Low-Dimensional Spreading Matrices for Multiuser Space-Time Modulation

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Abstract — **In [1] we proposed a multiuser space-time modulation scheme that leverages single-user space-time constellations and guarantees both, a full diversity order, as well as an asymptotic (high SNR) single-user like performance for every user. This is achieved by multiplying each user's space-time information matrix symbol by a low-dimensional "spreading matrix." For instance, for N_{T}-transmit antennas per user and if the single-user space-time constellation employed requires only the minimum dimension N_{T}, no more than $N_{\mathrm{T}}+1$ dimensions are required for the common signal space of all users, i.e., each user's spreading matrix is of size $(N_{\mathrm{T}}+1)\times N_{\mathrm{T}}$, independent of the number of users.**

In this paper, we present a simplified design criterion to obtain these spreading matrices by numerical optimization. Since the columns of each user's spreading matrix are constrained to be orthonormal, we propose to perform the optimization using a parameterization of the Grassmann manifold. A special feature of the simplified criterion, and thus of the resulting spreading matrices, is that they are independent of the particular single-user space-time constellations (of a given dimension), so that different spectral efficiencies can be attained without changing or redesigning the spreading matrices.

I. Introduction

In a multiuser space-time "narrowband" channel with several transmit antennas per user, the optimum detector achieves full (transmit and receive antenna) diversity order for every user provided each user employs a full diversity, single-user space-time constellation or (block-) code [1]. However, with an increase in the number of users, an increasing signal-to-noise ratio (SNR) is required of each user to achieve the same error probability.

The modulation method we proposed in [1] leverages existing single-user space-time constellations and uses spreading matrices to alleviate–and to even eliminate asymptotically (for high SNR)–the SNR penalty of the narrowband multiuser channel while requiring only a small increase in signal space dimensions. In particular, the single-user space-time symbol matrices of each user are multiplied by a low-dimensional spreading matrix with orthonormal columns. The spreading matrix has row dimension just greater than the (time)-dimension of the single-user space-time constellation so as to achieve high aggregate spectral efficiencies while markedly improving performance relative to the narrowband channel. For example, if each user employs N_{T} transmit antennas and a single-user space-time code with the minimum required dimensions N_{T} (cf. [2], [3]), our proposed modulation scheme requires just one more dimension to achieve asymptotic single-user like performance. By an "asymptotic single-user like" performance we mean that for high SNR the upper bound on the multiuser bit error rate (BER) converges to the upper bound on the single-user BER.

In this paper we present a simplified design criterion (relative to the one we proposed in [1]), so that spreading matrices designed according to it continue to provide asymptotic single-user like performance for every user. The additional advantage is that the simplified criterion, and hence the spreading matrices, depend only on the dimensionality of the chosen single-user space-time constellation. To obtain the spreading matrices by numerical optimization we characterize each matrix using a parameterization of the Grassmann manifold. Consequently, the optimization can be performed in an unconstrained manner using standard numerical optimization techniques.

In an example, we design spreading matrices for two and three users in $D=3$ dimensions, each employing the Alamouti scheme [4] for $N_{\mathrm{T}}=2$ transmit antennas. It is seen that at a BER of 10^{-3} the optimum detector can detect the three users within 1 dB of the single-user BER (while achieving twice the spectral efficiency) and outperforms a single user with the same spectral efficiency by roughly 3 dB.

In Section II, we describe the K user, N_{T} transmit, N_{R} receive antenna symbol-synchronous system model that incorporates the new spreading matrix-based modulation scheme. The optimum receiver and asymptotic pair-wise error probabilities are quickly obtained as special cases of [1] in Section III. In Section IV we present the design criterion and algorithm for obtaining the low-dimensional spreading matrices and give an example design when the users employ the Alamouti scheme [4]. We conclude in Section V.

II. Multi-Antenna, Multiuser Discrete Time System Model

We briefly describe a system model in which K users communicate simultaneously in a common D–dimensional signal space. Each of the K users employs N_{T} transmit antennas to send information symbol-synchronously to an N_{R} receive antenna array of the base-station. Since there are N_{T} transmit antennas and D dimensions, user k transmits one out of M possible $D\times N_{\mathrm{T}}$ complex-valued signal matrices $\mathbf{S}_{km}\in\mathbb{C}^{D\times N_{\mathrm{T}}}$. The transmitted matrices $\mathbf{S}_{km}$ are generated by multiplying a $D\times D_{\mathrm{SU}}$ signature spreading matrix $\mathbf{F}_k$ with a single-user $D_{\mathrm{SU}}\times N_{\mathrm{T}}$ space-time symbol $\mathbf{B}_m$, i.e., $\mathbf{S}_{km}=\mathbf{F}_k\mathbf{B}_m$. The single-user space-time information symbol matrices $\mathbf{B}_m$ can originate from, for example, the Alamouti scheme [4], orthogonal designs [5], or the algebraic codes of [6], or any other single-user space-time constellation that guarantees full transmit antenna diversity. The dimensions of the single-user space-time constellation are signified by D_{SU} and it is well

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known that to achieve full diversity order $N = N_{\mathrm{T}}N_{\mathrm{R}}$ at least N_{T} dimensions are necessary [2], [3]. For simplicity, we assume that each user employs the same single-user space-time constellation in this paper. To succinctly write the discrete-time model for this system we need more definitions.

Let H_i denote the i^{th} hypothesis with $1 \leq i \leq M^K$. Without loss of generality let i determine uniquely the K-tuple $(i_1, i_2, \ldots, i_K)$ according to $i = \sum_{k=1}^{K}(i_k - 1)M^{k-1} + 1$. We let hypothesis H_i denote that user k transmits the signal $\mathbf{S}_{ki_k} = \mathbf{F}_k\mathbf{B}_{i_k}$ for each k. Define the $D \times KD_{\mathrm{SU}}$ matrix of all signature matrices $\mathbf{F} = [\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_K]$, and the $KD_{\mathrm{SU}} \times KN_{\mathrm{T}}$ block diagonal matrix of all data symbols as $\mathbf{D}_i$, where the k^{th} $D_{\mathrm{SU}} \times N_{\mathrm{T}}$ diagonal block of $\mathbf{D}_i$ is $\mathbf{B}_{i_k}$. Furthermore, define the "script" versions of these matrices as $\boldsymbol{\mathcal{F}}_k = \mathbf{I}_{N_{\mathrm{R}}} \otimes \mathbf{F}_k$, $\boldsymbol{\mathcal{F}} = [\boldsymbol{\mathcal{F}}_1, \boldsymbol{\mathcal{F}}_2, \ldots, \boldsymbol{\mathcal{F}}_K]$, $\boldsymbol{\mathcal{B}}_{i_k} = \mathbf{I}_{N_{\mathrm{R}}} \otimes \mathbf{B}_{i_k}$ (thus $\boldsymbol{\mathcal{S}}_{ki_k} = \boldsymbol{\mathcal{F}}_k\boldsymbol{\mathcal{B}}_{i_k}$), and $\boldsymbol{\mathcal{D}}_i$ as the block-diagonal matrix with $\boldsymbol{\mathcal{B}}_{i_k}$ as diagonal elements ($\otimes$ denotes the Kronecker or tensor product). Then, following [1], the discrete-time model can be written as

$$\mathbf{y} = \boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{D}}_i\boldsymbol{\mathcal{W}}^{1/2}\mathbf{h} + \boldsymbol{\eta}, \tag{1}$$

where $\mathbf{y}$ is the DN_{R}-length vector of observations, $\boldsymbol{\mathcal{W}} = \operatorname{diag}\{w_1, w_2, \ldots, w_K\} \otimes \mathbf{I}_N$ ($N = N_{\mathrm{T}}N_{\mathrm{R}}$), and w_k is the k^{th} user's average received energy, $\mathbf{h}$ is a KN-length vector of $\mathcal{CN}(\mathbf{0}, \boldsymbol{\Sigma})$ distributed fading coefficients, with $\boldsymbol{\Sigma}_{kk}$ as the k^{th} diagonal block, and $\boldsymbol{\eta}$ is the DN_{R}-length $\mathcal{CN}(\mathbf{0}, \sigma^2\mathbf{I})$ distributed additive noise vector. The signals and fading processes are normalized so that $\bar{\gamma}_k = w_k/\sigma^2$ represents the average received signal-to-noise ratio (SNR) of the k^{th} user per receiver antenna and per (super-) symbol. For details on the normalization see [1]; as an example, in i.i.d. fading our normalization leads to $N_{\mathrm{T}}\boldsymbol{\Sigma}_{kk} = \mathbf{I}_N$.

III. Optimum Receiver and Analysis

Defining the new $(KN_{\mathrm{T}} + D)N_{\mathrm{R}}$–dimensional sufficient statistic $\mathbf{z} = \sigma^{-1}\begin{bmatrix} \mathbf{h}^{\mathsf{T}} & \mathbf{y}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$ and the matrix

$$\mathbf{Q}_i = \begin{bmatrix} \boldsymbol{\mathcal{W}}^{1/2}\boldsymbol{\mathcal{D}}_i^{\dagger}\boldsymbol{\mathcal{F}}^{\dagger}\boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{D}}_i\boldsymbol{\mathcal{W}}^{1/2} & -\boldsymbol{\mathcal{W}}^{1/2}\boldsymbol{\mathcal{D}}_i^{\dagger}\boldsymbol{\mathcal{F}}^{\dagger} \\ -\boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{D}}_i\boldsymbol{\mathcal{W}}^{1/2} & \mathbf{0}_{DN_{\mathrm{R}}} \end{bmatrix}, \tag{2}$$

the jointly optimum coherent receiver Φ can be expressed as

$$\Phi : \hat{i} = \arg\min_{1 \leq i \leq M^K} \mathbf{z}^{\dagger}\mathbf{Q}_i\mathbf{z} = \arg\min_{1 \leq i \leq M^K} \delta_i, \tag{3}$$

where δ_i is defined implicitly. Note that the sufficient statistics $\mathbf{z}$ are $\mathcal{CN}(\mathbf{0}, \mathbf{K}_{\mathbf{zz}|H_i})$ distributed, where $\mathbf{K}_{\mathbf{zz}|H_i} = E\left[\mathbf{z}\mathbf{z}^{\dagger}\right]$. This fact is used in [1] to find the asymptotics of the pair-wise error probabilities $\Pr\{\delta_j < \delta_i\}$ by applying the techniques of [3] ($\{\delta_j < \delta_i\}$ is the event that the decision statistic δ_j is smaller than δ_i given that H_i is the true hypothesis). For convenience, we restate the result on the asymptotic pair-wise error probabilities here, but need to introduce some assumptions and notation first.

We assume that the users are ordered such that users $1, 2, \ldots, e$ suffer from an error, if the receiver would erroneously decide for hypothesis H_j when hypothesis H_i is transmitted. To avoid a complication in notation, we do not denote this user-ordering with any special symbols, but assume it implicitly. Another notational convenience is to partition the matrices $\mathbf{F}$ (that contains the signature spreading matrices of all users) and $\mathbf{D}_i$ (the block-diagonal matrix of all information symbols) into blocks that contain the spreading matrices/information symbols of the e users that suffer from an error relative to H_j, and the second part containing the $\bar{e} = K - e$ spreading matrices/information symbols corresponding to the correctly detected users, i.e.,

$$\mathbf{F} = [\mathbf{F}^e \; \mathbf{F}^c], \tag{4}$$

$$\mathbf{D}_i = \begin{bmatrix} \mathbf{D}_i^e & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^c \end{bmatrix}, \; \mathbf{D}_j = \begin{bmatrix} \mathbf{D}_j^e & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^c \end{bmatrix}, \tag{5}$$

where c signifies the common part in the two data matrices $\mathbf{D}_i$ and $\mathbf{D}_j$. The matrix $\mathbf{F}^e$ is $D \times eD_{\mathrm{SU}}$ and $\mathbf{F}^c$ is $D \times \bar{e}D_{\mathrm{SU}}$. $\mathbf{D}_i^e$ and $\mathbf{D}_j^e$ are $eD_{\mathrm{SU}} \times eN_{\mathrm{T}}$ and $\mathbf{D}^c$ is $\bar{e}D_{\mathrm{SU}} \times \bar{e}N_{\mathrm{T}}$. Similarly, we define $\boldsymbol{\mathcal{F}}^e$, $\boldsymbol{\mathcal{F}}^c$, $\boldsymbol{\mathcal{D}}_i^e$, $\boldsymbol{\mathcal{D}}_j^e$, and $\boldsymbol{\mathcal{D}}^c$ (whose sizes are multiplied by N_{R} when compared to the "non-script" quantities). Furthermore, we define $\boldsymbol{\Sigma}_{ee}$ and $\boldsymbol{\mathcal{W}}_{ee}$ as the $eN \times eN$ upper-left block of $\boldsymbol{\Sigma}$ and $\boldsymbol{\mathcal{W}}$, respectively (recall $N = N_{\mathrm{T}}N_{\mathrm{R}}$).

With this notation and the results of [1] one easily finds the asymptotic pair-wise error-probability given in the proposition, for which we assume that the fading correlation matrix $\boldsymbol{\Sigma}$ has full rank N. For ease of notation, we introduce $|\mathbf{X}|_{\mathsf{NZ}}$ as the product of the non-zero eigenvalues of $\mathbf{X}$.

Proposition 1: (Asymptotic Pair-Wise Error Probability) Let r be the rank of $\mathbf{F}^e\left(\mathbf{D}_j^e - \mathbf{D}_i^e\right)$. For coherent detection the pair-wise error probability $\Pr\{\delta_j < \delta_i\}$ of the optimum receiver Φ approaches arbitrarily closely $\Pr^{\mathrm{a}}\{\delta_j < \delta_i\} =$

$$\frac{\sigma^{2rN_{\mathrm{R}}}\binom{2rN_{\mathrm{R}}-1}{rN_{\mathrm{R}}}}{\left|\boldsymbol{\mathcal{W}}_{ee}^{1/2}\boldsymbol{\Sigma}_{ee}\boldsymbol{\mathcal{W}}_{ee}^{1/2}\left(\boldsymbol{\mathcal{D}}_j^e - \boldsymbol{\mathcal{D}}_i^e\right)^{\dagger}\boldsymbol{\mathcal{F}}^{e\dagger}\boldsymbol{\mathcal{F}}^e\left(\boldsymbol{\mathcal{D}}_j^e - \boldsymbol{\mathcal{D}}_i^e\right)\right|_{\mathsf{NZ}}}$$

as σ goes to zero.

IV. Signal Design

A key conclusion from the above proposition is that the jointly optimum receiver achieves the total order of diversity $N = N_{\mathrm{T}}N_{\mathrm{R}}$ for every user that simply employs a full transmit diversity single-user space-time constellation–without any spreading ($\mathbf{F}_k = \mathbf{I}_{D_{\mathrm{SU}}}\ \forall k$). However, for increasing number of users, it incurs a penalty in terms of SNR, when compared to single-user performance. As we suggested in [1], this SNR penalty can be alleviated for sufficiently high SNR by ensuring that the diversity order of $e \geq 2$ user error events is at least $(N_{\mathrm{T}} + 1)N_{\mathrm{R}}$ for which we have to ensure that $\mathbf{F}^e\left(\mathbf{D}_j^e - \mathbf{D}_i^e\right)$ has at least rank $N_{\mathrm{T}} + 1$. Then for high SNR, the contributions of the $e \geq 2$ user error events to the bound on k^{th} user's (symbol or bit) error probability can be neglected when compared to the single-user error events, whose diversity order is $N = N_{\mathrm{T}}N_{\mathrm{R}}$. Thus the bound on the multiuser system converges asymptotically to the bound on a single-user system that uses the space-time constellation $\{\mathbf{F}_k\mathbf{B}_m\}_{m=1}^{M}$. Moreover, if the columns of $\mathbf{F}_k$ are orthonormal, then the bound on the k^{th} user's error rate converges to the bound on a single user's error rate, who employs the space-time constellation $\{\mathbf{B}_m\}_{m=1}^{M}$, which we call "asymptotic single-user performance."

Using a rank inequality ([7, Section 0.4.5]), one can easily show that if any compound matrix $[\mathbf{F}_l, \mathbf{F}_k]$, $l \neq k$, is of rank greater than or equal $2D_{\text{SU}} - N_{\text{T}} + 1$, then it is guaranteed that $\mathbf{F}^e \left(\mathbf{D}_j^e - \mathbf{D}_i^e\right)$ has rank greater than or equal $N_{\text{T}} + 1$ for $e \geq 2$. To obtain this result, we just have to note that $\left(\mathbf{D}_j^e - \mathbf{D}_i^e\right)$ has always rank eN_{T}, due to the use of a single-user full rank space-time constellation for each user.

To fulfill the rank criterion for any matrix $[\mathbf{F}_l, \mathbf{F}_k]$, $l \neq k$, we need of course $D \geq 2D_{\text{SU}} - N_{\text{T}} + 1$. The spreading matrices requiring only the minimum number of dimensions $2D_{\text{SU}} - N_{\text{T}} + 1$ are of particular interest, because by the minimum increase in dimensionality every user can achieve asymptotically single-user like performance. For example, when the single-user constellation requires only the minimum dimensions $D_{\text{SU}} = N_{\text{T}}$, only one more dimension is required. In this paper, we suggest to generate the spreading matrices $\{\mathbf{F}_k\}_{k=1}^K$ by minimizing

$$c = \max_{1 \leq k \leq K} \sum_{\substack{l=1 \\ l \neq k}}^{K} \left| \mathbf{F}_k \mathbf{F}_k^\dagger + \mathbf{F}_l \mathbf{F}_l^\dagger \right|^{-N_{\text{R}}}, \qquad (6)$$

under the constraint that each matrix $\mathbf{F}_k$ has orthonormal columns. Note that the criterion (6) is easily evaluated, because it is independent of the specific single-user space-time constellation. It also ensures in a “fair” way that all pairs $[\mathbf{F}_l, \mathbf{F}_k]$, $l \neq k$, fulfill the rank criterion, which is necessary to achieve single-user like performance.

Since we want to constrain our signals to have orthonormal columns, the optimization can be performed unconstrained in the Grassmann manifold $G(D_{\text{SU}}, D)$, the space of all D_{SU}-dimensional subspaces of $\mathbb{C}^D$. A parameterization of $G(D_{\text{SU}}, D)$ which employs $2D_{\text{SU}}D - D_{\text{SU}}^2$ real parameters is explicitly detailed in [8] following, for example, [9]. However, with D^2 real variables we can over-parameterize $G(D_{\text{SU}}, D)$ and obtain a somewhat simpler characterization (cf. [8], [9]). The latter parameterization is built by expressing any rectangular matrix $\mathbf{U} \in \mathbb{C}^{D \times D_{\text{SU}}}$ as the product of a square $D \times D$ unitary matrix and a fixed rectangular $D \times D_{\text{SU}}$ matrix $\mathbf{W}$ (typically $\mathbf{W}$ is taken to be the first D_{SU} columns of $\mathbf{I}_D$). Any of the $D \times D$ unitary matrices can in turn be expressed as the product of a real diagonal matrix $\mathbf{\Phi}$ and “simple” unitary matrices $\mathbf{V}^{pq}(\phi_{pq}, \theta_{pq})$ (complex Givens rotation matrices, each parameterized by two angular parameters and two indices). For details, see [8], [9]. Each user's spreading matrix $\mathbf{F}_k$ is parameterized in this way and thus the numerical optimization of (6) can be performed without constraints.

For our examples, we choose the constituent space-time symbols $\mathbf{B}_{i_k}$ according to Alamouti's scheme [4] for $N_{\text{T}} = 2$ transmit antennas. For Figure 1, we designed $D = 3$-dimensional spreading matrices for up to three equal-energy users. From the figure we see that the simulated BER for a single-user employing 4-PSK symbols in the Alamouti scheme is close to its upper bound, to which the upper bounds on the multiuser systems ($K = 2$, $K = 3$) converge. The $K = 3$ user system in which each user employs 4-PSK symbols in the Alamouti scheme has a total spectral efficiency of 4 bps/Hz (three users each transmitting 2 bits from each of 2 antennas in 3 dimensions). If one would attempt to achieve such a spectral efficiency with orthogonal users, each user would have to employ 16-QAM modulation. From the figure, we see that at a BER of 10^{-3} there is roughly a 3 dB gap between the spread-matrix design and the orthogonal system. For larger systems with more users and/or antennas, this gap is expected to grow.

V. Conclusions

We present a simplified criterion for the design of low-dimensional spreading matrices for multiuser space-time communications and give example designs. The matrices are constrained to have orthonormal columns and are obtained by a numerical optimization that profits from a parameterization of the Grassmann manifold. As promised by the analysis of [1], the jointly optimum detector can detect each user with single-user like performance for sufficiently high SNR.

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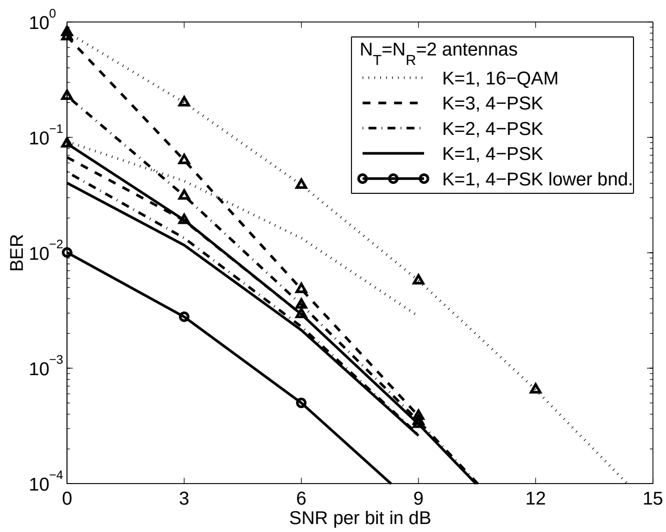

Fig. 1. Each user employs the Alamouti scheme and asymptotically achieves single-user like performance. The $K = 3$, 4-PSK system has the same spectral efficiency as the $K = 1$, 16-QAM system, but the latter has a 3 dB worse energy efficiency at BER 10^{-3}. The lines with triangles are upper bounds and plain lines are simulations.