow more good programs, while still filtering out the bad ones.
- Example of a good program that is filtered out by the type systems we've seen so far:

  $(\lambda$ r: {x:Nat}. r.x) {x=0,y=1}

Subtyping

- The **subtype** relation is a binary relation on types, usually written $T_1 <: T_2$, that is meant to capture when values of type T_1 can be safely used in contexts that expect values of type T_2.
- It is often helpful to think of types as describing sets of values, and subtyping as the subset relation between these sets.
- Different languages provide different definitions for which pairs of types are in the subtype relation.
- The type system is extended with a **subsumption** rule that allows an expression of type T_1 to be treated as if it has type T_2, provided $T_1 <: T_2$.

$$\frac{\Gamma \vdash e : T_1 \qquad T_1 <: T_2}{\Gamma \vdash e : T_2}$$

Defining the Subtype Relation

- For the moment, the setting is the simply typed lambda calculus with naturals, Booleans, and records.
- And of course, a value of type *Bool* can be used as a *Bool*, and similarly for *Nat*.

$$\frac{}{Bool <: Bool} \qquad \frac{}{Nat <: Nat}$$

- We want to allow records with more fields to be used in contexts that expect fewer fields. This is called **width subtyping**. Common fields must have the same type. In the following, we view a record type as a partial function from labels to types.

$$\frac{dom(R_2) \subseteq dom(R_1) \qquad \forall l \in dom(R_2).\ R_1(l) = R_2(l)}{R_1 <: R_2}$$

For example, $\{x : Nat, y : Nat\} <: \{x : Nat\}$.

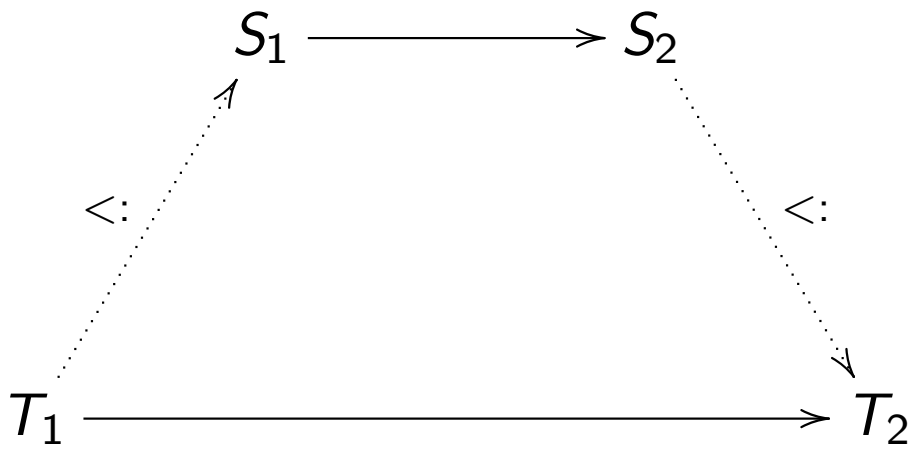
Subtyping for Records

- We also want to allow subtyping between records whose fields may differ according to the subtype relation. This is called **depth subtyping**.

$$\frac{dom(R_1) = dom(R_2) \qquad \text{for } l \in dom(R_1). \quad R_1(l) <: R_2(l)}{R_1 <: R_2}$$

For example,
$\{x : \{a : Nat, b : Nat\}, y : Nat\} <: \{x : \{a : Nat\}, y : Nat\}$.

Subtyping for Functions



$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \to S_2 <: T_1 \to T_2}$$

Functions are **contravariant** in their parameter types and **covariant** in the return types.

Properties of the Subtype Relation

Proposition (Basic Properties)

1. *Subtyping is reflexive: for any* T, $T <: T$.
2. *Subtyping is transitive: for any* R, S, T, *if* $R <: S$ *and* $S <: T$, *then* $R <: T$.

Proposition (Inversion)

1. *If* $S <: T_1 \to T_2$ *then* $S = S_1 \to S_2$, $T_1 <: S_1$, *and* $S_2 <: T_2$ *for some* S_1 *and* S_2.
2. *If* $S <: R$ *and* R *is a record type, then* S *is a record type and* $dom(R) \subseteq dom(S)$ *and for* $l \in dom(S)$, $S(l) <: R(l)$.

Properties of the Subtype Relation

Lemma (Lambda Typing)

If $\Gamma \vdash \lambda x : S_1.\ e : T_1 \to T_2$, *then* $T_1 <: S_1$, *and* $\Gamma, x : S_1 \vdash e : T_2$.

Let ρ range over record values and R over record types. A record value is a partial function from labels to values.

Lemma (Record Typing)

If $\Gamma \vdash \rho : R$, *then* $dom(R) \subseteq dom(\rho)$ *and for* $l \in dom(R)$, $\Gamma \vdash \rho(l) : R(l)$.

Properties of the Subtype Relation

Lemma (Substitution)

If $\Gamma, x : S \vdash e : T$ *and* $\Gamma \vdash e' : S$, *then* $\Gamma \vdash [x \mapsto e']e : T$.

Proof.

By induction on $\Gamma, x : S \vdash e : T$.

Case $\dfrac{\Gamma \vdash e : T_1 \qquad T_1 <: T_2}{\Gamma \vdash e : T_2}$:

From the induction hypothesis, we have $\Gamma \vdash [x \mapsto e']e : T_1$. Then by the subsumption rule we have $\Gamma \vdash [x \mapsto e']e : T_2$.

$\vdots$

$\square$

Proof of Type Safety

Lemma (Subject Reduction)

If $\Gamma \vdash e : T$ *and* $e \longrightarrow e'$, *then* $\Gamma \vdash e' : T$.

Proof. By cases on $e \longrightarrow e'$.

Case $\boxed{((\lambda x : S_1.\ e_b)\ v) \longrightarrow [x \mapsto v]e_b}$:

Because e is well typed, we have $T = T_1 \to T_2$, $\Gamma \vdash (\lambda x : S_1.\ e_b) : T_1 \to T_2$, and $\Gamma \vdash v : T_1$. From the lambda typing lemma, we have $T_1 <: S_1$ and $\Gamma, x : S_1 \vdash e_b : T_2$. Because $T_1 <: S_1$ we have $\Gamma \vdash v : S_1$. Then by the substitution lemma, $\Gamma \vdash [x \mapsto v]e_b : T_2$.

Subject Reduction, continued

Case $\boxed{\dfrac{\textit{for } l' \in \textit{dom}(\rho).\ \rho(l') \text{ is a value} \qquad l \in \textit{dom}(\rho)}{\rho.l \longrightarrow \rho(l)}$:

Because e (which is $\rho.l$) is well typed, we have $\Gamma \vdash \rho : R$, for some record type R, and $l \in \textit{dom}(R)$ and $T = R(l)$. Then by the record typing lemma, for every $j \in \textit{dom}(R)$, we have $\Gamma \vdash \rho(j) : R(j)$. Therefore $\Gamma \vdash \rho(l) : R(l)$.

$\vdots$