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Noncoherent Multiuser Space-Time Communications: Optimum Receivers and Signal Design

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Abstract — **The jointly optimum noncoherent receiver is obtained for multiuser communications in a frequency non-selective Rayleigh fading channel with N_{T} transmit antennas per user and N_{R} receive antennas. Based on a general analysis of quadratic receivers in zero-mean complex Gaussian vectors, we derive asymptotically (for high SNR) tight expressions for the pair-wise error probabilities for some cases of interest. This analysis yields the important result that the number of dimensions needed for each of the K users to achieve full diversity order $N = N_{\mathrm{T}}N_{\mathrm{R}}$ is $(K+1)N_{\mathrm{T}}$, which is independent of the size of the users' codebooks. Our asymptotic analysis also yields a performance criterion to design noncoherent multiuser signals for space-time channels. We give an example of such a design. Gains in spectral efficiency are more difficult to realize for noncoherent (as compared to coherent) channels.**

I. INTRODUCTION

In a companion paper [1], we obtained and analyzed coherent multiuser space-time receivers for linear and nonlinear modulation and developed code-design criteria for the design of CDMA signature sequences and multiuser block codes over the complex field. In that work, it was assumed that the fading channels for all the user-antenna pairs are independent and perfectly known at the base station receiver. These assumptions are reasonable for rich scattering environments and stationary or slow moving communicators where relatively short and infrequent pilot transmissions suffice for channel estimation algorithms to track the slow variations in the channel accurately.

However, in highly mobile communications, the channel fades rapidly and channel estimation may require long and frequent pilot transmissions that result in significant losses of spectral efficiency. Hence, noncoherent communications becomes attractive for these channels. As in the coherent case, the information theory of noncoherent communication points to significant gains over single-antenna communications [2], [3]. Motivated by these promises, several researches have proposed coding and modulation schemes for noncoherent, multi-antenna single-user channels [4]–[9].

In this paper, we present a theory of modulation and noncoherent detection for *multiuser* space-time communication. The K user, N_{T} transmit, N_{R} receive antenna system model of this paper is exactly the same as in [1] and is hence omitted. We obtain the optimum multiuser receiver in Section II and analyze it based on our general results on the asymptotic analysis of quadratic receivers in complex zero-mean Gaussian vectors [10]. In contrast to the coherent case, in which we undertook the analysis without placing any restrictions on the dimensionality on the signal space, we will assume in this paper that the number of dimensions D is greater than KN_{T}. However, the number of minimum dimensions is independent of M, the size of each user's constellation. In particular, it can be less than $2K$, thereby relaxing the assumption made in [11] for analyzing the optimum multiuser receiver for binary nonorthogonal modulation for a single transmit and receive antenna in frequency-flat Rayleigh fading channels ([12] extends this work to frequency-selective Rayleigh fading with possibly multiple receive antennas, under the same assumption). Assuming $D > KN_{\mathrm{T}}$, we find the exact asymptotic pair-wise error probabilities for the optimum receiver in Section II-B and show that at least $D \geq (K+1)N_{\mathrm{T}}$ dimensions are necessary to guarantee full order of diversity $N = N_{\mathrm{T}}N_{\mathrm{R}}$ for each user. Moreover, for $D \geq (K+1)N_{\mathrm{T}} + 1$ and binary signaling, the upper bound on the symbol error rate converges to the lower bound for high SNR; for M-ary signaling the interfering users' energies will not affect the asymptotic (union) upper bound on the symbol error rate, i.e., the receiver is *a fortiori* near-far resistant.

While the optimum receiver still requires a knowledge of the fading statistics and the average SNR of each user, the asymptotically optimum receiver introduced in Section III only requires a knowledge of the noise power and does not incur a penalty in asymptotic performance.

The asymptotic expressions on the pair-wise error probabilities can be used to design signals, that minimize the maximum asymptotic error rate over all users. We give an example of such a design in Section IV and conclude in Section V.

Notation: Besides the notation defined in [1] we define the projection matrices $\mathbf{P}_{\mathbf{M}} = \mathbf{M}\left(\mathbf{M}^{\dagger}\mathbf{M}\right)^{-1}\mathbf{M}^{\dagger}$ and $\mathbf{P}_{\mathbf{M}}^{\perp} = \mathbf{I} - \mathbf{P}_{\mathbf{M}}$.

II. OPTIMUM NONCOHERENT MULTIUSER RECEIVER AND ANALYSIS

In this section, we obtain the optimum noncoherent multiuser receiver and derive an asymptotic analysis of the pair-wise error probabilities. Our analysis applies our general results on quadratic receivers obtained in [10].

The optimum noncoherent receiver does not assume knowledge of the fading coefficients, but uses knowledge about the mean SNR of all users and the fading statistics. Defining

$$\mathbf{K}_{\mathbf{yy}|H_i} = E\left[\mathbf{y}\mathbf{y}^{\dagger}\right] = \mathcal{F}_i \mathcal{W}^{1/2} \boldsymbol{\Sigma} \mathcal{W}^{1/2} \mathcal{F}_i^{\dagger} + \sigma^2 \mathbf{I}, \quad (1)$$

the jointly optimum noncoherent receiver Φ^{NC} is simply

$$\Phi^{\mathrm{NC}} : \hat{i} \quad = \quad \arg \min_{1 \leq i \leq M^K} \mathbf{y}^{\dagger}\mathbf{K}_{\mathbf{yy}|H_i}^{-1}\mathbf{y} + \ln\left|\mathbf{K}_{\mathbf{yy}|H_i}\right|$$

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$$= \arg \min_{1 \leq i \leq M^K} \delta_i^{\mathrm{NC}}, \tag{2}$$

where we defined δ_i^{NC} implicitly.

A. Bounds on the Symbol Error Rate

Let $\mathcal{E}_k(\Phi^{\mathrm{NC}})$ denote the event that the receiver Φ^{NC} detects user k erroneously. The symbol error rate of the k^{th} user for equi-probable symbols can then be upper-bounded by

$$\Pr\left\{\mathcal{E}_k(\Phi^{\mathrm{NC}})\right\} \leq M^{-K} \sum_{i=1}^{M^K} \sum_{\forall j \in \Lambda_i(k)} \Pr\left\{\delta_j^{\mathrm{NC}} < \delta_i^{\mathrm{NC}}\right\}, \tag{3}$$

where $\Lambda_i(k)$ is the set of the $(M-1)M^{K-1}$ indices of hypotheses in which the k^{th} user's symbol differs from its symbol corresponding to the transmitted hypothesis H_i. $\Pr\left\{\mathcal{E}_k(\Phi^{\mathrm{NC}})\right\}$ can be lower-bounded by

$$\Pr\left\{\mathcal{E}_k(\Phi^{\mathrm{NC}})\right\} \geq M^{-K} \sum_{i=1}^{M^K} \Pr\left\{\delta_{\hat{j}}^{\mathrm{NC}} < \delta_i^{\mathrm{NC}}\right\}, \tag{4}$$

where $H_{\hat{j}}$ corresponds to one of the $M-1$ hypotheses H_j that result in an error only for user k when compared to H_i and can be chosen to maximize $\Pr\left\{\delta_{\hat{j}}^{\mathrm{NC}} < \delta_i^{\mathrm{NC}}\right\}$ to tighten the bound. Details for the straight-forward derivation of these bounds are given in [1].

B. Pair-wise Error Probabilities

As in the coherent case, the pair-wise error probabilities $\Pr\left\{\delta_j^{\mathrm{NC}} < \delta_i^{\mathrm{NC}}\right\}$ are crucial for the bounds on the symbol as well as the bit error rate. Note that the noncoherent optimum receiver (as opposed to the coherent receiver in [1]) involves in general a nonzero constant $c_i = \ln\left|\mathbf{K}_{\mathbf{yy}|H_i}\right|$ besides the quadratic form, complicating the analysis somewhat. In [10] we deal with this case, for both asymptotic and non-asymptotic scenarios. Before we come to the main results of this paper, which are the asymptotic analysis of the pair-wise error probabilities and the insights gained from that analysis, we state the pair-wise error probabilities for finite SNR in the following proposition, which can be easily obtained from, for example, [10].

Proposition 1 (Expression for $\Pr\left\{\delta_j^{\mathrm{NC}} < \delta_i^{\mathrm{NC}}\right\}$)
Let $\{\lambda_l\}_{l=1}^{L}$ be the distinct non-zero eigenvalues of $\mathbf{C}_{ij}^{\mathrm{NC}} = \mathbf{K}_{\mathbf{yy}|H_i}\mathbf{K}_{\mathbf{yy}|H_j}^{-1} - \mathbf{I}$ with multiplicities $\{\mu_l\}_{l=1}^{L}$, and let $\{\lambda_l\}_{l=1}^{L_n}$ be negative and $\{\lambda_l\}_{l=L_n+1}^{L}$ positive, respectively. With $c_{ij} = c_i - c_j = \ln \frac{|\mathbf{K}_{\mathbf{zz}|H_i}|}{|\mathbf{K}_{\mathbf{zz}|H_j}|}$ we have

$$\Pr\left\{\delta_j^{\mathrm{NC}} < \delta_i^{\mathrm{NC}}\right\} = -\sum_{k=1}^{L_n} \operatorname{Res}\left(\frac{\exp(s c_{ij})}{s \prod_{l=1}^{L} \lambda_l^{\mu_l}\left(s + \frac{1}{\lambda_l}\right)^{\mu_l}},\ s_k = \frac{-1}{\lambda_k}\right)$$

for $c_{ij} \leq 0$ and

$$\Pr\left\{\delta_j^{\mathrm{NC}} < \delta_i^{\mathrm{NC}}\right\} = 1 + \sum_{k=L_n+1}^{L} \operatorname{Res}\left(\frac{\exp(s c_{ij})}{s \prod_{l=1}^{L} \lambda_l^{\mu_l}\left(s + \frac{1}{\lambda_l}\right)^{\mu_l}},\ s_k = \frac{-1}{\lambda_k}\right)$$

for $c_{ij} > 0$.

The residue of a function $\mathrm{f}(s)$ in a pole a of multiplicity m is defined in [1].

The residue expressions are utterly unrevealing about the essential nature of the error probabilities. For example they fail to answer the basic question about the number of dimensions necessary such that each user is detected with an error rate that is asymptotically at least polynomial in the inverse SNR. The lowest order of the polynomial, i.e., the diversity order of the error rate when each user employs multiple transmit antennas, also warrants clarification, as does the question about the minimum of dimensions and the signals that achieve the maximum diversity $N = N_{\mathrm{T}}N_{\mathrm{R}}$. Some of the answers to these questions are given in the following proposition.

Proposition 2: (Minimum Number of Dimensions for Noncoherent Multiuser Communications)
To noncoherently detect K users, each transmitting one out of M signals from $N_{\mathrm{T}} = 1$ antenna, with an error probability that is asymptotically polynomial in the inverse SNR with lowest exponent, i.e., diversity order, N_{R}, the signal space dimensionality must be at least $K+1$. To ensure a diversity order of $N = N_{\mathrm{T}}N_{\mathrm{R}}$, a dimensionality of at least $(K+1)N_{\mathrm{T}}$ is necessary.

That at least $(K+1)N_{\mathrm{T}}$ dimensions are necessary to achieve full order of diversity for every user will become clear in hindsight of the asymptotic analysis of the pair-wise error probabilities. This analysis will show that the maximum diversity order over all pair-wise error probabilities in a fictitious $\widetilde{K} = KN_{\mathrm{T}}$ system in which each of the users employs only one transmit antenna is bounded by $(D - \widetilde{K})N_{\mathrm{R}}$. Since the admissible error events of the system in which K groups of N_{T} users co-operate (i.e., a K user system in which each user employs N_{T} transmit antennas), is a subset of the error events of the $\widetilde{K}$–user system, it is clear that if the number of dimensions is less than $(K+1)N_{\mathrm{T}}$ then it is not possible to achieve full order of diversity $N = N_{\mathrm{T}}N_{\mathrm{R}}$.

The proof of the first part of the proposition, that at least $K+1$ dimensions are necessary to achieve a diversity order of N_{R} for $N_{\mathrm{T}} = 1$ is somewhat involved and beyond the scope of this paper. But for some insights, consider

$$\mathbf{C}_{ij}^{\mathrm{NC}} = \mathbf{K}_{\mathbf{yy}|H_i}\left(\mathbf{K}_{\mathbf{yy}|H_j}^{-1} - \mathbf{K}_{\mathbf{yy}|H_i}^{-1}\right), \tag{5}$$

whose eigenvalues together with c_{ij} determine the symbol error rate. It is easy to see that if $\boldsymbol{\mathcal{F}}_i\boldsymbol{\Sigma}\boldsymbol{\mathcal{F}}_i^{\dagger}$ and $\boldsymbol{\mathcal{F}}_j\boldsymbol{\Sigma}\boldsymbol{\mathcal{F}}_j^{\dagger}$ are both invertible, then $\mathbf{C}_{ij}^{\mathrm{NC}}$ converges to a matrix independent of σ^2 (and c_{ij} converges to a constant). Consequently, the eigenvalues and the error probability become asymptotically independent of the SNR. We conclude that $\boldsymbol{\mathcal{F}}_i\boldsymbol{\Sigma}\boldsymbol{\mathcal{F}}_i^{\dagger}$ and $\boldsymbol{\mathcal{F}}_j\boldsymbol{\Sigma}\boldsymbol{\mathcal{F}}_j^{\dagger}$ should not be both invertible, which requires either $D > K$ dimensions or, if $D \leq K$, that all but one of the $\mathbf{F}_i$ are low rank. In the following we argue that the latter choice ($D \leq K$, $\mathbf{F}_i$ low rank) does not achieve an error probability that is asymptotically polynomial in the inverse SNR. Consequently, for this paper we will restrict ourselves to $D > K$ for $N_{\mathrm{T}} = 1$. Since for $N_{\mathrm{T}} > 1$ we can employ the argument of upper bounding the error probability of this channel by that of a fictitious

$\widetilde{K} = KN_\mathrm{T}$ channel, we assume that for the general analysis $D > KN_\mathrm{T}$[1]. For analytical tractability we will also assume that each $\mathbf{F}_i$ has full rank KN_T in the asymptotic analysis to come.

But before we present the asymptotic analysis with the stated assumptions, we want to give some intuition as to why for $D \leq K$, $N_\mathrm{T} = 1$, and all but one $\mathbf{F}_i$ low-rank the error probability is asymptotically not strictly a polynomial in the inverse SNR. Consider single-user noncoherent communications with a single transmit and receive antenna. As is already pointed out in [13], a single dimension is sufficient to communicate in a Rayleigh fading channel, if on–off keying is used (see also a recent treatment of on–off keying in Rician fading in [14]). The error probability for a Mark (i.e. the user transmits $\sqrt{2w}$ but the receiver decides Space) is seen in [13], [15] (and can also easily be obtained from Proposition 1) to be $P_M = 1 - \exp\left(-\frac{1}{2\bar{\gamma}} \ln(2\bar{\gamma}+1)\right)$, where $\bar{\gamma} = w/\sigma^2$. The error probability for a Space (i.e. the user transmits nothing, but the receiver decides Mark) is $P_S = (2\bar{\gamma}+1)^{-\left(1+\frac{1}{2\bar{\gamma}}\right)}$. Unlike the Gaussian case, where asymptotically P_M and P_S contribute equally to the average bit error rate $P^\mathrm{b} = \frac{1}{2}(P_M + P_S)$ (see the correction of an error in Stein's treatment of on–off keying in the Gaussian channel in [15]), in Rayleigh fading P^b is dominated by P_M, and, more importantly, the asymptotic decay of P^b is $P^\mathrm{a} = \frac{1}{4\bar{\gamma}} \ln(2\bar{\gamma})$ and thus not *quite* polynomial in $\bar{\gamma}^{-1}$, which shows in Figure 1 by an ever widening gap between P^b and P_S.

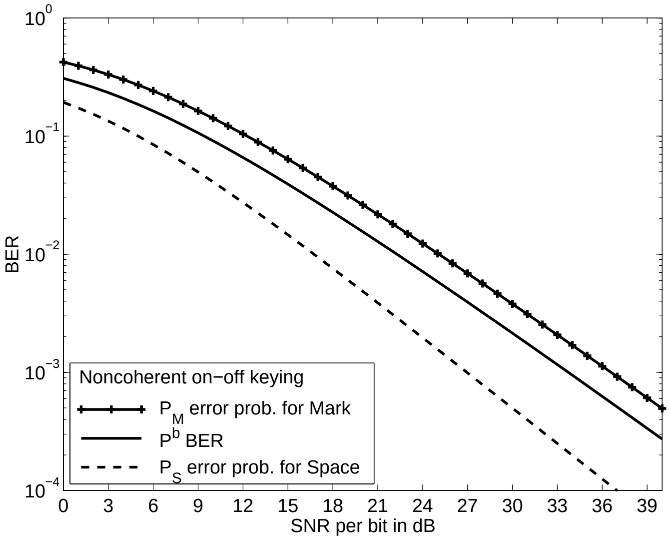

Fig. 1. Mark, Space, and bit error rate for noncoherent on–off keying in a single-user channel.

This example shows that for $K = 1$ and $N_\mathrm{R} = 1$ the above claim about the less than polynomial decline in error probability for $D \leq K$ is true. Note also that noncoherent pulse amplitude modulation (PAM) does not work for $M > 2$ on the fading channel:[2] It follows from the consideration of asymptotic eigenvalues of $\mathbf{C}_{ij}^\mathrm{NC}$ as discussed above, that the error probability floors if a users tries to distinguish two signals with different non-zero amplitudes $\sqrt{w_i}$ and $\sqrt{w_j}$. On–off keying works because one "amplitude" is zero and hence $\mathbf{F}_i\mathbf{F}_i^\dagger = 0$ is not invertible.

We now present the asymptotically tight pair-wise error probabilities for $D > KN_\mathrm{T}$ and assuming that each $\mathbf{F}_i$ has full rank KN_T. To find these exact asymptotic pair-wise error probabilities, we have to find the asymptotic eigenvalues of $\mathbf{C}_{ij}^\mathrm{NC}$ and $\hat{c}_{ij}^\mathrm{NC} = \lim_{\sigma\to 0} \ln \frac{\left|\mathbf{K}_{\mathbf{yy}|H_i}\right|}{\left|\mathbf{K}_{\mathbf{yy}|H_j}\right|}$. For the latter constant, one easily sees that $\mathbf{K}_{\mathbf{yy}|H_i}$ and $\mathbf{K}_{\mathbf{yy}|H_j}$ have the same number of eigenvalues ($KN_\mathrm{T} - D$) that tend to zero (and cancel in the ratio), and one easily finds

$$\hat{c}_{ij}^\mathrm{NC} = \lim_{\sigma\to 0} \ln \frac{\left|\mathbf{K}_{\mathbf{yy}|H_i}\right|}{\left|\mathbf{K}_{\mathbf{yy}|H_j}\right|} = N_\mathrm{R} \ln \frac{\left|\mathbf{F}_i^\dagger \mathbf{F}_i\right|}{\left|\mathbf{F}_j^\dagger \mathbf{F}_j\right|}, \qquad (6)$$

where the factor N_R arises from the fact that $|\boldsymbol{\mathcal{F}}_i^\dagger \boldsymbol{\mathcal{F}}_i| = |\mathbf{F}_i^\dagger \mathbf{F}_i|^{N_\mathrm{R}}$.

To simplify the analysis of the asymptotic eigenvalues, we introduce an effective signal matrix $\mathbf{H}_i$ defined as

$$\boldsymbol{\mathcal{H}}_i = \boldsymbol{\mathcal{F}}_i \boldsymbol{\mathcal{W}}^{1/2} \mathbf{U}^\dagger, \qquad (7)$$

where $\mathbf{U}$ is an upper triangular matrix resulting from the Cholesky factorization of $\mathbf{\Sigma} = \mathbf{U}^\dagger \mathbf{U}$. With this we can write the covariance matrix of the observations as $\mathbf{K}_{\mathbf{yy}|H_i} = \boldsymbol{\mathcal{H}}_i \boldsymbol{\mathcal{H}}_i^\dagger + \sigma^2 \mathbf{I}$. In stating the final results for the asymptotic pair-wise error probability, we will switch back to the original system parameters $\boldsymbol{\mathcal{F}}_i$ ($\mathbf{F}_i$) and $\mathbf{\Sigma}$.

As in the coherent case [1], we assume that the users are ordered such that users $1, 2, \ldots, e$ suffer from a (symbol-)error, if the detector would erroneously decide for hypothesis H_j when hypothesis H_i is transmitted. Obviously, such an ordering can always be obtained. To avoid a complication in notation, we usually do not denote this user-ordering by any special symbols, but assume it implicitly. Another notational convenience is to split up the transmitted signal into two parts, the first containing the signals of the e users that suffer from an error relative to H_j, and the second part containing the $\bar{e} = K - e$ signals corresponding to the correctly detected users, i.e.,

$$\mathbf{F}_i = \left[\mathbf{F}_i^e \; \mathbf{F}^c\right], \; \mathbf{F}_j = \left[\mathbf{F}_j^e \; \mathbf{F}^c\right],$$

where c signifies the common part in the two signals $\mathbf{F}_i$ and $\mathbf{F}_j$. The matrices $\mathbf{F}_i^e$ and $\mathbf{F}_j^e$ are $D \times eN_\mathrm{T}$ and $\mathbf{F}^c$ is $D \times \bar{e}N_\mathrm{T}$. Similarly, we define $\boldsymbol{\mathcal{F}}_i^e$, $\boldsymbol{\mathcal{H}}_i^e$, $\boldsymbol{\mathcal{F}}_j^e$, $\boldsymbol{\mathcal{H}}_j^e$, and $\boldsymbol{\mathcal{F}}^c$, $\boldsymbol{\mathcal{H}}^c$ (whose sizes are multiplied by N_R when compared to $\mathbf{F}_i^e$, $\mathbf{F}_j^e$, and $\mathbf{F}^c$, respectively). Furthermore, we define $\mathbf{\Sigma}_{ee}$ and $\boldsymbol{\mathcal{W}}_{ee}$ as the $eN \times eN$ upper-left block of $\mathbf{\Sigma}$ and $\boldsymbol{\mathcal{W}}$, respectively (recall $N = N_\mathrm{T}N_\mathrm{R}$). $\mathbf{\Sigma}_{ec}$ and $\mathbf{\Sigma}_{cc}$ ($\boldsymbol{\mathcal{W}}_{cc}$) are the corresponding upper- and lower-right blocks of $\mathbf{\Sigma}$ ($\boldsymbol{\mathcal{W}}$).

We have to find the asymptotic eigenvalues of $\mathbf{C}_{ij}^\mathrm{NC}$ to calculate the asymptotic pair-wise error probability. Applying Woodbury's identity (inverse of a small rank adjustment [16,

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[1] Note that any intelligently designed K user N_T transmit antenna system should certainly perform better than a fictitious $\widetilde{K}$ user system and this justifies the choice of the upper bound.

[2] At least as long as there are no instantaneous amplitudes available to the receiver, but only average energies and second order statistics about the fading, as we assume in this paper.

Section 0.7.4]) twice allows us to write

$$\mathbf{C}_{ij}^{\mathrm{NC}} = \left(\sigma^{-2}\boldsymbol{\mathcal{H}}_i\boldsymbol{\mathcal{H}}_i^{\dagger} + \mathbf{I}\right) \left[\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp} + \sigma^2\boldsymbol{\mathcal{H}}_j \left(\boldsymbol{\mathcal{H}}_j^{\dagger}\boldsymbol{\mathcal{H}}_j\right)^{-1} \times \right.$$
$$\left. \left(\mathbf{I} + \sigma^2 \left(\boldsymbol{\mathcal{H}}_j^{\dagger}\boldsymbol{\mathcal{H}}_j\right)^{-1}\right)^{-1} \left(\boldsymbol{\mathcal{H}}_j^{\dagger}\boldsymbol{\mathcal{H}}_j\right)^{-1} \boldsymbol{\mathcal{H}}_j^{\dagger}\right] - \mathbf{I}.$$

For small σ^2 we can approximate $\mathbf{C}_{ij}^{\mathrm{NC}}$ by

$$\mathbf{C}_{ij}^{\mathrm{NC}} \to \widetilde{\mathbf{C}_{ij}^{\mathrm{NC}}} = \tag{8}$$
$$\sigma^{-2}\boldsymbol{\mathcal{H}}_i\boldsymbol{\mathcal{H}}_i^{\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp} + \boldsymbol{\mathcal{H}}_i\boldsymbol{\mathcal{H}}_i^{\dagger}\boldsymbol{\mathcal{H}}_j \left(\boldsymbol{\mathcal{H}}_j^{\dagger}\boldsymbol{\mathcal{H}}_j\right)^{-2} \boldsymbol{\mathcal{H}}_j^{\dagger} - \mathbf{P}_{\boldsymbol{\mathcal{H}}_j}.$$

$\widetilde{\mathbf{C}_{ij}^{\mathrm{NC}}}$ can be written as the product of the matrix $\mathbf{A} = \left[\boldsymbol{\mathcal{H}}_i^e\ \boldsymbol{\mathcal{H}}_j^e\ \boldsymbol{\mathcal{H}}^c\right]$ and $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 - \mathbf{B}_3$, where

$$\mathbf{B}_1 = \begin{bmatrix} \sigma^{-2}\boldsymbol{\mathcal{H}}_i^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp} \\ \mathbf{0} \\ \sigma^{-2}\boldsymbol{\mathcal{H}}^{c\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp} \end{bmatrix}, \tag{9}$$

$$\mathbf{B}_2 = \begin{bmatrix} \boldsymbol{\mathcal{H}}_i^{e\dagger}\boldsymbol{\mathcal{H}}_j \left(\boldsymbol{\mathcal{H}}_j^{\dagger}\boldsymbol{\mathcal{H}}_j\right)^{-2} \boldsymbol{\mathcal{H}}_j^{\dagger} \\ \mathbf{0} \\ \boldsymbol{\mathcal{H}}^{c\dagger}\boldsymbol{\mathcal{H}}_j \left(\boldsymbol{\mathcal{H}}_j^{\dagger}\boldsymbol{\mathcal{H}}_j\right)^{-2} \boldsymbol{\mathcal{H}}_j^{\dagger} \end{bmatrix}, \tag{10}$$

$$\mathbf{B}_3 = \begin{bmatrix} \mathbf{0} \\ \left(\boldsymbol{\mathcal{H}}_j^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}^c}^{\perp}\boldsymbol{\mathcal{H}}_j^e\right)^{-1} \boldsymbol{\mathcal{H}}_j^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}^c}^{\perp} \\ [\mathbf{B}_3]_3 \end{bmatrix}, \tag{11}$$

and $[\mathbf{B}_3]_3 =$
$\left(\boldsymbol{\mathcal{H}}^{c\dagger}\boldsymbol{\mathcal{H}}^c\right)^{-1} \boldsymbol{\mathcal{H}}^{c\dagger} \left(\mathbf{I} + \boldsymbol{\mathcal{H}}_j^e \left(\boldsymbol{\mathcal{H}}_j^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}^c}^{\perp}\boldsymbol{\mathcal{H}}_j^e\right)^{-1} \boldsymbol{\mathcal{H}}_j^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}^c}^{\perp}\right)$.

Since the non-zero eigenvalues of $\mathbf{AB}$ are equal to the non-zero eigenvalues of $\mathbf{BA}$, we calculate the latter product and find after some algebra that the bottom $(K-e)N$ rows are zero. Deleting these rows and the corresponding columns and applying some more algebra, one is finally left with a Hermitian matrix $\mathbf{M}_{ij}$, whose $eN \times eN$ upper-left, -right, and lower-right blocks are

$$\begin{aligned}
[\mathbf{M}_{ij}]_{\mathrm{ul}} &= \sigma^{-2}\boldsymbol{\mathcal{H}}_i^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp}\boldsymbol{\mathcal{H}}_i^e + \\
&\quad + \boldsymbol{\mathcal{H}}_i^{e\dagger}\boldsymbol{\mathcal{H}}_j \left(\boldsymbol{\mathcal{H}}_j^{\dagger}\boldsymbol{\mathcal{H}}_j\right)^{-2} \boldsymbol{\mathcal{H}}_j^{\dagger}\boldsymbol{\mathcal{H}}_i^e, \\
[\mathbf{M}_{ij}]_{\mathrm{ur}} &= \boldsymbol{\mathcal{H}}_i^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}^c}^{\perp}\boldsymbol{\mathcal{H}}_j^e \left(\boldsymbol{\mathcal{H}}_j^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}^c}^{\perp}\boldsymbol{\mathcal{H}}_j^e\right)^{-1}, \\
[\mathbf{M}_{ij}]_{\mathrm{lr}} &= -\mathbf{I}_{eN}.
\end{aligned}$$

For small σ and if $\boldsymbol{\mathcal{H}}_i^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp}\boldsymbol{\mathcal{H}}_i^e$ has full rank the asymptotic eigenvalues of $\mathbf{M}_{ij}$ are the positive real eigenvalues of $\sigma^{-2}\boldsymbol{\mathcal{H}}_i^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp}\boldsymbol{\mathcal{H}}_i^e$ and minus unity with multiplicity eN (cf. [10, Appendix C]). For $\boldsymbol{\mathcal{H}}_i^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp}\boldsymbol{\mathcal{H}}_i^e$ to have full rank, $eN \le DN_{\mathrm{R}} - KN$ is necessary (hence $eN_{\mathrm{T}} \le D - KN_{\mathrm{T}}$). Should $\boldsymbol{\mathcal{H}}_i^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp}\boldsymbol{\mathcal{H}}_i^e$ only have rank $rN_{\mathrm{R}} < eN$ (r is the rank of $\mathbf{F}_i^{e\dagger}\mathbf{P}_{\mathbf{F}_j}^{\perp}\mathbf{F}_i^e$), then the positive eigenvalues linear in σ^{-2} of $\mathbf{C}_{ij}^{\mathrm{NC}}$ are still the positive eigenvalues of $\sigma^{-2}\boldsymbol{\mathcal{H}}_i^{e\dagger}\mathbf{P}_{\boldsymbol{\mathcal{H}}_j}^{\perp}\boldsymbol{\mathcal{H}}_i^e$. Furthermore, it can be shown that the eigenvalue minus unity shows up with multiplicity rN_{R}. However, $\mathbf{M}_{ij}$ may have other real eigenvalues asymptotically independent of σ. We haven't yet been able to find expressions for these eigenvalues in closed form (numerical calculation is straight forward). Even if such expressions were found, it remains unclear as to how the asymptotic error probability can be expressed in closed form: although the results of [10] can be easily extended to take care of these "extra" eigenvalues in the sense of writing the asymptotic error probability using residues, for most cases these residues involve poles of high multiplicity and lack the structure previously present, so that they could not be expressed in closed form. (Calculating these residues with a computer, however, is no problem at all.) For this reason, we present the asymptotic error probabilities only for the case that $\mathbf{F}_i^{e\dagger}\mathbf{P}_{\mathbf{F}_j}^{\perp}\mathbf{F}_i^e$ has full rank eN_{T}.

Proposition 3 (Asymptotic Pair-Wise Error Probability) For noncoherent detection and assuming $\mathbf{F}_i^{e\dagger}\mathbf{P}_{\mathbf{F}_j}^{\perp}\mathbf{F}_i^e$ has full rank (requiring $eN_{\mathrm{T}} \le D - KN_{\mathrm{T}}$ or, equivalently $D \ge (e+K)N_{\mathrm{T}}$), the sum of the two corresponding pair-wise error probabilities of the optimum detector Φ^{NC} approaches arbitrarily closely

$$\begin{aligned}
P_{ij}^{\mathrm{NC}} &= \Pr\nolimits^{\mathrm{a}}\left\{\delta_j^{\mathrm{NC}} < \delta_i^{\mathrm{NC}}\right\} + \Pr\nolimits^{\mathrm{a}}\left\{\delta_i^{\mathrm{NC}} < \delta_j^{\mathrm{NC}}\right\} \\
&= \frac{\sigma^{2eN}\sum_{n=0}^{eN}\binom{2eN-n}{eN}\frac{\left(\hat{c}_{ij}^{\mathrm{NC}}\right)^n}{n!}}{\left|\mathbf{\Sigma}_{ee}\right| \left|\mathbf{W}_{ee}\mathbf{F}_i^{e\dagger}\mathbf{P}_{\mathbf{F}_j}^{\perp}\mathbf{F}_i^e\right|^{N_{\mathrm{R}}}},
\end{aligned}$$

where $\hat{c}_{ij}^{\mathrm{NC}} = N_{\mathrm{R}} \ln \frac{\left|\mathbf{F}_i^{\dagger}\mathbf{F}_i\right|}{\left|\mathbf{F}_j^{\dagger}\mathbf{F}_j\right|} \ge 0$, which can always be assured by renumbering the hypotheses accordingly.

III. ASYMPTOTICALLY OPTIMUM RECEIVER

The optimum noncoherent receiver Φ^{NC} does not require either instantaneous phases nor the instantaneous amplitudes of the users. However, the average SNRs of the users and the statistics of the fading coefficients, the correlation matrix $\mathbf{\Sigma}$, are assumed to be known at the receiver. It turns out that the asymptotic expansion of Φ^{NC} does not even require knowledge about these statistics. Moreover, it is not hard to prove that the asymptotic pair-wise error probabilities of the asymptotic expansion receiver are identical to those of the optimum receiver. Thus, the asymptotic expansion receiver is asymptotically optimum. It is easily found by the approximation

$$\begin{aligned}
\mathbf{K}_{\mathbf{yy}|H_i}^{-1} &= \sigma^{-2}\left(\mathbf{I} - \boldsymbol{\mathcal{H}}_i\left(\sigma^2\mathbf{I} + \boldsymbol{\mathcal{H}}_i^{\dagger}\boldsymbol{\mathcal{H}}_i\right)^{-1}\boldsymbol{\mathcal{H}}_i^{\dagger}\right) \\
&\approx \sigma^{-2}\mathbf{P}_{\boldsymbol{\mathcal{H}}_i}^{\perp} = \sigma^{-2}\mathbf{P}_{\boldsymbol{\mathcal{F}}_i}^{\perp}, \qquad (12)
\end{aligned}$$

valid for small σ and realizing that $\mathbf{K}_{\mathbf{yy}|H_i}$ has $D - KN_{\mathrm{T}}$ eigenvalues σ^2 for each i, so that the receiver asymptotically need not consider this part of $\ln\left|\mathbf{K}_{\mathbf{yy}|H_i}\right|$ and we are left with

$$\begin{aligned}
\Phi^{\mathrm{NC-AO}} : \hat{\imath} &= \arg\min_{1\le i\le M^K} \sigma^{-2}\mathbf{y}^{\dagger}\mathbf{P}_{\boldsymbol{\mathcal{F}}_i}^{\perp}\mathbf{y} + N_{\mathrm{R}}\ln\left|\mathbf{F}_i^{\dagger}\mathbf{F}_i\right|, \\
&= \arg\max_{1\le i\le M^K} \sigma^{-2}\sum_{n=1}^{N_{\mathrm{R}}}\mathbf{y}_n^{\dagger}\mathbf{P}_{\mathbf{F}_i}\mathbf{y}_n - N_{\mathrm{R}}\ln\left|\mathbf{F}_i^{\dagger}\mathbf{F}_i\right|,
\end{aligned}$$

where the last equation is most easily obtained by writing $\mathbf{K}_{\mathbf{yy}|H_i}$ as $\left(\mathbf{I}_{N_{\mathrm{R}}} \otimes \mathbf{F}_i \mathbf{W}^{1/2}\right) \widehat{\boldsymbol{\Sigma}} \left(\mathbf{I}_{N_{\mathrm{R}}} \otimes \mathbf{F}_i \mathbf{W}^{1/2}\right)^{\dagger} + \sigma^2 \mathbf{I}$. $\Phi^{\mathrm{NC-AO}}$ has a geometric interpretation: The receiver compares the energy of the sufficient statistics in the subspaces spanned by $\mathbf{F}_i$ and chooses, after some normalizations, the hypothesis in whose corresponding sub-space the sufficient statistics have the largest energy. Note that while the receiver does not require any knowledge about the fading statistics or the users' mean energies, knowledge about the noise level is still required.

IV. INTERPRETATIONS AND SIGNAL DESIGN

The asymptotic union bound gives a closed-form expression for the system performance depending on the employed signals and can thus be exploited to design signals that minimize it. The design techniques for this minimization are very similar to the ones described in [9], [8], where signals are designed for non-coherent single-user space-time communications. The space-time problem can be considered a multiuser problem with cooperating users.

We consider a $K = 8$ user example, with $N_{\mathrm{T}} = 1$ transmit antenna per user and $N_{\mathrm{R}} = 2$ receive antennas. The users communicate in a common 10 dimensional signal space using $M = 2$ signals each. Thus the aggregate spectral efficiency is 0.8 bps/Hz. Figure 2 shows the bounds and simulated bit error rate for one user in i.i.d. fading ($\boldsymbol{\Sigma} = \mathbf{I}$) and equal-energy interfering users ($\mathbf{W} = \mathbf{I}$); the signal design algorithm returns a signal set such that the performances of the users are undistinguishable from each other. Since $D = 10 \geq (K+1)N_{\mathrm{T}} + 1$ the upper bound converges to the lower bound. As a reference we give the BER of binary orthogonal signaling and the BER of a single user, employing $M = 4$ signals in $D = 2$ dimensions that were designed with the techniques of [9], [8]. Note that the single user achieves a spectral efficiency of 1 bps/Hz with a roughly 3 dB lower SNR per bit than the multiuser system, whose aggregate spectral efficiency is 0.8 bps/Hz.

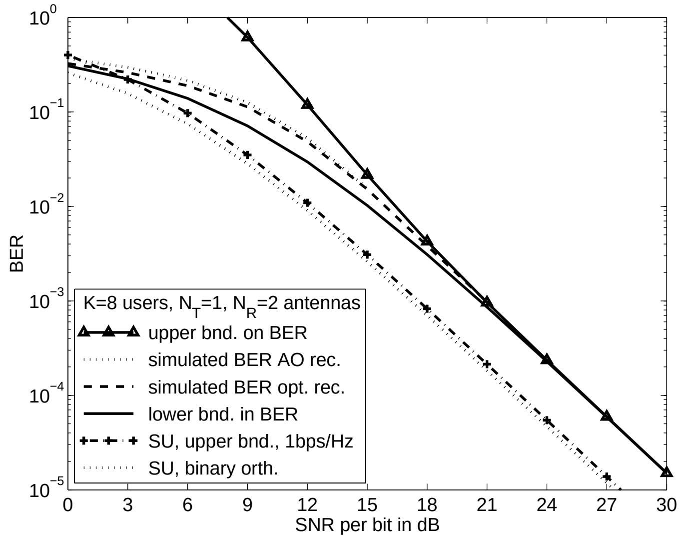

Fig. 2. The spectral efficiency (SE) of noncoherent modulation can be increased from 0.5 bps/Hz to 1 bps/Hz without a penalty in SNR per bit. On the other hand, increasing the total SE of a multiuser system by adding users in signal space of dimensionality less than $2K$ seems to be dis-advantageous in terms of SNR per bit.

V. CONCLUSIONS

We analyze multiuser multi-antenna noncoherent communications for $D > KN_{\mathrm{T}}$ and find exact pair-wise error probabilities. Based on the analysis, we propose a signal design algorithm that minimizes the maximum of the asymptotic union bounds on the users' symbol error rates. However, it seems to be more difficult in noncoherent multiuser communications to improve the spectral efficiency by adding users in a common signal space than in coherent communications. In our example, increasing the spectral efficiency by increasing the constellation size of one user in two dimensions and (and subsequently assigning orthogonal signal spaces to each user for a multiuser system) was more energy efficient than communicating in a common signal space.

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