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Coherent Multiuser Space-Time Communications: Optimum Receivers and Signal Design

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Abstract — **The jointly optimum receiver is obtained for multiuser communications in a frequency non-selective Rayleigh fading channel with N_{T} transmit antennas per user and N_{R} receive antennas. Based on a general analysis of quadratic receivers in zero-mean complex Gaussian vectors, asymptotically tight expressions (for high SNR) for the pair-wise error probabilities are derived. Consequently, it is shown that N_{T}-dimensional single-user signaling suffices to provide full diversity order $N = N_{\mathrm{T}}N_{\mathrm{R}}$ for all the users. In other words, the presence of other users does not increase the minimum dimension required beyond what is needed for the single-user space-time channel.**

For the special case of low-rank "CDMA" signaling with $N_{\mathrm{T}} = 1$ and provided the signatures of any two users are linearly independent, it is shown that the error probability of a K-user system asymptotically approaches single-user like performance for every user. Remarkably therefore, an increase in the number of users, and hence an increase in the aggregate spectral efficiency, does not require the users to transmit with more power to achieve single-user like performance asymptotically. A signal design algorithm is proposed to illustrate this point and examples are given. These results are then generalized to the multiple transmit antenna case. A new $(N_{\mathrm{T}} + 1)$-dimensional signaling strategy is proposed for the multiuser channel that leverages existing single-user space-time signal designs while ensuring a full diversity order and single-user like performance asymptotically for every user.

I. Introduction

Multiple antenna communication has received considerable attention in recent years due in large part to the information theoretic work in [1], [2], which showed that the use of multiple transmit and receive antennas could achieve considerable gains on the Rayleigh fading channel when the receiver has perfect side information about the channel state. Motivated by these promises, several researchers have recently proposed multi-antenna coding and modulation schemes for coherent single-user channels (cf. [3]–[7]) to show that diversity communication systems, when designed intelligently, can yield significant improvements over single antenna channels. The information theory of the single-user space-time channel easily extends to the multiuser multi-antenna channel [1] and it can be inferred that the gains in the capacity region are every bit as dramatic for the multiuser channel as well.

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In this paper, we present a theory of modulation and detection for *multiuser* space-time communication. In Section II, we describe a very general K user, N_{T} transmit, N_{R} receive antenna system model. Based on our general results on the asymptotic analysis of quadratic receivers in complex zero-mean Gaussian vectors [8], we analyze, in Section III, the optimum multiuser receiver and obtain asymptotically tight expressions for the pair-wise error probabilities. In Section IV we interpret these probabilities for the single transmit (IV-A) and multiple transmit antennas (IV-B) cases in terms of the minimum dimension needed to achieve full order of diversity for all users. We also propose new optimized multiuser signal design strategies that leverage single-user space-time designs in order to deliver single-user like performance in the high SNR regime.

Notation: Throughout the paper $^{\mathsf{T}}$ denotes transpose and † complex conjugate transpose. The multi-variate circularly symmetric, complex Gaussian distribution with mean-vector $\mathbf{m}$ (and covariance matrix $\mathbf{K}$) is denoted by $\mathcal{CN}(\mathbf{m})$ ($\mathcal{CN}(\mathbf{m},\mathbf{K})$). $E[\cdot]$ denotes the expected value of the expression in brackets. For any matrix $\mathbf{A}$ we write its determinant as $|\mathbf{A}|$ and its trace as $\mathrm{tr}\,(\mathbf{A})$. For any vector $\mathbf{a}$, we write its ℓ_2 norm as $\sqrt{\mathbf{a}^{\dagger}\mathbf{a}} = \|\mathbf{a}\|$. The logarithm to the base b is denoted by $\log_b$, the natural logarithm by $\ln$. The Kronecker product of two matrices is denoted by $\otimes$.

II. Multi-Antenna, Multiuser Discrete Time System Model

We describe a system model for K users communicating simultaneously in a common D–dimensional signal space.[1] Each of the K users employs N_{T} transmit antennas to send information symbol-synchronously to an N_{R} receive antenna array of the base-station. Since there are N_{T} transmit antennas and D dimensions, each user transmits one out of M possible $D \times N_{\mathrm{T}}$ complex-valued signal matrices, drawn from the set $\mathcal{S}_k = \{\mathbf{S}_{k1}, \mathbf{S}_{k2}, \ldots, \mathbf{S}_{kM}\}$ with $\mathbf{S}_{km} \in \mathbb{C}^{D\times N_{\mathrm{T}}}$. The signal matrices $\mathbf{S}_{km}$ may be thought of as space-time (block-) codewords where each element of the matrix is drawn from a finite, QAM-like constellation with $\mathcal{S}_k$ being the k^{th} user's codebook, or they may be thought of as super-symbols of some arbitrary constellation $\mathcal{S}_k$. Hence, the receiver is a decoder or a detector in the two cases, respectively. We will refer to the matrices $\mathbf{S}_{km}$ as (super-) symbols or signals or codewords as is appropriate and use the general term receiver when the terms

[1] We suggest the term "space-dimension" communication rather than "space-time" communications. The latter implies a basis of time-translates of a single waveform (so that D corresponds to the length of the coherence interval in symbol durations), which is too restrictive as pointed out in [8].

detector or decoder are both applicable. To succinctly write the discrete-time model for this system we need more definitions.

Let H_i denote the i^{th} hypothesis with $1 \le i \le M^K$. Without loss of generality let i determine uniquely the K-tuple $(i_1, i_2, \ldots, i_K)$ according to $i = \sum_{k=1}^{K}(i_k - 1)M^{k-1} + 1$. We let hypothesis H_i denote that user k transmits the signal $\mathbf{S}_{ki_k}$ for each k. Define the $D \times KN_{\mathrm{T}}$ matrix of signals corresponding to hypothesis H_i as $\mathbf{F}_i = [\mathbf{S}_{1i_1}, \mathbf{S}_{2i_2}, \ldots, \mathbf{S}_{Ki_K}]$. Thus the discrete-time model for the n^{th} receive antenna can be written as

$$\mathbf{y}_n = \mathbf{F}_i \mathbf{W}^{1/2} \mathbf{h}_n + \boldsymbol{\eta}_n, \tag{1}$$

where $\mathbf{y}_n$ is the D–dimensional vector of observations, $\mathbf{W} = \operatorname{diag}\{w_1, w_2, \ldots, w_K\} \otimes \mathbf{I}_{N_{\mathrm{T}}}$ with w_k being the k^{th} user's average energy, $\mathbf{h}_n = \left[\mathbf{h}_{1n}^{\mathsf{T}}, \mathbf{h}_{2n}^{\mathsf{T}}, \ldots, \mathbf{h}_{Kn}^{\mathsf{T}}\right]^{\mathsf{T}}$ is a KN_{T}–dimensional vector of $\mathcal{CN}(\mathbf{0})$ distributed fading coefficients with $\mathbf{h}_{kn}$ containing the N_{T} fading coefficients from the k^{th} user's transmit antennas to receive antenna n, and $\boldsymbol{\eta}_n$ is the D–dimensional $\mathcal{CN}(\mathbf{0}, \sigma^2\mathbf{I}_D)$ distributed additive noise vector. To obtain the sufficient statistics for all N_{R} receive antennas, we simply stack the $\mathbf{y}_n$ to obtain

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{N_{\mathrm{R}}} \end{bmatrix} = \left(\mathbf{I}_{N_{\mathrm{R}}} \otimes \mathbf{F}_i \mathbf{W}^{1/2}\right) \widehat{\mathbf{h}} + \boldsymbol{\eta}, \tag{2}$$

where $\widehat{\mathbf{h}} = \left[\mathbf{h}_1^{\mathsf{T}}, \ldots, \mathbf{h}_{N_{\mathrm{R}}}^{\mathsf{T}}\right]^{\mathsf{T}}$ contains the fading coefficients in "receiver-antenna" order. For the analysis to come, it will be more convenient to organize the fading coefficients user-wise in the vector $\mathbf{h} = \left[\mathbf{h}_{11}^{\mathsf{T}}, \ldots, \mathbf{h}_{1N_{\mathrm{R}}}^{\mathsf{T}}, \ldots, \mathbf{h}_{K1}^{\mathsf{T}}, \ldots, \mathbf{h}_{KN_{\mathrm{R}}}^{\mathsf{T}}\right]^{\mathsf{T}}$, which also requires the introduction of $\boldsymbol{\mathcal{S}}_{km} = \mathbf{I}_{N_{\mathrm{R}}} \otimes \mathbf{S}_{km}$, $\boldsymbol{\mathcal{F}}_i = [\boldsymbol{\mathcal{S}}_{1i_1}, \boldsymbol{\mathcal{S}}_{2i_2}, \ldots, \boldsymbol{\mathcal{S}}_{Ki_K}]$, and $\boldsymbol{\mathcal{W}} = \mathbf{W} \otimes \mathbf{I}_{N_{\mathrm{R}}}$. With these definitions the DN_{R} sufficient statistics can be written as

$$\mathbf{y} = \boldsymbol{\mathcal{F}}_i \boldsymbol{\mathcal{W}}^{1/2} \mathbf{h} + \boldsymbol{\eta}. \tag{3}$$

We denote the correlation matrix of the fading coefficients as $\boldsymbol{\Sigma} = E\left[\mathbf{h}\mathbf{h}^{\dagger}\right]$. $\boldsymbol{\Sigma}_{kk}$ denotes the k^{th} diagonal $N \times N$ block of $\boldsymbol{\Sigma}$ and thus is the k^{th} user's fading correlation matrix. The signals and fading processes are normalized so that $\bar{\gamma}_k = w_k/\sigma^2$ represents the average received signal-to-noise ratio (SNR) of the k^{th} user per receiver antenna and per (super-) symbol. Each user's signals are normalized such that their average energy over the transmit antennas is unity, i.e.,

$$E\left[\operatorname{tr}\left(\mathbf{S}_{km}^{\dagger}\mathbf{S}_{km}\right)\right] = N_{\mathrm{T}} \;\forall\, k, \tag{4}$$

where the expected value is taken over m. The fading coefficients are normalized such that

$$E\left[\widetilde{\mathbf{h}}_k^{\dagger} \boldsymbol{\mathcal{S}}_{km}^{\dagger} \boldsymbol{\mathcal{S}}_{km} \widetilde{\mathbf{h}}_k\right] = N_{\mathrm{R}} \;\forall\, k, \tag{5}$$

where $\widetilde{\mathbf{h}}_k = \left[\mathbf{h}_{k1}^{\mathsf{T}}, \ldots, \mathbf{h}_{kN_{\mathrm{R}}}^{\mathsf{T}}\right]^{\mathsf{T}}$ contains all of the k^{th} user's fading coefficients. For equi-probable symbols this condition can be written as

$$\sum_{m=1}^{M} \operatorname{tr}\left(\boldsymbol{\Sigma}_{kk} \boldsymbol{\mathcal{S}}_{km}^{\dagger} \boldsymbol{\mathcal{S}}_{km}\right) = MN_{\mathrm{R}}. \tag{6}$$

Note that with these normalizations the average received SNR is independent of the number of transmit antennas. For example in i.i.d. fading the stated conditions lead to $N_{\mathrm{T}}\boldsymbol{\Sigma}_{kk} = \mathbf{I}_N$.

III. Optimum Receiver and Analysis

In this section we specify the optimum receiver in terms of a quadratic form in the observations and the fading coefficients. This allows us to make use of the general results of [8] for the asymptotically tight analysis of the pair-wise error probabilities.

A. Maximum Likelihood Receiver

The likelihood function of the sufficient statistics $\mathbf{y}$ given the fading coefficients $\mathbf{h}$ and the true hypothesis H_i (i.e. $\mathbf{F}_i$, is transmitted) is

$$p(\mathbf{y}|\mathbf{h}, H_i) = \frac{1}{\pi^{DN_{\mathrm{R}}}\sigma^{2DN_{\mathrm{R}}}} \exp\left(-\sigma^{-2}\|\mathbf{y} - \boldsymbol{\mathcal{F}}_i \boldsymbol{\mathcal{W}}^{1/2}\mathbf{h}\|^2\right). \tag{7}$$

Defining the new $(KN_{\mathrm{T}}+D)N_{\mathrm{R}}$–dimensional sufficient statistic $\mathbf{z} = \sigma^{-1}\begin{bmatrix} \mathbf{h}^{\mathsf{T}} & \mathbf{y}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$ and the matrix

$$\mathbf{Q}_i = \begin{bmatrix} \boldsymbol{\mathcal{W}}^{1/2}\boldsymbol{\mathcal{F}}_i^{\dagger}\boldsymbol{\mathcal{F}}_i\boldsymbol{\mathcal{W}}^{1/2} & -\boldsymbol{\mathcal{W}}^{1/2}\boldsymbol{\mathcal{F}}_i^{\dagger} \\ -\boldsymbol{\mathcal{F}}_i\boldsymbol{\mathcal{W}}^{1/2} & \mathbf{0}_{DN_{\mathrm{R}}} \end{bmatrix}, \tag{8}$$

the jointly optimum coherent receiver Φ^{C} can be expressed as

$$\Phi^{\mathrm{C}} : \hat{i} = \arg\min_{1\le i\le M^K} \mathbf{z}^{\dagger}\mathbf{Q}_i^{\mathrm{C}}\mathbf{z} = \arg\min_{1\le i\le M^K} \delta_i^{\mathrm{C}}, \tag{9}$$

where δ_i^{C} is defined implicitly. Note that the sufficient statistics $\mathbf{z}$ are $\mathcal{CN}(\mathbf{0}, \mathbf{K}_{\mathbf{zz}|H_i})$ distributed, where

$$\begin{aligned} \mathbf{K}_{\mathbf{zz}|H_i} &= E\left[\mathbf{z}\mathbf{z}^{\dagger}\right] \qquad (10) \\ &= \begin{bmatrix} \sigma^{-2}\boldsymbol{\Sigma} & \sigma^{-2}\boldsymbol{\Sigma}\boldsymbol{\mathcal{W}}^{1/2}\boldsymbol{\mathcal{F}}_i^{\dagger} \\ \sigma^{-2}\boldsymbol{\mathcal{F}}_i\boldsymbol{\mathcal{W}}^{1/2}\boldsymbol{\Sigma} & \sigma^{-2}\boldsymbol{\mathcal{F}}_i\boldsymbol{\mathcal{W}}^{1/2}\boldsymbol{\Sigma}\boldsymbol{\mathcal{W}}^{1/2}\boldsymbol{\mathcal{F}}_i^{\dagger} + \mathbf{I} \end{bmatrix}. \end{aligned}$$

B. Bounds on Symbol and Bit Error Rate

Let $\mathcal{E}_k(\Phi^{\mathrm{C}})$ denote the event that the receiver Φ^{C} detects user k erroneously. Then $\Pr\{\mathcal{E}_k(\Phi^{\mathrm{C}})|H_i\}$ is the symbol error probability of the k^{th} user detected by receiver Φ^{C} conditioned on the hypothesis H_i. It is the probability of the union of the corresponding $(M-1)M^{K-1}$ possible events of the form $\left\{\delta_j^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right\}$. Since the probability of the union is usually not computable, consider the union upper bound which is the sum of the pair-wise error probabilities $\Pr\left\{\delta_j^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right\}$. [2] A lower bound is obtained by considering the pair-wise probability $\Pr\left\{\delta_{\hat{j}}^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right\}$, where $H_{\hat{j}}$ corresponds to one of the $M-1$ hypotheses H_j that result in an error only for user k when compared to H_i. The lower bound can be tightened by choosing $H_{\hat{j}}$ such that $\Pr\left\{\delta_{\hat{j}}^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right\}$ is maximized.

[2] $\Pr\left\{\delta_j^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right\}$ is only an *error* probability in a binary hypothesis test. However, the term "pair-wise error probability" is customarily used in the literature.

The k^{th} user's symbol error rate (SER) $\Pr\{\mathcal{E}_k(\Phi^c)\}$ for equi-probable symbols is then bounded as

$$\begin{aligned}\Pr\{\mathcal{E}_k(\Phi^c)\} &= M^{-K}\sum_{i=1}^{M^K}\Pr\{\mathcal{E}_k(\Phi^c)|H_i\} &(11)\\ &\leq M^{-K}\sum_{i=1}^{M^K}\sum_{\forall j\in\Lambda_i(k)}\Pr\left\{\delta_j^{\mathrm{C}}<\delta_i^{\mathrm{C}}\right\}, &(12)\\ \Pr\{\mathcal{E}_k(\Phi^c)\} &\geq M^{-K}\sum_{i=1}^{M^K}\Pr\left\{\delta_{\hat{j}}^{\mathrm{C}}<\delta_i^{\mathrm{C}}\right\}, &(13)\end{aligned}$$

where $\Lambda_i(k)$ is the set of the $(M-1)M^{K-1}$ indices of hypotheses in which the k^{th} user's symbol differs from its symbol corresponding to the true hypothesis H_i.

To obtain bounds on the average bit error rate (BER) P_k^{b} of the k^{th} user, we introduce the event $H_i \to H_j$ that hypothesis H_i is detected as H_j (in the presence of all other hypotheses). Since the events $H_i \to H_j$ are mutually exclusive, the average bit error rate can be written as

$$P_k^{\mathrm{b}} = M^{-K}\sum_{i=1}^{M^K}\sum_{\forall j\in\Lambda_i(k)}\frac{b_{ij}(k)}{\log_2 M}\Pr\{H_i\to H_j\}, \quad (14)$$

where $b_{ij}(k)$ is the number of erroneously detected bits of user k, when hypothesis H_i is detected as H_j. An upper bound on P_k^{b} is obtained by upper-bounding the probabilities $\Pr\{H_i\to H_j\}$ by $\Pr\left\{\delta_j^{\mathrm{C}}<\delta_i^{\mathrm{C}}\right\}$. A lower bound on P_k^{b} is obtained by lower bounding $b_{ij}(k)$ by one and using the fact that the inner sum of probabilities is equal to $\Pr\{\mathcal{E}_k(\Phi^c)|H_i\}$, which in turn can be lower bounded by $\Pr\left\{\delta_{\hat{j}}^{\mathrm{C}}<\delta_i^{\mathrm{C}}\right\}$, as in (13).

C. Pair-wise Error Probabilities

The pair-wise error probabilities $\Pr\left\{\delta_j^{\mathrm{C}}<\delta_i^{\mathrm{C}}\right\}$ are crucial for the bounds on the symbol as well as the bit error rate. They can be obtained via the calculation of residues (cf. [8]–[11]). However, the residues depend on the eigenvalues of $\mathbf{C}_{ij}^{\mathrm{C}} = \mathbf{K}_{\mathbf{zz}|H_i}(\mathbf{Q}_j-\mathbf{Q}_i)$ and do not in general give any insight into the dependencies on the system parameters of interest, such as the signal and fading correlations. A remedy for this is offered by the asymptotic (high SNR) analysis of the pair-wise error probabilities in [8] where we examined the asymptotic analysis of quadratic receivers in Rayleigh fading channels and found formulas for the asymptotic error-probabilities that require "only" the evaluation of the asymptotic eigenvalues of $\mathbf{C}_{ij}^{\mathrm{C}}$. The structure of these asymptotic eigenvalues follows the structure observed in [8]: half of the non-zero eigenvalues are positive and linear in σ^{-2}, and the other half converge to minus unity.

We state next the pair-wise error probabilities for finite SNR in the following proposition, which can be easily obtained from, for example, [8].

Proposition 1 (Expression for $\Pr\left\{\delta_j^{\mathrm{C}}<\delta_i^{\mathrm{C}}\right\}$)
Let $\{\lambda_l\}_{l=1}^{L}$ be the distinct non-zero eigenvalues of $\mathbf{C}_{ij}^{\mathrm{C}} = \mathbf{K}_{\mathbf{zz}|H_i}(\mathbf{Q}_j-\mathbf{Q}_i)$ with multiplicities $\{\mu_l\}_{l=1}^{L}$, and let $\{\lambda_l\}_{l=1}^{L_n}$ be negative and $\{\lambda_l\}_{l=L_n+1}^{L}$ positive, respectively. Then

$$\Pr\left\{\delta_j^{\mathrm{C}}<\delta_i^{\mathrm{C}}\right\} = -\sum_{k=1}^{L_n}\mathrm{Res}\left(\frac{1}{s\prod_{l=1}^{L}\lambda_l^{\mu_l}\left(s+\frac{1}{\lambda_l}\right)^{\mu_l}},\ s_k=\frac{-1}{\lambda_k}\right).$$

The residue of a function $\mathrm{f}(s)$ in a pole a of multiplicity m is defined to be

$$\mathrm{Res}\,(\mathrm{f}(s),\ a) = \frac{1}{(m-1)!}\lim_{s\to a}\frac{d^{m-1}}{ds^{m-1}}\left[(s-a)^m\mathrm{f}(s)\right].$$

For rational functions the limit is trivial, because the poles cancel with the $(s-a)^m$ terms.

Note that the calculation of the residues is numerically unstable for high-multiplicities of eigenvalues, so that for these cases one must use, for example, a saddle point integration technique [12].

As discussed in detail in [8], we must find the asymptotic eigenvalues of $\mathbf{C}_{ij}^{\mathrm{C}}$ to obtain the asymptotic expression for $\Pr\left\{\delta_j^{\mathrm{C}}<\delta_i^{\mathrm{C}}\right\}$. To this end, we introduce some assumptions and notation. We assume that the users are ordered such that users $1,2,\ldots,e$ suffer from an error, if the receiver would erroneously decide for hypothesis H_j when hypothesis H_i is transmitted. To avoid a complication in notation, we do not denote this user-ordering with any special symbols, but assume it implicitly. Another notational convenience is to split up the transmitted signal into two parts, the first containing the signals of the e users that suffer from an error relative to H_j, and the second part containing the $\bar{e}=K-e$ signals corresponding to the correctly detected users, i.e.,

$$\mathbf{F}_i = \left[\mathbf{F}_i^e\ \mathbf{F}^c\right],\ \mathbf{F}_j = \left[\mathbf{F}_j^e\ \mathbf{F}^c\right], \quad (15)$$

where c signifies the common part in the two signals $\mathbf{F}_i$ and $\mathbf{F}_j$. The matrices $\mathbf{F}_i^e$ and $\mathbf{F}_j^e$ are $D\times eN_{\mathrm{T}}$ and $\mathbf{F}^c$ is $D\times\bar{e}N_{\mathrm{T}}$. Similarly, we define $\boldsymbol{\mathcal{F}}_i^e$, $\boldsymbol{\mathcal{F}}_j^e$, and $\boldsymbol{\mathcal{F}}^c$ (whose sizes are multiplied by N_{R} when compared to $\mathbf{F}_i^e$, $\mathbf{F}_j^e$, and $\mathbf{F}^c$, respectively). Furthermore, we define $\boldsymbol{\Sigma}_{ee}$ and $\boldsymbol{\mathcal{W}}_{ee}$ as the $eN\times eN$ upper-left block of $\boldsymbol{\Sigma}$ and $\boldsymbol{\mathcal{W}}$, respectively (recall $N=N_{\mathrm{T}}N_{\mathrm{R}}$). $\boldsymbol{\Sigma}_{ec}$ and $\boldsymbol{\Sigma}_{cc}$ ($\boldsymbol{\mathcal{W}}_{cc}$) are the corresponding upper- and lower-right blocks of $\boldsymbol{\Sigma}$ ($\boldsymbol{\mathcal{W}}$).

With these and the definitions of

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix}\boldsymbol{\Sigma}_{ee}\\ \boldsymbol{\Sigma}_{ce}\\ \boldsymbol{\mathcal{F}}_i^e\boldsymbol{\mathcal{W}}_{ee}^{1/2}\boldsymbol{\Sigma}_{ee}+\boldsymbol{\mathcal{F}}^c\boldsymbol{\mathcal{W}}_{cc}^{1/2}\boldsymbol{\Sigma}_{ce}\end{bmatrix}, &(16)\\ \mathbf{B} &= \boldsymbol{\mathcal{W}}_{ee}^{1/2}\left(\boldsymbol{\mathcal{F}}_j^e-\boldsymbol{\mathcal{F}}_i^e\right)^{\dagger}, &(17)\\ \mathbf{C} &= \begin{bmatrix}\boldsymbol{\mathcal{F}}_j^e\boldsymbol{\mathcal{W}}_{ee}^{1/2} & \boldsymbol{\mathcal{F}}^c\boldsymbol{\mathcal{W}}_{cc}^{1/2} & -\mathbf{I}_{DN_{\mathrm{R}}}\end{bmatrix}, &(18)\\ \mathbf{Z} &= \begin{bmatrix}\mathbf{0}_{eN} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_{\bar{e}N} & \mathbf{0}\\ -\left(\boldsymbol{\mathcal{F}}_j^e-\boldsymbol{\mathcal{F}}_i^e\right)\boldsymbol{\mathcal{W}}_{ee}^{1/2} & \mathbf{0} & \mathbf{0}_{DN_{\mathrm{R}}}\end{bmatrix}, &(19)\end{aligned}$$

one finds after some tedious algebra that

$$\mathbf{C}_{ij}^{\mathrm{C}} = \sigma^{-2}\mathbf{ABC}+\mathbf{Z}. \quad (20)$$

Since the eigenvalues of $\mathbf{C}_{ij}^{\mathrm{C}}$ are the same as the eigenvalues of $\widehat{\mathbf{C}}_{ij}^{\mathrm{C}} = \mathbf{T}\mathbf{C}_{ij}^{\mathrm{C}}\mathbf{T}^{-1}$ for any invertible matrix $\mathbf{T}$, we are free to choose

$$\mathbf{T} = \begin{bmatrix} \mathbf{\Sigma}_{ee}^{-1/2} & \mathbf{0} & \mathbf{0} \\ -\mathbf{\Sigma}_{ce}\mathbf{\Sigma}_{ee}^{-1} & \mathbf{I}_{\bar{e}N} & \mathbf{0} \\ -\boldsymbol{\mathcal{F}}_i^e \boldsymbol{\mathcal{W}}_{ee}^{1/2} & -\boldsymbol{\mathcal{F}}^c \boldsymbol{\mathcal{W}}_{cc}^{1/2} & \mathbf{I}_{DN_{\mathrm{R}}} \end{bmatrix}, \quad (21)$$

$$= \begin{bmatrix} \mathbf{\Sigma}_{ee}^{1/2} & \mathbf{0} & \mathbf{0} \\ \mathbf{\Sigma}_{ce}\mathbf{\Sigma}_{ee}^{-1/2} & \mathbf{I}_{\bar{e}N} & \mathbf{0} \\ \left[\mathbf{T}^{-1}\right]_{31} & \boldsymbol{\mathcal{F}}^c \boldsymbol{\mathcal{W}}_{cc}^{1/2} & \mathbf{I}_{DN_{\mathrm{R}}} \end{bmatrix}^{-1}, \quad (22)$$

where $\left[\mathbf{T}^{-1}\right]_{31} = \boldsymbol{\mathcal{F}}_i^e \boldsymbol{\mathcal{W}}_{ee}^{1/2}\mathbf{\Sigma}_{ee}^{1/2} + \boldsymbol{\mathcal{F}}^c \boldsymbol{\mathcal{W}}_{cc}^{1/2}\mathbf{\Sigma}_{ce}\mathbf{\Sigma}_{ee}^{-1/2}$. Defining $\boldsymbol{\mathcal{F}}_{ji}^e = \boldsymbol{\mathcal{F}}_j^e - \boldsymbol{\mathcal{F}}_i^e$, we can calculate $\widehat{\mathbf{C}}_{ij}^{\mathrm{C}}$ as

$$\widehat{\mathbf{C}}_{ij}^{\mathrm{C}} = \begin{bmatrix} \left[\widehat{\mathbf{C}}_{ij}^{\mathrm{C}}\right]_{11} & \mathbf{0} & -\sigma^{-2}\mathbf{\Sigma}_{ee}^{1/2}\boldsymbol{\mathcal{W}}_{ee}^{1/2}\boldsymbol{\mathcal{F}}_{ji}^{e\dagger} \\ \mathbf{0} & \mathbf{0}_{\bar{e}N} & \mathbf{0} \\ -\boldsymbol{\mathcal{F}}_{ji}^e \boldsymbol{\mathcal{W}}_{ee}^{1/2}\mathbf{\Sigma}_{ee}^{1/2} & \mathbf{0} & \mathbf{0}_{DN_{\mathrm{R}}} \end{bmatrix}, \quad (23)$$

where $\left[\widehat{\mathbf{C}}_{ij}^{\mathrm{C}}\right]_{11} = \sigma^{-2}\mathbf{\Sigma}_{ee}^{1/2}\boldsymbol{\mathcal{W}}_{ee}^{1/2}\boldsymbol{\mathcal{F}}_{ji}^{e\dagger}\boldsymbol{\mathcal{F}}_{ji}^{e}\boldsymbol{\mathcal{W}}_{ee}^{1/2}\mathbf{\Sigma}_{ee}^{1/2}$.

Let $\mathbf{USV}^\dagger$ be the economy-size singular value decomposition (SVD) of $\boldsymbol{\mathcal{F}}_{ji}^e \boldsymbol{\mathcal{W}}_{ee}^{1/2}\mathbf{\Sigma}_{ee}^{1/2}$, whose rank is easily shown to be rN_{R}, where r is the rank of $\mathbf{F}_j^e - \mathbf{F}_i^e$. Hence $\mathbf{S}$ is of size $rN_{\mathrm{R}} \times rN_{\mathrm{R}}$. Then it is not hard to show that the non-zero eigenvalues of $\widehat{\mathbf{C}}_{ij}^{\mathrm{C}}$ are the non-zero eigenvalues of

$$\mathbf{M} = \begin{bmatrix} \sigma^{-2}\mathbf{S}^\dagger\mathbf{S} & -\sigma^{-2}\mathbf{S}^\dagger\mathbf{U}^\dagger \\ -\mathbf{US} & \mathbf{0}_{DN_{\mathrm{R}}} \end{bmatrix}. \quad (24)$$

The eigenvalues of $\mathbf{M}$ are found by applying a determinantal equality [13, Section 0.8.5], i.e.,

$$|\mathbf{M} - \lambda\mathbf{I}| = \left|\sigma^{-2}\mathbf{S}^\dagger\mathbf{S} - \lambda\mathbf{I}\right| \quad (25)$$
$$\times \left|-\lambda\mathbf{I} - \mathbf{US}\left(\mathbf{S}^\dagger\mathbf{S} - \sigma^2\lambda\mathbf{I}\right)^{-1}\mathbf{S}^\dagger\mathbf{U}^\dagger\right|.$$

For small σ the second determinant of the product can be approximated by $\left|-\lambda\mathbf{I} - \mathbf{UU}^\dagger\right|$, so that we finally arrive at the following proposition, for which we made use of [13, Theorem 1.3.20], which states that the non-zero eigenvalues of $\mathbf{XY}$ are equal to those of $\mathbf{YX}$.

Proposition 2 (Asymptotic Eigenvalues of $\mathbf{C}_{ij}^{\mathrm{C}}$)
The asymptotic non-zero eigenvalues of $\mathbf{C}_{ij}^{\mathrm{C}} = \mathbf{K}_{\mathbf{zz}|H_i}(\mathbf{Q}_j - \mathbf{Q}_i)$ are arbitrarily close to the rN_{R} non-zero eigenvalues of $\sigma^{-2}\boldsymbol{\mathcal{W}}_{ee}^{1/2}\mathbf{\Sigma}_{ee}\boldsymbol{\mathcal{W}}_{ee}^{1/2}\boldsymbol{\mathcal{F}}_{ji}^{e\dagger}\boldsymbol{\mathcal{F}}_{ji}^{e}$ and minus unity with multiplicity rN_{R}.

With these eigenvalues and the results of [8], one easily finds the asymptotic pair-wise error-probability given in the next proposition. For ease of notation we introduce $|\mathbf{X}|_{\mathsf{NZ}}$ as the product of the non-zero eigenvalues of $\mathbf{X}$.

Proposition 3 (Asymptotic Pair-Wise Error Probability)
For coherent detection the pair-wise error probability $\Pr\left\{\delta_j^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right\}$ of the optimum receiver Φ^{C} approaches arbitrarily closely

$$\Pr^{\mathrm{a}}\left\{\delta_j^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right\} = \frac{\sigma^{2rN_{\mathrm{R}}}\binom{2rN_{\mathrm{R}}-1}{rN_{\mathrm{R}}}}{\left|\boldsymbol{\mathcal{W}}_{ee}^{1/2}\mathbf{\Sigma}_{ee}\boldsymbol{\mathcal{W}}_{ee}^{1/2}\boldsymbol{\mathcal{F}}_{ji}^{e\dagger}\boldsymbol{\mathcal{F}}_{ji}^{e}\right|_{\mathsf{NZ}}}$$

as σ goes to zero.

IV. Interpretations and Signal Designs

The asymptotic result of the previous section encompasses many special cases of interest, some of which we explore in this section. For example, we specialize to $N_{\mathrm{T}} = 1$ in Section IV-A and gain some insights into this case, which help understand the multiple transmit antenna case. In Section IV-B we consider multiple transmit antennas but specialize to $K = 1$ first, before we discuss the general K-user, N_{T}-antenna problem. We focus on giving specific results and interpretations for the asymptotic pair-wise error probabilities and assume that it is understood that the corresponding optimum receiver can be obtained by applying the specifics to Φ^{C} as defined above.

A. One Transmit-Antenna per User

We distinguish between linear versus general M-ary/block-coded modulation. In linear modulation, each user modulates its signature sequence by a symbol drawn from a fixed alphabet, like a QAM or PSK constellation. In M-ary or block-coded modulation the k^{th} user's m^{th} signal vector may be a block-code over a finite alphabet or a super-symbol drawn from an arbitrary constellation.

A.1 Linear Modulation – Multiuser Detection and CDMA Signature Sequence Design

If we specialize $\mathbf{F}_i = \mathbf{FB}_i$, where $\mathbf{B}_i$ is a $K \times K$ diagonal matrix containing the users' constellation symbols, the system model of (1) corresponds to a synchronous code division multiple access (CDMA) model. Note that we do not make any assumptions on the number of dimensions D, so that overloaded systems with $D < K$ are included in the analysis. We define $\mathbf{F}^e$ to be made up from the columns (signatures sequences) of the e users that suffer from an error. $\mathbf{B}_i^e$, $\mathbf{B}_j^e$ are diagonal matrices that contain the information symbols of these users.

Corollary 1: (Asymptotic Pair-Wise Error Probability for Coherent CDMA Detection)
Assuming $\mathbf{\Sigma} = \mathbf{I}_{KN_{\mathrm{R}}}$[3] and that any subset of D columns of $\mathbf{F}$ span the D-dimensional signal space, we have for $e \leq D$ that the pair-wise error probability of the optimum detector Φ^{C} approaches arbitrarily closely

$$\Pr^{\mathrm{a}}\left(\delta_j^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right) = \frac{\sigma^{2eN_{\mathrm{R}}}\binom{2eN_{\mathrm{R}}-1}{eN_{\mathrm{R}}}}{\left|\mathbf{W}_{ee}\left(\mathbf{B}_j^e - \mathbf{B}_i^e\right)^\dagger\mathbf{F}^{e\dagger}\mathbf{F}^e\left(\mathbf{B}_j^e - \mathbf{B}_i^e\right)\right|^{N_{\mathrm{R}}}}$$

as σ goes to zero.

For $D \leq e$ we have

$$\Pr^{\mathrm{a}}\left(\delta_j^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right) = \frac{\sigma^{2DN_{\mathrm{R}}}\binom{2DN_{\mathrm{R}}-1}{DN_{\mathrm{R}}}}{\left|\mathbf{F}^e\left(\mathbf{B}_j^e - \mathbf{B}_i^e\right)\mathbf{W}_{ee}\left(\mathbf{B}_j^e - \mathbf{B}_i^e\right)^\dagger\mathbf{F}^{e\dagger}\right|^{N_{\mathrm{R}}}}.$$

[3] In this and some of the following corollaries, the assumption of i.i.d. fading is only made to avoid having to introduce more notation.

We note that for $D = 1$ and BPSK modulation the second bound specializes to the one given in [14]. Note that as the number of users increases, the aggregate spectral efficiency $K \log_2 M$ increases linearly since all the users employ the same "signature signal" which, without being wasteful of bandwidth, can be taken to be the minimum bandwidth sinc pulse or a raised cosine pulse with sufficient roll-off to ensure robustness to timing jitter and user quasi-synchronism. The second bound of Corollary 1 implies that there is no loss of order of diversity compared to a single-user channel for any of the users. Without any bandwidth expansion compared to a single-user channel, multiple users can be accommodated with no loss of diversity order. There would be, however, a loss of energy efficiency in that each user would have to transmit at a somewhat higher power to achieve the performance it would have in the absence of other users, and this loss would increase with the number of users.

Consider the case $D = 2$ with aggregate spectral efficiency of $\frac{K}{2} \log_2 M$ bps/Hz which also linearly increases with an increase in the number of users albeit at half the rate of the narrowband channel. In this case, it is easy to design two-dimensional signature sequences over the field of complex numbers by simply ensuring that any two users are assigned linearly independent signals so that not only do all the users achieve full order of diversity but even the above-mentioned loss of asymptotic effective energy relative to single-user performance is eliminated. Hence, with a bandwidth expansion by a factor of two relative to a single-user channel, an increasing number of users can be accommodated and received with a reliability that is asymptotically equivalent to "single-user like" performance for every user, in the sense that the upper bound on the multiuser BER converges to the single-user upper bound.

To obtain such spreading signals for $D > 1$ dimensions we suggest a signal design algorithm that minimizes the maximum of a per-user asymptotic performance criterion over all users. This criterion is derived from the upper bound on the k^{th} user's asymptotic bit error rate, which in turn results from (14) by upper-bounding $\Pr\{H_i \to H_j\}$ by $\Pr^{\text{a}}\{\delta_j^{\text{C}} < \delta_i^{\text{C}}\}$ for small σ, i.e.,

$$\begin{aligned} P_k^{\text{b}} &\leq M^{-K} \sum_{i=1}^{M^K} \sum_{\forall j \in \Lambda_i(k)} \frac{b_{ij}(k)}{\log_2 M} \Pr^{\text{a}}\{\delta_j < \delta_i\} \\ &= \sum_{d=1}^{D} c_d(k) \sigma^{2dN_{\text{R}}}, \end{aligned} \tag{26}$$

where $c_d(k)$ contains all the coefficients with diversity order d in the upper bound on the k^{th} user's BER. The signal design algorithm must minimize $\max_{1 \leq k \leq K} c_2(k)$. While the terms $c_d(k)$ with $d > 1$ asymptotically do not influence the BER, we conjecture that by minimizing $\max_{1 \leq k \leq K} c_2(k)$, the convergence of the upper-bound to the lower-bound is improved, so that the BER of a system employing optimized signals is improved at finite SNRs (asymptotically the BER does not depend on the signals provided that at least any two signature sequences are linearly independent).

Figure 1 shows the performance of multiuser systems employing optimized signature sequences in $D = 2$ dimensions. The equal-energy users ($\mathbf{W} = \mathbf{I}$) transmit spreaded BPSK symbols from one transmit antenna to one receive antenna in i.i.d. fading ($\mathbf{\Sigma} = \mathbf{I}$). The design algorithm yielded signal sets for which the users' performances are identical for finite SNR, so that we plot the upper bound and simulated BER of one user only. We see that asymptotically single-user performance is achieved. However, for increasing number of users the asymptote is reached for increasing SNR only.

Figure 2 shows the performance of the signature sequences for $K = 10$ users of the previous plot in comparison with narrow-band communications ($D = 1$) and a single user in one dimension with a spectral efficiency of $5bps/Hz$ (thus the single user employs 32-QAM). As before we choose $\mathbf{W} = \mathbf{I}_K$ and $\mathbf{\Sigma} = \mathbf{I}_K$ for one transmit and receive antenna. Although an asymptotic performance criterion is optimized in the design process for the signature sequences, the advantage over narrow-band signaling at a BER of 10^{-2} is over 15 dB for the $K = 10$ narrowband system and about 4 dB for the $K = 5$ narrowband system, which has the same spectral efficiency as the $K = 10, D = 2$ CDMA system. While for a BER 10^{-2} the gap to the single user employing 32-QAM is about 1 dB, the single user is asymptotically out-performed by roughly 6 dB.

Figure 3 shows bounds and simulated BERs for a $N_{\text{R}} = 2$ receive antenna, $K = 3$ user system in which each user employs the most energy efficient 8-QAM constellation. The designed signals have an identical absolute value of the cross-correlation of 0.5 and the resulting BER of each user is almost indistinguishable from that of a single-user channel. On the other hand, at a BER of 10^{-2}, the gap to narrow-band signaling is roughly 5 dB. In all cases the bounds on the BERs are not tight, due to the use of $M = 8$-ary signaling.

A.2 M-ary or Block Coded Modulation

In this section, we interpret the k^{th} column of $\mathbf{F}_i$ as a super-symbol of user k which can be thought of as belonging to some dense lattice (or more generally to an arbitrary non-lattice constellation), whose individual scalar elements may be drawn from a regular QAM-like alphabet or may be arbitrary complex numbers, not necessarily restricted to be part of a finite alphabet. When the "codeword" interpretation is appropriate, the receiver may be thought of as a decoder.

We rewrite the general expression of the asymptotic pair-wise error probability from Proposition 3.

Corollary 2: (Asymptotic Pair-Wise Error Probability for Coherent Decoding)
Assuming $\mathbf{\Sigma} = \mathbf{I}_{KN_{\text{R}}}$ and that $\left(\mathbf{F}_j^e - \mathbf{F}_i^e\right)$ has rank $r \leq \min(D, e)$, the pair-wise error probability of the optimum decoder Φ^{C} approaches arbitrarily closely

$$\begin{aligned} &\Pr^{\text{a}}\{\delta_j^{\text{C}} < \delta_i^{\text{C}}\} = \\ &= \frac{\sigma^{2rN_{\text{R}}} \binom{2rN_{\text{R}}-1}{rN_{\text{R}}}}{\left|\mathbf{W}_{ee}\left(\mathbf{F}_j^e - \mathbf{F}_i^e\right)^{\dagger}\left(\mathbf{F}_j^e - \mathbf{F}_i^e\right)\right|_{\text{NZ}}^{N_{\text{R}}}} \end{aligned}$$

as σ goes to zero.

Let us reconcile this result for the fictitious case where all K users co-operate so that we have an equivalent single-user, K-transmit, N_{R}-receive antenna channel. In this case, the maximum diversity order for a given K could be achieved if $r = K = e$ and the proposition corresponds to the well-known rank criterion [6], for which of course we need $D \geq K$. By the use of space-time codes such as the orthogonal designs of [5] or the algebraic codes of [15] which satisfy this rank criterion, one can achieve full diversity order (namely KN_{R}).

However, the multiuser rank criterion is very different from the single-user criterion because while in the single-user channel with K transmit antennas, signals transmitted over the different transmit antennas can be dependent (i.e., an super-information symbol is encoded into a $D \times N_{\mathrm{T}}$ matrix), the columns of this matrix in the multiuser channel arise from the independent transmission of vectors of length D each from the K different users.

B. Multiple Transmit-Antennas per User

The classical single-user multiple transmit antenna space-time coding analysis also profits from our general analysis: in contrast to the earlier, Chernoff bound based approaches, our analysis provides asymptotically tight expressions for the pair-wise error probability and considers possibly correlated fading. Finally, for the multi-transmit antenna, multiuser space time channel, we propose a signal design algorithm that ensures single-user like performance asymptotically.

B.1 One User

In this case one user transmits a $D \times N_{\mathrm{T}}$ signal matrix $\{\mathbf{S}_m\}_{m=1}^{M}$ with average energy w. $\boldsymbol{\Sigma}$ simplifies to the $N \times N$ fading correlation matrix associated with all the antennas. Recalling that we defined $\boldsymbol{\mathcal{S}}_m = \mathbf{I}_{N_{\mathrm{R}}} \otimes \mathbf{S}_m$, Proposition 3 easily simplifies to the following corollary.

Corollary 3: (Asymptotic Pair-Wise Error Probability for Single User Reception)
Assuming that $(\mathbf{S}_j - \mathbf{S}_i)$ has rank $r \leq \min(D, N_{\mathrm{T}})$, the pair-wise error probability of the optimum receiver Φ^{C} approaches arbitrarily closely

$$\mathrm{Pr}^{\mathrm{a}}\left\{\delta_j^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right\} = \frac{\left(\frac{w}{\sigma^2}\right)^{-rN_{\mathrm{R}}}\binom{2rN_{\mathrm{R}}-1}{rN_{\mathrm{R}}}}{\left|\boldsymbol{\Sigma}\left(\boldsymbol{\mathcal{S}}_j - \boldsymbol{\mathcal{S}}_i\right)^{\dagger}\left(\boldsymbol{\mathcal{S}}_j - \boldsymbol{\mathcal{S}}_i\right)\right|_{\mathsf{NZ}}}$$

as σ goes to zero.

Note that in addition to revealing the rank and determinant criterion of [4], [6] for i.i.d. fading, this formula is is also asymptotically tight and considers the more general case of correlated fading. As a consequence of the asymptotic tightness, asymptotically tight lower bounds on symbol and bit error rates can be obtained. For correlated fading and full-diversity space-time codes, the fading correlation does not affect the determinant criterion, because $\left|\boldsymbol{\Sigma}\left(\boldsymbol{\mathcal{S}}_j - \boldsymbol{\mathcal{S}}_i\right)^{\dagger}\left(\boldsymbol{\mathcal{S}}_j - \boldsymbol{\mathcal{S}}_i\right)\right| = |\boldsymbol{\Sigma}|\left|(\mathbf{S}_j - \mathbf{S}_i)^{\dagger}(\mathbf{S}_j - \mathbf{S}_i)\right|^{N_{\mathrm{R}}}$. Consequently, full-diversity space-time codes that were optimized for i.i.d. are also asymptotically optimal for correlated fading. The analysis presented here also improves on our work in [8] by providing exact expressions for the asymptotic pair-wise error probabilities in case $\mathbf{S}_i - \mathbf{S}_j$ is low rank.

B.2 Multiple Users

The observations we made for the various special cases allow us to finally draw some conclusions about Proposition 3 for multiuser communication when each user employs N_{T} transmit antennas. Most importantly, if every user employs a full-diversity space-time code/constellation (requiring $D \geq N_{\mathrm{T}}$) in the same D dimensional signal space, the multiuser system still achieves asymptotically a diversity order of $N = N_{\mathrm{R}}N_{\mathrm{T}}$, i.e., no loss in diversity order occurs when compared to the single-user case, without any bandwidth expansion (this was also realized independently in [15] by using the weaker Chernoff analysis that does not yield asymptotically tight bounds on pair-wise error rates). However, a loss in energy-efficiency occurs when more users are added. This behavior mirrors exactly the $N_{\mathrm{T}} = 1$, $D = 1$ narrow-band case discussed above. We saw that we could improve on this behavior by expanding the signal space to $D = 2$ dimensions to design signals such that the optimum receiver achieves single-user like performance asymptotically. One way to generalize this idea to the multiple transmit antenna case is signal according to

$$\mathbf{F}_i = \mathbf{F}\begin{bmatrix} \mathbf{B}_{1i_1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{2i_2} & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{B}_{Ki_K} \end{bmatrix}, \qquad (27)$$

where $\mathbf{F}$ is a $D \times KD_{\mathrm{SU}}$ fixed "signature" matrix and $\mathbf{B}_{ki_k}$ are $D_{\mathrm{SU}} \times N_{\mathrm{T}}$ single user full-diversity code matrices. Denote the D_{SU} columns of $\mathbf{F}$ that correspond to user k as $\mathbf{F}_k$, i.e., $\mathbf{F} = [\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_K]$. Using a rank inequality ([13, Section 0.4.5]), one can show that if any compound matrix $[\mathbf{F}_l, \mathbf{F}_k]$, $l \neq k$, is of rank greater than or equal $2D_{\mathrm{SU}} - N_{\mathrm{T}} + 1$, then it is guaranteed that the asymptotic probability of an event $\left(\delta_j^{\mathrm{C}} < \delta_i^{\mathrm{C}}\right)$ involving $e = 2$ users has a diversity order of at least $(N_{\mathrm{T}} + 1)N_{\mathrm{R}}$, so that this probability (and all pair-wise error probabilities involving more than two users) can be asymptotically neglected when compared to the single-user pair-wise error probabilities whose diversity order is $N_{\mathrm{T}}N_{\mathrm{R}}$. Thus, even in the multi-transmit antenna case single-user like performance can be asymptotically achieved.

To find such a suitable signature matrix $\mathbf{F}$ we need of course $D \geq 2D_{\mathrm{SU}} - N_{\mathrm{T}} + 1$. Furthermore, the design algorithm discussed in Section IV-A.1 can be adapted to design an optimized signature matrix $\mathbf{F}$.

V. CONCLUSIONS

The general analysis of [8] is applied to coherent multiuser space-time reception. Consequently, asymptotically tight expressions for the pair-wise error probabilities are obtained. Several conclusions can be drawn from this analysis:

- In a one transmit antenna per user CDMA system, all users can be detected with asymptotic single-user like performance, if the common signal space has at least dimensionality two. An algorithm to design "optimum" spreading sequences is presented.

- For M-ary/block coded modulation with one transmit antenna per user a signal/code design criterion is presented.
- For the "classical" single-user N_T transmit and N_R receive antenna space-time communications, we improve on the previous approaches by providing asymptotically tight bounds while including channels with correlated fading.
- For the multiuser space-time problem it is established that every user achieves the total order of diversity $N = N_T N_R$ when communicating with D-dimensional single-user space-time codes in a common D-dimensional signal space. To achieve asymptotically single-user like performance for multiple users, at least $N_T + 1$ dimensions are necessary, as opposed to N_T in the single-user channel. A signal design algorithm is given, that generalizes the algorithm for $N_T = 1$.

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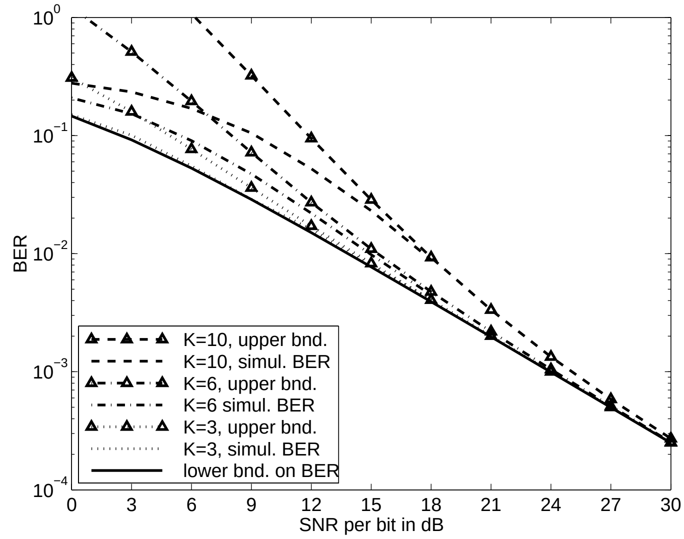

Fig. 1. While an increasing number of users can be accommodated in $D = 2$ dimensions with asymptotic single-user performance, the asymptote is reached for higher SNR as K increases.

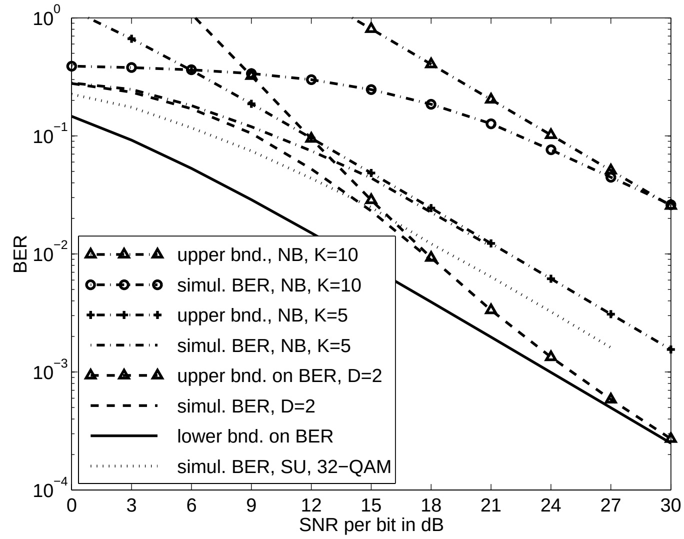

Fig. 2. With only $D = 2$ dimensions the 10 user system can asymptotically achieve single-user performance and out-perform the $K = 10$ narrow-band system by 15 dB at a BER of 10^{-2}.

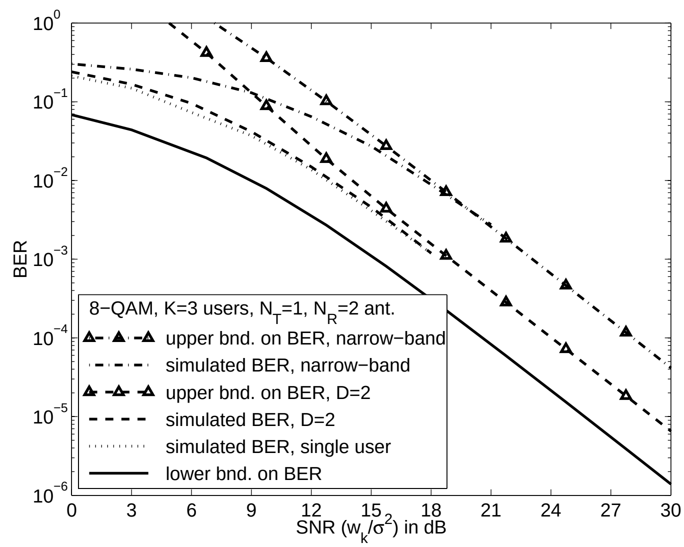

Fig. 3. For increasing constellation size, the multiuser system can still asymptotically achieve single-user like performance.