Optimum Noncoherent Multiuser Decision Feedback Detection

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***Abstract*—A theory of noncoherent decision feedback multiuser detection for nonorthogonal binary modulation is developed that parallels that of coherent decision feedback multiuser detection for single-pulse modulation. In particular, an optimum noncoherent decision feedback detector is obtained that maximizes *symmetric energy* over a newly defined class of decision feedback detectors. Unlike the usual per-user performance metrics such as asymptotic efficiency or near–far resistance, the symmetric energy measure captures, with a single number, the asymptotic (high signal-to-noise ratio (SNR)) bit-error performance of all users at once. Several properties of the optimum decision feedback detector are established, one of which is that it outperforms the decision feedback generalized-likelihood ratio (GLR) detector in symmetric energy. It is also shown that, regardless of the order in which users are detected, the optimum noncoherent decision feedback detector outperforms its non-decision feedback counterpart in symmetric energy. Furthermore, two simple rules are obtained for determining the order in which users must be detected to guarantee that the optimum decision feedback detector outperforms its non-decision feedback counterpart (which in turn, is superior to the decorrelative GLR detector presented earlier) in terms of asymptotic effective energy for every user. In fact, one of the two (greedy) ordering rules also maximizes symmetric energy among all possible orderings. Such ordering rules are not available for the noncoherent decision feedback GLR detector in earlier work of the authors. Feasible sets of received energies are characterized in which it is possible, with power control, to achieve quality-of-service objectives for each user. None of the results in this paper make simplifying assumptions about the effects of error propagation. The term "noncoherent" in this work is used to denote that the receiver has no knowledge of the carrier phases and received signal energies of any of the users.**

***Index Terms*—Decision feedback, generalized-likelihood ratio (GLR), multipulse modulation, multiuser detection, noncoherent detection.**

I. Introduction

THE need for noncoherent detection arises in applications such as mobile communications where there may be fading, oscillator phase instability, uncertain and rapid changes in propagation delay of the transmitted signal, etc. A common modulation technique that lends itself to noncoherent detection is orthogonal multipulse modulation (OMM), a simple example of which is frequency shift keying (FSK). An M-ary symbol is transmitted by sending one of M orthogonal, equi-energy waveforms. OMM is appropriate for power-efficient communication where the energy per bit required to achieve a given error rate can be lowered with increasing values of the size of the symbol alphabet [5], [6].

The problem of noncoherent single- and multiuser detection for nonorthogonal multipulse modulation (NMM) over the Gaussian channel was first studied in [3]. One reason for considering nonorthogonal modulation is that it allows the system designer to exercise greater control over bandwidth (with OMM requiring maximum bandwidth). This paper introduces the notion of decorrelation for the NMM multiuser channel and obtains and analyzes three decorrelative detection rules.

The idea of noncoherent decorrelating decision feedback detection was introduced in [7], where the decision feedback versions of the decorrelating optimal (minimum error rate), asymptotically optimal, and the GLR detectors of [3] were developed and analyzed. Users are detected in some order and since signal amplitudes are unknown at the receiver, decision feedback for a particular user is achieved by projecting the received signal onto the orthogonal complement of the subspace spanned by the signals that correspond to the decisions of the users already detected and all possible signals of users yet to be detected.[1] It was shown that decision feedback could be achieved with no increase in computational complexity (but with increased storage requirement), and could result in significant gains in performance over the noncoherent NMM decorrelating detectors. Among those three noncoherent decision feedback (NDF) detectors, only the generalized-likelihoos ratio (GLR)-based NDF detector of [2] (referred to henceforth as the G-NDF detector for brevity) has the feature of requiring the knowledge of neither the phases nor the received energies of any of the users' signals. However, that detector does not possess the nice analytical properties of the coherent decision feedback detectors for linear modulation as obtained in [1]. This motivates the search for a better theory of NDF detectors.

We define in this paper a new K-map class of NDF detectors for a K-user synchronous channel with nonorthogonal binary

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[1]Note that in contrast, the idea of decision feedback in the sequential decision projection (SDP) detector of [12] does not involve decorrelation of the signals of the as yet undetected users. The undetected user signals are essentially treated as additive noise. Consequently, the decision variables are contaminated by interference and performance guarantees such as those derived in this work for the optimal noncoherent decision feedback (O-NDF) (e.g., nonzero symmetric energy) cannot be claimed for the SDP detector.

modulation and solve the problem of finding the optimal detector from within that class that maximizes symmetric energy without making simplifying assumptions about error propagation. This optimum NDF detector (referred to in the rest of the paper as the O-NDF detector), like the G-NDF detector, does not require the energies and phases of any of the users' signals. Moreover, since the G-NDF detector is a member of the K-map class of NDF detectors, the O-NDF detector outperforms it in symmetric energy. As a by-product of our work, we also obtain the non-decision feedback version of the optimum NDF detector which outperforms the decorrelating GLR detector of [3], [4] for binary modulation.

The performance analysis shows that unlike the G-NDF detector, the O-NDF detector outperforms its non-decision feedback counterpart in symmetric energy regardless of the order in which users are detected. It is also shown that under certain conditions, the effects of error propagation in the decision feedback detector can be neglected, i. e., the decision feedback detector achieves *genie-aided* performance where all past users' decisions are assumed perfect. With the knowledge of user energy estimates, it is possible to order users such that the user-wise performance of the O-NDF detector can be made superior to that of its non-decision feedback counterpart in terms of asymptotic effective energy. This result does not hold for the G-NDF detector. In other words, we have a theory of noncoherent decision feedback detection for binary modulation that parallels that of coherent decision feedback detection for single-pulse modulation as obtained in [1]. The extension to M-ary modulation is a subject for future research.

While the framework of [1] guides the problem formulations in this paper and [2] develops and analyzes a particular noncoherent decision feedback strategy (the G-NDF), this paper seeks to develop a general theory of noncoherent decision feedback that is as rich as its counterpart for coherent decision feedback [1]. There are a few general results regarding error probability analysis with error propagation that do not depend on whether the signaling is single pulse or binary or whether the detection is coherent or noncoherent, and these we borrow from [1]. However, the primary focus of this paper is to develop the analytical results that are necessary to place the theory of noncoherent decision feedback on par with that of coherent decision feedback. Such results cannot be inferred from either [1] or [2].

Several other papers have been published in noncoherent multiuser detection for Gaussian and fading channels (cf. [8]–[15]). These and other papers are cited in [2] with a summary of the results therein. The text [18] contains a brief treatment of the subject based on some of those references. More recent papers on the subject also include [16] and [17].

The rest of this paper is organized as follows. In Section II, we introduce the system model. In Section III, we define a class of noncoherent detectors with and without decision feedback. Section IV contains a definition of symmetric energy, and the formulation of, and the solution to, the optimum symmetric energy problems for detectors with and without feedback. Per-user performance bounds are obtained in terms of asymptotic effective energy for the optimum NDF in Section V along with results on quality-of-service (QoS)-based power control. Section VI specifies ordering rules and their properties. Section VII extends the results of this paper to the case where the two signals of each user have distinct energies. Section IX concludes the paper.

II. System Model

Consider a synchronous multiuser additive Gaussian noise channel with each user employing nonorthogonal binary modulation where one of two possibly nonorthogonal signals is transmitted to send one bit of information. The superposition of the K signals arrive in symbol synchronism at the receiver so that the lowpass received signal can be modeled as

$$r(t)=\sum_{k=1}^{K}\sqrt{E_k}e^{j\theta_{ki_k}}s_{ki_k}(t)+z(t), \qquad t\in[0,T] \tag{1}$$

with $z(t)$ having a noise power spectral density (one-sided) of σ^2, and $s_{ki_k}(t), i_k\in\{1,2\}$ being the two linearly independent, complex, equiprobable, unit-energy, and time-limited signature signals of user k. E_k denotes the kth user's received energy per bit, assumed equal for both signals of the same user, and θ_{ki_k} denotes the phase of the kth user's i_kth signal. It is assumed that the phases remain constant over one signal interval, and are independent and uniformly distributed random variables on $[0,2\pi]$.

The received signal $r(t)$ is first passed through a bank of $2K$ matched filters, matched to each of the signature signals $s_{ki_k}(t)$. The output of the filter bank is a $2K$-dimensional sufficient statistic

$$\boldsymbol{y}=\int_0^T r(t)\boldsymbol{s}^*(t)dt \tag{2}$$

where $*$ denotes complex conjugation

$$\boldsymbol{s}(t)=\left[\boldsymbol{s}_1^T(t),\boldsymbol{s}_2^T(t),\ldots,\boldsymbol{s}_K^T(t)\right]^T$$

with

$$\boldsymbol{s}_k^T(t)=[s_{k1}(t)\ s_{k2}(t)], \qquad k\in\{1,\ldots K\}$$

(the integral of a vector of signals in (2) is simply the vector of integrals of the elements of that vector). The elements in $\boldsymbol{y}$ are denoted and arranged according to

$$\boldsymbol{y}=[y_1(1)y_1(2)\cdots y_K(1)y_K(2)]^T.$$

The idea in [3] was to view the binary modulator (which is referred to in [12] and [18] as a "nonlinear" modulator) as a linear modulator in a $2K$-dimensional signal space. This is achieved by representing user k's information by the vector

$$\boldsymbol{b}_k^T\triangleq[b_{k1},b_{k2}]\in\{(1\ 0),(0\ 1)\} \tag{3}$$

so that (1) can be expressed as

$$r(t)=\sum_{k=1}^{K}\sum_{m=1}^{2}b_{km}\sqrt{E_k}e^{j\theta_{km}}s_{km}(t)+z(t), \qquad t\in[0,T]. \tag{4}$$

Substituting (4) into (2) we obtain the model

$$\boldsymbol{y}=\boldsymbol{R}A\boldsymbol{b}+\boldsymbol{n} \tag{5}$$

with

$$\boldsymbol{R}=\int_0^T\boldsymbol{s}^*(t)\boldsymbol{s}^T(t)dt\in\boldsymbol{C}^{2K\times 2K} \tag{6}$$

$$\boldsymbol{R}_{ij} = \int_0^T \boldsymbol{s}_i^*(t)\boldsymbol{s}_j^T(t)dt \in \boldsymbol{C}^{2\times 2},\ i,j \in \{1,\dots,K\} \quad (7)$$

$$\boldsymbol{b}^T = \left[\boldsymbol{b}_1^T,\dots,\boldsymbol{b}_k^T,\dots,\boldsymbol{b}_K^T\right] \quad (8)$$

$$\boldsymbol{A} = diag(\boldsymbol{A}_1,\boldsymbol{A}_2,\dots,\boldsymbol{A}_K) \in \boldsymbol{C}^{2K\times 2K} \quad (9)$$

$$\boldsymbol{A}_k = diag(\sqrt{E_k}e^{j\theta_{k1}},\ \sqrt{E_k}e^{j\theta_{k2}}). \quad (10)$$

Moreover, $\boldsymbol{n}$ is a zero-mean complex Gaussian random vector with covariance matrix $\sigma^2\boldsymbol{R}$, which we denote as $\boldsymbol{n}$ being $N(0,\sigma^2\boldsymbol{R})$. For simplicity, all $s_{km}(t)$ are assumed to be linearly independent so that $\boldsymbol{R}$ is positive definite. The results of this paper can be extended to the more general case of linearly dependent signaling. In this regard, the reader is referred to [2] for a derivation and analysis of the G-NDF detector when $\boldsymbol{R}$ is rank deficient.

A conventional detector has come to be defined in the multiuser detection literature [18] as the optimum single-user detector when used in a multiuser channel. Therefore, the noncoherent conventional detector for binary modulation would make the bit decisions according to (cf. [6])

$$\hat{i}_k = \begin{cases} 1, & \text{if } |y_k(1)|^2 \geq |y_k(2)|^2 \\ 2, & \text{else.} \end{cases} \quad (11)$$

Unfortunately, this detector is near–far limited [3] in multiuser channels since it does not account for multiaccess interference.

III. Noncoherent Detection With and Without Decision Feedback

We define a K-parameter class of decorrelating detectors that includes the GLR detector of [3] as a particular case. By optimizing performance over the K-parameter class, we will obtain a detector that is superior in performance.

Let $\boldsymbol{z}_k$ be the $(2k-1)$th and $(2k)$th elements of the decorrelator outputs $\boldsymbol{z} = \boldsymbol{R}^{-1}\boldsymbol{y}$, such that

$$\boldsymbol{z}_k = \boldsymbol{A}_k\boldsymbol{b}_k + \tau_k \quad (12)$$

where τ_k is the corresponding filtered additive noise vector. The decision for user k is made solely based on $\boldsymbol{z}_k \triangleq [z_k(1)z_k(2)]^T$ as follows:

$$\hat{i}_k = \begin{cases} 1, & \text{if } |z_k(1)|^2 \geq \alpha_k|z_k(2)|^2 \\ 2, & \text{if } |z_k(1)|^2 < \alpha_k|z_k(2)|^2 \end{cases} \quad (13)$$

so that the decisions of the K users are made in parallel with the parameters $\{\alpha_k\}_{k=1}^K$ being real valued and positive.

The reader can easily verify that the noncoherent parallel decorrelating GLR detector (henceforth referred to as the G-NPD detector) of [3] for binary modulation is a member of the K-parameter class of detectors defined above. In particular, the G-NPD detector corresponds to $\alpha_k = \frac{\boldsymbol{P}_k(2,2)}{\boldsymbol{P}_k(1,1)}$, where $\boldsymbol{P}_k = \boldsymbol{Q}_k^{-1}$ and $\boldsymbol{Q}_k$ is the kth 2×2 block along the diagonal of $\boldsymbol{R}^{-1}$.

Next, we define decision feedback detectors. The idea of noncoherent decorrelating decision feedback was introduced in [7] based on which we define a new K-map class of NDF detectors. As in coherent decision feedback detection, users are detected sequentially and we assume without loss of generality that the users are indexed in the order in which they are detected so that the kth user's decision is made after the decisions of the users $1,\dots,k-1$ have been made. We let $\hat{i}_m \in \{1,2\}$ denote the decision as to which of the two signals was transmitted by user m, and $\hat{\boldsymbol{i}}_1^m = (\hat{i}_1,\dots,\hat{i}_m)^T$ denotes the symbol decisions of users indexed 1 to m.

Consider for user k an *expurgated* version of the matched filter outputs

$$\boldsymbol{y}_{ex}^{(k)} = \left(\boldsymbol{y}_1(\hat{i}_1),\dots,\boldsymbol{y}_{k-1}(\hat{i}_{k-1}),\boldsymbol{y}_k^T,\dots,\boldsymbol{y}_K^T\right)^T.$$

Consider also the matrix $\boldsymbol{R}_{ex}^{(k)}$ which is a $2K-k+1$ principal submatrix of $\boldsymbol{R}$ obtained by retaining its rows and columns corresponding to the same indices that are used to retain the elements of $\boldsymbol{y}$ to form the subvector $\boldsymbol{y}_{ex}^{(k)}$. Apply a decorrelative transformation to the expurgated statistic $\boldsymbol{y}_{ex}^{(k)}$, and obtain $\boldsymbol{z}_{ex}^{(k)} = (\boldsymbol{R}_{ex}^{(k)})^{-1}\boldsymbol{y}_{ex}^{(k)}$. The 2-length subvector comprising the kth and $(k+1)$th elements of $\boldsymbol{z}_{ex}^{(k)}$ form the decision statistic $\boldsymbol{z}_{df}^{(k)} = (\boldsymbol{z}_{df}^{(k)}(1),\boldsymbol{z}_{df}^{(k)}(2))^T$, for user k in the decision feedback strategy. Under the assumption of perfect "previous" decisions (i.e., perfect decisions for users $1,\dots,k-1$), $\boldsymbol{z}_{df}^{(k)}$ admits the simple model

$$\boldsymbol{z}_{df}^{(k)} = \boldsymbol{A}_k\boldsymbol{b}_k + \gamma^{(k)} \quad (14)$$

where $\gamma^{(k)}$ is filtered additive noise. Thus, $\boldsymbol{z}_{df}^{(k)}$ is free of multiple-access interference from both previous and future users (i.e., users $k+1,\dots,K$). The noise vector $\gamma^{(k)}$ has a covariance matrix $\sigma^2\boldsymbol{D}_k$ where $\boldsymbol{D}_k$ is the 2×2 principal submatrix of $(\boldsymbol{R}_{ex}^{(k)})^{-1}$ obtained by retaining rows k to $k+1$ and columns k to $k+1$ of $\left(\boldsymbol{R}_{ex}^{(k)}\right)^{-1}$. Note that $\boldsymbol{D}_k$ depends on the past decisions $\hat{\boldsymbol{i}}_1^{k-1}$. It now remains to specify the decision rule for any user k based on $\boldsymbol{z}_{df}^{(k)}$.

Consider a K-map NDF detector that is parametrized by a set of K functions or mappings

$$\beta_k : \{1,2\}^{k-1} \longrightarrow \mathcal{R}^+, \qquad k = 1,2,\dots,K \quad (15)$$

where the domain of β_k is a set of $(k-1)$-length vectors with binary-valued elements ($\in \{1,2\}$) and the range is the positive real line $\mathcal{R}^+$. These mappings decide the decision rules for the K users defined recursively starting from user 1 to user K according to

$$\hat{i}_k = \begin{cases} 1, & \text{if } |z_{df}^{(k)}(1)|^2 \geq \beta_k(\hat{\boldsymbol{i}}_1^{k-1})|z_{df}^{(k)}(2)|^2 \\ 2, & \text{otherwise.} \end{cases} \quad (16)$$

Note that decision feedback occurs at two distinct levels. The past decisions $\hat{\boldsymbol{i}}_1^{k-1}$ decide how the decision statistics $\boldsymbol{z}_{df}^{(k)}$ are obtained and they also influence the decision rule (16) through the map β_k. We will refer to an arbitrary NDF detector as $\phi^{\boldsymbol{\beta}}$, with the K maps $\{\beta_k\}$ denoted collectively as $\boldsymbol{\beta}$. Note that the characterization of the map β_k for a particular detector requires the specification of its range $S_k = \left\{\beta_k(\boldsymbol{j})\ ;\ \boldsymbol{j} \in \{1,2\}^{k-1}\right\}$ with cardinality 2^{k-1}. Hence, the collection of maps β specified by the sets S_k for $k = 1,2,\dots,K$, requires $2^K - 1$ parameters.

The GLR-based NDF detector of [2], denoted here as $\phi^{\text{G-NDF}}$, can be shown to be a K-map NDF detector. The implementation of ϕ^β is, therefore, identical to that of $\phi^{\text{G-NDF}}$, described in detail in [2]. In particular, when $\phi^{\text{G-NDF}}$ is specialized to binary modulation, the decision rule for user

k can be manipulated into the form in (16) with the K maps determined as

$$\beta_k(\hat{\boldsymbol{i}}_1^{k-1}) = \frac{H_k(2,2)}{H_k(1,1)} \tag{17}$$

with $H_k(i,i)$ being the ith diagonal element of the 2×2 matrix $\boldsymbol{H}_k = \boldsymbol{D}_k^{-1}$ with $\boldsymbol{D}_k$ as defined earlier in this section. The dependence of $\boldsymbol{D}_k$ (and, hence, $\boldsymbol{H}_k$) on the past decisions $\hat{\boldsymbol{i}}_1^{k-1}$ is suppressed for notational simplicity. Note that the range of β_k consists of the quantities $H_k(2,2)/H_k(1,1)$ which in turn are independent of the signal energies. Hence, $\phi^{\text{G-NDF}}$ can be implemented without the knowledge of both carrier phases as well as signal energies.

IV. Optimum Symmetric Energy

In this section, we define an asymptotic (high signal-to-noise ratio (SNR)) performance measure called *symmetric energy* that captures with a single parameter the performance of a multiuser detector by taking into account the bit-error rate (BER) of all users at once.

Definition: If the bit-error probability in the multiuser channel for user k of any noncoherent detector ϕ (denoted as $P_k(\sigma,\phi)$) is expressed in the form of the minimum error probability for noncoherent detection for binary orthogonal signaling over a single-user channel, so that we let

$$P_k(\sigma,\phi) = 2^{-1}\exp(-e_k(\sigma,\phi)/2\sigma^2)$$

then $e_k(\sigma,\phi)$ is the effective energy whose limit as $\sigma \to 0$ is defined as asymptotic effective energy and denoted as $\mathcal{E}_k(\phi)$. The *symmetric energy* is defined as the worst case asymptotic effective energy so that

$$\mathcal{E}(\phi) = \min_{1\le k\le K} \mathcal{E}_k(\phi)\,. \tag{18}$$

Consequently, a nonzero value of symmetric energy for a given detector implies that the BER decays exponentially as $\sigma \to 0$ for *all* K users with the symmetric energy being the slowest rate of exponential decay among the BERs of the K users. Therefore, symmetric energy characterizes the performance of a multiuser detector with a single parameter by taking the performance of all users into account at once, unlike the per-user measures such as asymptotic efficiency or near–far resistance. In fact, it is the asymptotic (high SNR) indicator of the joint error rate that any one of the users is detected erroneously [1].

Next, from the K-parameter class of decorrelating detectors (without decision feedback), we seek the parameter set α that maximizes the symmetric energy or

$$\alpha^* \in \arg\max_{\alpha} \mathcal{E}(\phi^{\boldsymbol{\alpha}}). \tag{19}$$

Since the asymptotic effective energy of user k depends only on α_k, the above optimization decouples into K independent problems of optimizing asymptotic effective energy per user, i.e.,

$$\alpha_k{}^* \in \arg\max_{\alpha_k} \mathcal{E}_k(\phi^{\boldsymbol{\alpha}}). \tag{20}$$

We refer to the above optimum noncoherent decorrelating detector as $\phi^{\text{O-NPD}}$.

The solution to the above problem (without feedback) can be obtained as a by-product of the solution to the more general problem of maximizing symmetric energy over the K-map class of NDF detectors

$$\beta^* \in \arg\max_{\beta} \mathcal{E}(\phi^{\beta})\,. \tag{21}$$

To solve this problem, we must contend with the error propagation issue. For the "present" user, error propagation effects arise from making incorrect decisions for "past" users. The following lemma from [1] is a key enabling result.

Lemma 1: For an arbitrary K-map noncoherent decision feedback detector, the symmetric energy (for any k) is equal to that of the same detector which is assisted by a genie that ensures that past user decisions are perfect (referred to as the genie-aided detector).

Proof: The proof of Lemma 1 (stated completely in [1] in the coherent detection context but applicable here as well) hinges on two facts: a) the symmetric energy is equal to the asymptotic effective energy corresponding to the union of error events $\cup_{i=1}^{K}\delta_i$ where δ_i is the error event that denotes the decision for user i being erroneous and b) with $\delta_{i,g}$ denoting the event that the ith user's decision is erroneous for the corresponding genie-aided decision feedback detector, that the union of error events $\cup_{i=1}^{K}\delta_i$ and $\cup_{k=1}^{K}\delta_{i,g}$ are equal. □

Lemma 1 allows a key simplification of the problem in (21). The optimization of symmetric energy described in (21) can be equivalently performed over the detectors $\phi_g^{\boldsymbol{\beta}}$ which are the genie-aided versions of $\phi^{\boldsymbol{\beta}}$.

Lemma 2: The optimization of symmetric energy in (21) over the collection of maps β can be decomposed into K decoupled optimizations for each of the K users. For the kth user, the optimization is over the map β_k and of the asymptotic effective energy of the kth user over the class of genie-aided versions of the K-map class of NDF detectors.

Proof: Using Lemma 1 and the definition of symmetric energy in (18), the problem in (21) can be written as

$$\beta^* \in \arg\max_{\beta} \min_{1\le k\le K} \mathcal{E}_k(\phi_g^{\beta}). \tag{22}$$

The asymptotic effective energy of the genie-aided detector for user k is determined by analyzing its decision rule in (16) under the assumption of perfect feedback. Under this assumption, we noted earlier that the multiuser interference-free model for $\boldsymbol{z}_{df}^{(k)}$ in (14) is exact. Consequently, the rule in (16) is unaffected by maps β_m for all $m \neq k$, which implies that its bit-error probability and, hence, its asymptotic effective energy $\mathcal{E}_k(\phi_g^{\beta})$ depends only on β_k. This allows us to obtain β^* or equivalently the K optimum maps $\{\beta_k^*\}_{k=1}^{K}$ each as a solution to the decoupled optimization

$$\beta_k^* \in \arg\max_{\beta_k} \mathcal{E}_k(\phi_g^{\beta}). \tag{23}$$

□

It can also be shown using the arguments in [1] that the collection of optimal maps as defined in (23) will also optimize the group symmetric energy $\Psi_k(\phi) \triangleq \min_{1\leq m\leq k}\mathcal{E}_m(\phi)$ for each k. This stronger optimality assures us that in maximizing the worst user's performance, the optimum detector will not give a poor performance for some users (relative to what could be achieved).

The goal is to solve the problem in (23). To this end, we obtain the asymptotic effective energy for user k for $\phi_g^{\boldsymbol{\beta}}$. Assuming equiprobable signaling for each user $P_k(\sigma,\phi_g^{\boldsymbol{\beta}})$, the probability of error for user k for $\phi_g^{\boldsymbol{\beta}}$ can be written as

$$P_k(\sigma,\phi_g^{\boldsymbol{\beta}}) = \frac{1}{2^{k-1}}\sum_{\boldsymbol{i}_1^{k-1}} P_k(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1})$$

where $P_k(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1})$ is the genie-aided conditional probability of error for user k conditioned on past user transmitted signals $\boldsymbol{i}_1^{k-1}$. Letting $\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}})$ and $\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1})$ denote the genie-aided asymptotic effective energies corresponding to $P_k(\sigma,\phi_g^{\boldsymbol{\beta}})$ and $P_k(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1})$, respectively, we have

$$\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}}) = \min_{\boldsymbol{i}_1^{k-1}}\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1}) \tag{24}$$

since the term in the sum of conditional error probabilities that has the slowest rate of exponential decay dominates the BER for high SNR and decides the value of $\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}})$.

Lemma 3: The optimum β defined by (21) can be obtained as solutions for the following optimizations for each $k=1,2,\ldots,K$ and $\boldsymbol{j}\in\{1,2\}^{k-1}$

$$\beta_k^*(\boldsymbol{j}) \in \arg\max_{\beta_k(\boldsymbol{j})}\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j}). \tag{25}$$

Proof: Substituting (24) into (23), we have that the optimum map β_k^* is a solution of

$$\beta_k^* \in \arg\max_{\beta_k}\min_{\boldsymbol{j}}\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j}). \tag{26}$$

Thus, one solution for the optimum map β_k^* is that which assigns to each $\boldsymbol{j}\in\{1,2\}^{k-1}$ a value in $\mathcal{R}^+$ such that each of the conditional asymptotic effective energies $\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$ is individually maximized. □

Consider next, a finer characterization of the conditional asymptotic effective energy $\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$. Averaging over the two types of errors, we have

$$P_k(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{j}) = \frac{1}{2}\sum_{\substack{i_k,j\\ i_k\neq j}} P_k^{i_k,j}(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$$

where $P_k^{i_k,j}(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$ is the genie-aided conditional probability of error for user k, conditioned further on signal i_k of user k being detected erroneously as signal j. Let $\mathcal{E}_k^{i_k,j}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$ denote the conditional asymptotic effective energy that corresponds to $P_k^{i_k,j}(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$. Consequently, we have

$$\mathcal{E}_k(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j}) = \min\left\{\mathcal{E}_k^{1,2}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j}),\mathcal{E}_k^{2,1}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})\right\}. \tag{27}$$

Therefore, the result of Lemma 3 can be restated as

$$\beta_k^*(\boldsymbol{j}) \in \arg\max_{\beta_k(\boldsymbol{j})}\min\left\{\mathcal{E}_k^{1,2}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j}),\mathcal{E}_k^{2,1}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})\right\}. \tag{28}$$

It now remains to obtain an explicit formula for $\mathcal{E}_k^{i_k,j}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$. Using the fact that the model for $\boldsymbol{z}_{df}^{(k)}$ in (14) is exact for the genie-aided detector ϕ_g^{β}, and noting hence that the decision rule in (16) conditioned on the realization of the perfect past user decisions $\boldsymbol{i}_1^{k-1}=\boldsymbol{j}$ depends only on $\beta_k(\boldsymbol{j})$, we have that $\mathcal{E}_k^{i_k,j}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$ is independent of all other values in the range of β_k. Moreover, the conditional probability of error of the decision rule in (16) can be obtained as a special case of the general result on error probability for multichannel binary signaling in [6, Appendix B] to obtain

$$P_k^{i_k,j}(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{j}) = Q\left(\frac{F^{i_kj}_{\beta_k(\boldsymbol{j})1}}{\sigma},\frac{F^{i_kj}_{\beta_k(\boldsymbol{j})2}}{\sigma}\right) - \frac{1}{2}I_0\left(\frac{F^{i_kj}_{\beta_k(\boldsymbol{j})1}F^{i_kj}_{\beta_k(\boldsymbol{j})2}}{\sigma^2}\right)\exp\left(-\frac{(F^{i_kj}_{\beta_k(\boldsymbol{j})1})^2+(F^{i_kj}_{\beta_k(\boldsymbol{j})2})^2}{2\sigma^2}\right) \tag{29}$$

where

$$F^{i_kj}_{\beta_k(\boldsymbol{j})1} = \sqrt{\frac{2\left(v_1^{i_kj}\right)^2 v_2^{i_kj}\left(\lambda_1^{i_kj}v_2^{i_kj}-\lambda_2^{i_kj}\right)}{(v_1^{i_kj}+v_2^{i_kj})^2}}$$

and

$$F^{i_kj}_{\beta_k(\boldsymbol{j})2} = \sqrt{\frac{2v_1^{i_kj}\left(v_2^{i_kj}\right)^2\left(\lambda_1^{i_kj}v_1^{i_kj}+\lambda_2^{i_kj}\right)}{(v_1^{i_kj}+v_2^{i_kj})^2}} \tag{30}$$

with $\lambda_1^{i_kj}=\beta_k(\boldsymbol{j})E_kD_k(j,j)$, $\lambda_2^{i_kj}=E_k$. Furthermore

$$v_1^{i_kj} = \sqrt{(w^{i_kj})^2+\frac{1}{|\boldsymbol{D}_k|\beta_k(\boldsymbol{j})}} - w^{i_kj}$$

and

$$v_2^{i_kj} = \sqrt{(w^{i_kj})^2+\frac{1}{|\boldsymbol{D}_k|\beta_k(\boldsymbol{j})}} + w^{i_kj} \tag{31}$$

where

$$w^{i_kj} = \frac{D_k(i_k,i_k)-\beta_kD_k(j,j)}{2\beta_k(\boldsymbol{j})|\boldsymbol{D}_k|}$$

and

$$|\boldsymbol{D}_k| = D_k(i_k,i_k)D_k(j,j)-|D_k(i_k,j)|^2. \tag{32}$$

In (29), the function $Q(.,.)$ is the Marcum Q-function with the integral representation

$$Q(a,b) = \int_b^\infty x\exp\left(-\frac{x^2+a^2}{2}\right)I_0(ax)dx.$$

Lemma 4: The conditional asymptotic efficiency $\mathcal{E}_k^{i_k,j}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$ admits the formula

$$\mathcal{E}_k^{i_k,j}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j}) = \max{}^2\left\{0,F^{i_kj}_{\beta_k(\boldsymbol{j})2}-F^{i_kj}_{\beta_k(\boldsymbol{j})1}\right\}. \tag{33}$$

Proof: The asymptotic analysis of probabilities which have the general form of the error probability in (29) was obtained in [8, the Appendix] through asymptotically tight upper and lower bounds. The result of this lemma results from a direct application of that analysis. □

We want to solve the optimization problem in (28) with $\mathcal{E}_k^{1,2}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$ and $\mathcal{E}_k^{2,1}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})$ given in (33).

Lemma 5: A necessary and sufficient condition for the optimum solution $\beta_k^*(\boldsymbol{j})$ of (28) is that the two conditional asymptotic effective energies be equal

$$\mathcal{E}_k^{1,2}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})\Big|_{\beta_k(\boldsymbol{j})=\beta_k^*(\boldsymbol{j})} = \mathcal{E}_k^{2,1}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{j})\Big|_{\beta_k(\boldsymbol{j})=\beta_k^*(\boldsymbol{j})}. \quad (34)$$

Proof: Note from the decision rule in (16) that while $P_k^{1,2}(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1})$ increases monotonically with increasing β_k, $P_k^{2,1}(\sigma,\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1})$ decreases monotonically. This relation between β_k and the two types of error probabilities holds for any σ and, in particular, it holds in the limit as $\sigma \to 0$. By the definition of asymptotic effective energy, therefore, $\mathcal{E}_k^{1,2}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1})$ decreases monotonically and $\mathcal{E}_k^{2,1}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1})$ increases monotonically with increasing β_k. Since both conditional probablities take on values in the interval 0 to 1 and are monotonic, we are assured of a single point of intersection for the two conditional asymptotic effective energies. It follows, therefore, that a necessary and sufficient condition for β_k to maximize

$$\min\left\{\mathcal{E}_k^{1,2}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1}),\mathcal{E}_k^{2,1}(\phi_g^{\boldsymbol{\beta}},\boldsymbol{i}_1^{k-1})\right\}$$

can be stated as in (34). □

Proposition 1: The optimum NDF decision rule is described for each $k=1,2,\ldots,K$ as

$$\hat{i}_k = \begin{cases} 1, & \text{if } |z_{df}^{(k)}(1)|^2 \geq |z_{df}^{(k)}(2)|^2 \\ 2, & \text{otherwise.} \end{cases} \quad (35)$$

Proof: Using Lemma 4, the result of Lemma 5 can now be explicitly stated as

$$\max{}^2\left\{0, F_{\beta_k(\boldsymbol{j})2}^{12} - F_{\beta_k(\boldsymbol{j})1}^{12}\right\}\Big|_{\beta_k(\boldsymbol{j})=\beta_k^*(\boldsymbol{j})} = \max{}^2\left\{0, F_{\beta_k(\boldsymbol{j})2}^{21} - F_{\beta_k(\boldsymbol{j})1}^{21}\right\}\Big|_{\beta_k(\boldsymbol{j})=\beta_k^*(\boldsymbol{j})}. \quad (36)$$

The present proposition is equivalent to the statement that the solution of the nonlinear equation in (36) is $\beta_k^*(\boldsymbol{j})=1$ for any k and $\boldsymbol{j}$. Consider the equations in (30) in order to examine the terms in the nonlinear equation in (36). At $\beta_k(\boldsymbol{j})=1$, let $F_{\beta_k(\boldsymbol{j})p}^{i_kj} \triangleq F_p^{i_kj}$, $p \in \{1,2\}$. It must be proved that the terms $F_2^{12}-F_1^{12}$ and $F_2^{21}-F_1^{21}$ are both positive and equal to each other. We note that for $\beta_k(\boldsymbol{j})=1$, the following equalities hold for the various terms defined in (31) and (32):

$$w^{12} = -w^{21}$$
$$v_1^{12}v_2^{12} = v_1^{21}v_2^{21}$$
$$\left(v_1^{12}+v_2^{12}\right)^2 = \left(v_1^{21}+v_2^{21}\right)^2.$$

After some algebraic manipulation using the above equalities we obtain

$$F_2^{12}-F_1^{12} = F_2^{21}-F_1^{21} = C(\sqrt{a+b}-\sqrt{a-b}) \quad (37)$$

where

$$C = \frac{4|\boldsymbol{D}_k|}{(\boldsymbol{D}_k(2,2)-\boldsymbol{D}_k(1,1))^2+2|\boldsymbol{D}_k|} \quad (38)$$

and

$$a = E_1\left(\frac{\boldsymbol{D}_k(2,2)+\boldsymbol{D}_k(1,1)}{2|\boldsymbol{D}_k|}\right)$$

and

$$b = E_1\left(\sqrt{\frac{(\boldsymbol{D}_k(2,2)-\boldsymbol{D}_k(1,1))^2}{4|\boldsymbol{D}_k|}+\frac{1}{|\boldsymbol{D}_k|}}\right). \quad (39)$$

Clearly, $C>0$ in the above equation and it is easy to show that $(a+b)>(a-b)>0$. Therefore, the left-hand side of the equation in (36) is $(F_2^{12}-F_1^{12})^2$ and the right-hand side is $(F_2^{21}-F_1^{21})^2$ and they are both equal. □

The surprising implication of the above proposition is that β^* is such that for every k, every possible realization of past user decisions $\boldsymbol{i}_1^{k-1}$ is mapped to 1. In other words, the optimum NDF decision rule in (35) is an unbiased comparison of the kth user's outputs from the decorrelation $(\boldsymbol{R}_{ex}^{(k)})^{-1}\boldsymbol{y}_{ex}^{(k)}$. Further, among the detectors constrained to be of the simple quadratic form in (16), the G-NDF detector (which seems to account for noise correlations in the decision statistic through its $\beta_k(\boldsymbol{i}_1^{k-1})$) compromises its symmetric energy performance by using nonunity values of $\beta_k(\boldsymbol{i}_1^{k-1})$ to alleviate the lack of knowledge of user energies and phases.

Proposition 2: The symmetric energy of the optimum N-DF detector (denoted as $\phi^{\text{O-NDF}}$) is given as

$$\mathcal{E}(\phi^{\text{O-NDF}}) = \min_{1\leq k\leq K}\min_{\boldsymbol{i}_1^{k-1}}\left\{\frac{2E_k}{(\boldsymbol{D}_k(1,1)+\boldsymbol{D}_k(2,2)+2|\boldsymbol{D}_k(1,2)|)}\right\}. \quad (40)$$

Proof: After some algebra, we get the following expressions for the conditional asymptotic efficiencies $\mathcal{E}_k^{12}(\phi_g^{\beta^*},\boldsymbol{i}_1^{k-1})$ and $\mathcal{E}_k^{21}(\phi_g^{\beta^*},\boldsymbol{i}_1^{k-1})$ defined in (33):

$$\mathcal{E}_k^{12}(\phi_g^{\beta^*},\boldsymbol{i}_1^{k-1}) = \mathcal{E}_k^{21}(\phi_g^{\beta^*},\boldsymbol{i}_1^{k-1}) = \left\{\frac{2E_k}{\boldsymbol{D}_k(1,1)+\boldsymbol{D}_k(2,2)+2|\boldsymbol{D}_k(1,2)|}\right\} \quad (41)$$

where $\boldsymbol{D}_k$ is defined in Section III and the dependency of $\boldsymbol{D}_k$ on $\boldsymbol{i}_1^{k-1}$ is suppressed for notational simplicity.

Therefore, using (24), the genie-aided asymptotic effective energy for user k for the optimum N-DF detector is given by

$$\mathcal{E}_k(\phi_g^{\text{O-NDF}}) = \min_{\boldsymbol{i}_1^{k-1}}\left\{\frac{2E_k}{(\boldsymbol{D}_k(1,1)+\boldsymbol{D}_k(2,2)+2|\boldsymbol{D}_k(1,2)|)}\right\}. \quad (42)$$

Finally, (40) follows from (42) and Lemma 1 which implies that the symmetric energy of the O-NDF detector is equal to that for its genie-aided version. □

Proposition 3: The symmetric energy of the G-NDF detector (denoted as $\phi^{\text{G-NDF}}$) is given as (43) at the top of the following page, where

$$\mathcal{E}_k^{i_kj}(\phi_g^{\text{G-NDF}},\boldsymbol{i}_1^{k-1}) = E_k\left\{\boldsymbol{H}_k(i_k,i_k)-\sqrt{\frac{\boldsymbol{H}_k(i_k,i_k)}{\boldsymbol{H}_k(j,j)}}|\boldsymbol{H}_k(i_k,j)|\right\}. \quad (44)$$

The proof is left to the reader. Equation (44) can be obtained by using

$$\beta_k(\boldsymbol{i}_1^{k-1}) = \frac{H_k(2,2)}{H_k(1,1)}$$

in the general expression for the conditional asymptotic effective energy for an arbitrary NDF detector in the K-map class.

$$\mathcal{E}(\phi^{\text{G-NDF}}) = \min_{1\le k\le K} \min_{\boldsymbol{i}_1^{k-1}} \min\{\mathcal{E}_k^{1,2}(\phi_g^{G-NDF}, \boldsymbol{i}_1^{k-1}), \mathcal{E}_k^{2,1}(\phi_g^{G-NDF}, \boldsymbol{i}_1^{k-1})\} \tag{43}$$

We turn to the problem of finding the nondecision feedback detector $\phi^{\text{O-NPD}}$ posed in (20). In terms of the conditional asymptotic effective energies, that problem can be equivalently rewritten as

$$\alpha_k^* \in \arg\max_{\alpha_k} \min\left\{\mathcal{E}_k^{1,2}(\phi^\alpha),\ \mathcal{E}_k^{2,1}(\phi^\alpha)\right\}. \tag{45}$$

Proposition 4: The optimum noncoherent decorrelating detector (denoted as $\phi^{\text{O-NPD}}$) is described for each $k = 1, 2, \ldots, K$ as

$$\hat{i}_k = \begin{cases} 1, & \text{if } |z_k(1)|^2 \ge |z_k(2)|^2 \\ 2, & \text{otherwise.} \end{cases} \tag{46}$$

Proof: We can apply Lemma 5 to obtain the necessary and sufficient condition to solve for the optimum α_k^* which is

$$\mathcal{E}_k^{1,2}(\phi^{\boldsymbol{\alpha}})\Big|_{\alpha_k=\alpha_k^*} = \mathcal{E}_k^{2,1}(\phi^{\boldsymbol{\alpha}})\Big|_{\alpha_k=\tilde{\alpha}_k^*}. \tag{47}$$

The solution to the above equation for $k = 1$ is essentially the same as that for the optimum map β_1^* for the first user in the decision feedback case in Proposition 1. Therefore, applying the implication of Proposition 1 repeatedly for each user yields $\alpha_k^* = 1$ for all k for ϕ^{O-NPD}. □

Proposition 5: The asymptotic effective energy for user k for $\phi^{\text{O-NPD}}$ is given by

$$\mathcal{E}_k(\phi^{\text{O-NPD}}) = \frac{2E_k}{\boldsymbol{Q}_k(1,1) + \boldsymbol{Q}_k(2,2) + 2|\boldsymbol{Q}_k(1,2)|} \tag{48}$$

where $\boldsymbol{Q}_k$ is the kth 2×2 block along the diagonal of $\boldsymbol{R}^{-1}$.

V. Performance Bounds, Power Control, and Quality of Service

In this section, we obtain per-user bounds on BER and asymptotic effective energy that take into account the error propagation effects. In cellular systems that employ power control, the base station must, in the ideal case, command the transmitters to transmit at the least possible powers (provided such powers are known) in order to ensure that they can achieve a prespecified quality of service (QoS). We characterize (or bound) the set of admissible powers that ensure that the noncoherent decision feedback detectors will achieve given QoS objectives in terms of asymptotic effective energies without making simplifying assumptions about the effects of error propagation. Consequently, we obtain (or bound) the component-wise minimal powers needed to achieve the QoS specification.

A. Per-User Performance Bounds

An exact per-user error probability analysis of $\phi^{\text{O-NDF}}$ is intractable because of error propagation effects. Bounds on BER and asymptotic effective energy can, however, be obtained as was done for the case of the G-NDF detector in [2].

For notational simplicity, we will denote the BER of the kth user $P_k(\sigma, \phi^{\text{O-NDF}})$ as P_k, the corresponding genie-aided BER $P_k(\sigma, \phi_g^{\text{O-NDF}})$ as $P_{k,g}$ and the conditional error probability $P_k^{i_k j}(\sigma, \phi_g^{\text{O-NDF}}, \boldsymbol{i}_1^{k-1})$ as $P_{k,g}^{i_k j}(\boldsymbol{i}_1^{k-1})$. Noting that the results of [2] on bounds for the BER of the G-NDF detector apply to any ϕ^β and in particular to $\phi^{\text{O-NDF}}$, we have

$$P_{k,g} \le P_k \le \sum_{m=1}^{k} P_{m,g} \tag{49}$$

where $P_{m,g}$, obtained in Section IV, can be written as

$$P_{m,g} = \frac{1}{2^m} \sum_{\substack{\boldsymbol{i}_1^m, j \\ j \ne i_m}} P_{m,g}^{i_m j}(\boldsymbol{i}_1^{m-1}) \tag{50}$$

with the pair-wise error rate $P_{m,g}^{i_m j}(\boldsymbol{i}_1^{m-1})$ obtained by substituting the optimum β^* for β in (29). The upper bound in (49) includes the effects of error propagation and the lower bound is simply the genie-aided detector's BER.

Using the BER bounds in (49) and the definition of asymptotic effective energy, we have the following simple but useful result.

Proposition 6: The asymptotic effective energy for user k for $\phi^{\text{O-NDF}}$ is bounded from above by its genie-aided asymptotic effective energy and from below by the group symmetric energy $\Psi_k(\phi^{\text{O-NDF}})$, i. e.,

$$\begin{aligned}\mathcal{E}_k(\phi_g^{\text{O-NDF}}) &\ge \mathcal{E}_k(\phi^{\text{O-NDF}}) \\ &\ge \min_{m=\{1,\ldots,k\}} \left\{\mathcal{E}_m(\phi_g^{\text{O-NDF}})\right\}\end{aligned} \tag{51}$$

where the expression for $\mathcal{E}_k(\phi_g^{\text{O-NDF}})$ is given in (42).

A simple consequence of the above proposition is the following.

Corollary 1: A sufficient condition for $\phi^{\text{O-NDF}}$ to achieve genie-aided performance for user k is given as

$$\mathcal{E}_k(\phi_g^{\text{O-NDF}}) \le \min_{m=\{1,\ldots,k-1\}} \mathcal{E}_m(\phi_g^{\text{O-NDF}}). \tag{52}$$

Hence, if the genie-aided asymptotic effective energies are in nonincreasing order, i.e.,

$$\mathcal{E}_1(\phi_g^{\text{O-NDF}}) \ge \mathcal{E}_2(\phi_g^{\text{O-NDF}}) \ge \cdots \ge \mathcal{E}_K(\phi_g^{\text{O-NDF}})$$

then the upper and lower bounds on asymptotic effective energy coincide for all users and error propagation effects can be ignored for high SNR. This sufficient condition is satisfied when the signal energies of the users are sufficiently disparate. To quantify this, we define the amplitude ratios $\kappa_k \triangleq \sqrt{E_k/E_{k+1}}$ and the genie-aided asymptotic efficiencies

$$\eta_k(\phi_g^{\text{O-NDF}}) \triangleq \mathcal{E}_k(\phi_g^{\text{O-NDF}})/E_k, \qquad \text{for } k \in \{1,\ldots,K\}.$$

Notice from (42) that, for a given order of detection, $\eta_k(\phi_g^{\text{O-NDF}})$ depends only on the signal correlations. Hence, $\phi^{\text{O-NDF}}$ achieves genie-aided performance for all users if

$$\kappa_k^2 \ge \frac{\eta_{k+1}(\phi_g^{\text{O-NDF}})}{\eta_k(\phi_g^{\text{O-NDF}})}, \qquad \forall k. \tag{53}$$

As a simple example, consider the signal correlation matrix $\boldsymbol{R}$ with $R_{ij} = \rho^{|i-j|}$. For $\rho = 0.2$, it can be evaluated that

$$\eta_1(\phi^{\text{O-NPD}}) = \eta_2(\phi_g^{\text{O-NDF}}) = 0.7869.$$

The condition in (53) requires that $\kappa_1^2 = E_1/E_2 \geq 1$. In this example then, if the stronger user is detected first and the weaker user last, the second user will always achieve genie-aided performance.

B. Power Control and QoS

Corollary 2: A sufficient condition on the user energies that guarantees QoS constraints in the form of required asymptotic effective energies $\chi_1, \ldots, \chi_K$ for users 1 to K, respectively (with users detected in the same order 1 to K), is that

$$E_k \geq \max\left\{\frac{\chi_k}{\eta_{k,g}}, \ldots, \frac{\chi_K}{\eta_{k,g}}\right\}, \qquad \forall k \in \{1, \ldots, K\} \quad (54)$$

where $\eta_{1,g} = \eta_1$ as there is no decision feedback for user 1.

The proof is easily deduced from (51). The conditions in (54) quantify (including error propagation effects) the intuitive idea that, in general, higher asymptotic effective energies require higher received energies. Note that (54) does not, however, guarantee genie-aided performance. Consider, for instance, a two-user channel where the required asymptotic effective energies are such that $\chi_2 > \chi_1$. Clearly, if $E_1 = \chi_2/\eta_1$ and $E_2 > \chi_2/\eta_{2,g}$, the condition in (54) is satisfied whereas that in (52) is not, and, therefore, genie-aided performance is not ensured.

Of the energies that satisfy the sufficient condition in (54), those that satisfy the inequalities with equality are not only component-wise minimal but also result in genie-aided performance for all users. The reader can verify that the upper and lower bounds on asymptotic effective energies given by (51) coincide in this case. A stronger result is possible when users are detected in the decreasing order of the QoS constraints.

Corollary 3: If users are detected in the decreasing order of the QoS constraints, i.e., if $\chi_1 \geq \chi_2 \geq \cdots \geq \chi_K$, then

$$E_k \geq \chi_k/\eta_{k,g}, \qquad \forall k \in \{1, \ldots, K\} \quad (55)$$

is a necessary and sufficient condition to achieve the QoS constraints. Moreover, when $E_k = \chi_k/\eta_{k,g}$, genie-aided performance is achieved for each user and these energies are component-wise minimal among all sets of energies that satisfy the QoS constraints.

The proof is straightforward and left to the reader. Corollary 3 is deceptively simple. It gives easy formulas for the set of admissible received energies and the component-wise minimal received energies such that the optimal decision feedback detector achieves the given QoS constraints *without* making any simplifying assumptions about the effects of error propagation. Note that when the QoS constraints are all equal, such a result was obtained for the coherent decorrelating decision feedback detector in [19]. In that case, the order of detection need not be specified although the minimal energies will in general depend on it. Note also that the minimal energies needed for the optimal decision feedback detector are the least (component-wise) among energies required to achieve the given QoS requirements for any other detector in the K-map class of decision feedback detectors.

Corollary 2 is more general but less sharp. For an arbitrary order of detection, it specifies an inner bound on the set of admissible received energies for which the QoS constraints are satisfied *without* making any simplifying assumptions on the effects of error propagation. The component-wise minimal received energies within this inner bounding region is therefore an upper bound on the component-wise minimal energies needed to achieve the QoS objectives. The problem of characterizing the exact set of admissible received energies or even a larger inner bound than the one given by Corollary 2 appears to be a difficult one.

VI. Rules for Ordering Users

In this section, we prove that the symmetric energy of the O-NDF detector is greater than or equal to the optimum decorrelating detector *independently* of the order of detection. It would also be desirable to ensure a per-user performance gain with decision feedback. This is indeed possible for the O-NDF detector, and we will provide simple ordering rules by which this can be guaranteed. The computations required to implement these ordering rules can be done off-line but they depend on a knowledge of the user energies. However, our energy-dependent ordering rules suggest data-dependent rules that can be implemented without knowledge of the received energies.

A key result that is required in proving the above claims is that the genie-aided version of the O-NDF detector should user-wise outperform its decorrelating (or non-decision feedback) counterpart in asymptotic effective energy for any order of detection. Such a result is very easily proved in the case of coherent decision feedback detectors such as the successive canceler, the decorrelating decision feedback detector, and the optimum decision feedback detector [1]. In the case of noncoherent decision feedback detection, however, the problem is less obvious. In fact, such a result is not true for the G-NDF detector (cf. [2]).

As a matter of notation, $\boldsymbol{X} > \boldsymbol{Y}$ ($\boldsymbol{X} \geq \boldsymbol{Y}$) denotes that the matrix $\boldsymbol{X} - \boldsymbol{Y}$ is positive (nonnegative) definite. $\boldsymbol{X} \circ \boldsymbol{Y}$ denotes the element-wise product or *Schur* product of the matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ [20]. Let $\boldsymbol{0}$ denote a matrix of zeros. Let $\sqrt{\boldsymbol{X}}$ denote a matrix obtained from $\boldsymbol{X}$ by replacing each element of $\boldsymbol{X}$ with its square root.

Lemma 6: a) If $\boldsymbol{A} > \boldsymbol{B} > \boldsymbol{0}$, and $\boldsymbol{C} > \boldsymbol{D} > \boldsymbol{0}$, where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are Hermitian symmetric matrices, then $\boldsymbol{A} \circ \boldsymbol{C} > \boldsymbol{B} \circ \boldsymbol{D} > \boldsymbol{0}$. b) If $\boldsymbol{A} \geq \boldsymbol{B} \geq \boldsymbol{0}$ then $\sqrt{\boldsymbol{A}} \geq \sqrt{\boldsymbol{B}}$.

Proof:

a) Using the Schur product theorem [20], which states that the Schur product of two positive-definite matrices is positive definite, we have $(\boldsymbol{A}-\boldsymbol{B}) \circ \boldsymbol{C} > \boldsymbol{0}$ and $\boldsymbol{B} \circ (\boldsymbol{C}-\boldsymbol{D}) > \boldsymbol{0}$. Hence, $\boldsymbol{A} \circ \boldsymbol{C} > \boldsymbol{B} \circ \boldsymbol{C} > \boldsymbol{B} \circ \boldsymbol{D} > \boldsymbol{0}$, where the last inequality also follows because of the Schur product theorem.

b) (By Contradiction) Suppose $\sqrt{\boldsymbol{B}} > \sqrt{\boldsymbol{A}}$ when, in fact, $\boldsymbol{A} \geq \boldsymbol{B} \geq 0$. By part a) of this lemma, $\sqrt{\boldsymbol{B}} \circ \sqrt{\boldsymbol{B}} > \sqrt{\boldsymbol{A}} \circ \sqrt{\boldsymbol{A}}$, or $\boldsymbol{B} > \boldsymbol{A}$, a contradiction. □

Recall that for perfect past user decisions, the inverse of the covariance matrix of the effective noise term in the kth user's decision statistic in (14) for the genie-aided decision feedback strategy is $\boldsymbol{D}_k^{-1}$ and the inverse of the covariance matrix of the effective noise term in the kth user's decision statistic in (12) for the decorrelating strategy is $\boldsymbol{Q}_k^{-1}$. Moreover, the matrix $\boldsymbol{D}_k^{-1}$ depends on the past symbols $\boldsymbol{i}_1^{k-1}$. The next lemma gives a general relationship between $\boldsymbol{D}_k^{-1}$ and $\boldsymbol{Q}_k^{-1}$.

Lemma 7: $\boldsymbol{D}_k^{-1} \geq \boldsymbol{Q}_k^{-1}$ for any $\boldsymbol{i}_1^{k-1} \in \{1,2\}^{k-1}$.

Proof: Note that $\boldsymbol{D}_k$ and $\boldsymbol{Q}_k$ are covariance matrices, and, hence, they are positive definite. Consider the two quadratic forms $\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x}$ and $\boldsymbol{x}_{ex}^{(k)\dagger} \boldsymbol{R}_{ex}^{(k)} \boldsymbol{x}_{ex}^{(k)}$ where $\boldsymbol{x} \in \boldsymbol{C}^{2K}$, $\boldsymbol{x} = \left[\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_K^T\right]^T$, and $\boldsymbol{x}_k \in \boldsymbol{C}^2$. Moreover, $\boldsymbol{R}_{ex}^{(k)}$ is as defined in Section III,

$$\boldsymbol{x}_{ex}^{(k)} = \left[\boldsymbol{x}_1(i_1), \ldots, \boldsymbol{x}_{k-1}(i_{k-1}), \boldsymbol{x}_k^T, \ldots, \boldsymbol{x}_K^T\right]^T,$$

and i_l is user l's information symbol, $i_l \in \{1,2\}$. Clearly

$$\boldsymbol{x}_{ex}^{(k)\dagger} \boldsymbol{R}_{ex}^{(k)} \boldsymbol{x}_{ex}^{(k)} = \boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x}$$

if $\boldsymbol{x}_l(j) = 0$ for $j \neq i_l$ for each $1 \leq l \leq (k-1)$. Hence, we have

$$\min_{\boldsymbol{x}_{ex}^{(k)},\, \boldsymbol{x}_k \text{ fixed}} \boldsymbol{x}_{ex}^{(k)\dagger} \boldsymbol{R}_{ex}^{(k)} \boldsymbol{x}_{ex}^{(k)} \geq \min_{\boldsymbol{x};\, \boldsymbol{x}_k \text{ fixed}} \boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x}. \tag{56}$$

Using a result from [21], the minimum in the right-hand side of (56) is equal to $\boldsymbol{x}_k^\dagger \boldsymbol{Q}_k^{-1} \boldsymbol{x}_k$ and the minimum in the left-hand side is identically equal to $\boldsymbol{x}_k^\dagger \boldsymbol{D}_k^{-1} \boldsymbol{x}_k$. Hence, $\boldsymbol{x}_k^\dagger \boldsymbol{D}_k^{-1} \boldsymbol{x}_k \geq \boldsymbol{x}_k^\dagger \boldsymbol{Q}_k^{-1} \boldsymbol{x}_k$. Since this inequality is true for any $\boldsymbol{x}_k$, we have $\boldsymbol{D}_k^{-1} \geq \boldsymbol{Q}_k^{-1}$. The proof is, of course, valid for any $\boldsymbol{i}_1^{k-1}$. □

Proposition 7: The asymptotic effective energy of the optimum decorrelating detector $\phi^{\text{O-NPD}}$ can be no greater than that of the genie-aided detector $\phi_g^{\text{O-NDF}}$ for every user, i.e.,

$$\mathcal{E}_k(\phi_g^{\text{O-NDF}}) \geq \mathcal{E}_k(\phi^{\text{O-NPD}}), \qquad k = 1, 2, \ldots, K. \tag{57}$$

Proof: It is sufficient to show that for any $\boldsymbol{i}_1^{k-1}$

$$\mathcal{E}_k(\phi_g^{\text{O-NDF}}, \boldsymbol{i}_1^{k-1}) \geq \mathcal{E}_k(\phi^{\text{O-NPD}}). \tag{58}$$

Using the expressions for the asymptotic effective energies of the genie-aided optimum NDF detector and the optimum decorrelating detector in (41) and (48), respectively, and with the *Frobenius norm* [20] of a matrix $\boldsymbol{X}$ defined as

$$\|\boldsymbol{X}\|_2 \triangleq \sum_{i,j=1}^{n} |X_{ij}|^2$$

it follows that (58) is equivalent to

$$\frac{\|\sqrt{\boldsymbol{D}_k}\|_2^2}{\|\sqrt{\boldsymbol{Q}_k}\|_2^2} \leq 1. \tag{59}$$

Using Lemma 7, we have $\boldsymbol{Q}_k \geq \boldsymbol{D}_k$ which implies by Lemma 6 that $\sqrt{\boldsymbol{Q}_k} \geq \sqrt{\boldsymbol{D}_k}$. If the matrices $\sqrt{\boldsymbol{Q}_k}$ and $\sqrt{\boldsymbol{D}_k}$ were positive definite, then using a result from [20], we would have $\|\sqrt{\boldsymbol{Q}_k}\|_2^2 \geq \|\sqrt{\boldsymbol{D}_k}\|_2^2$ and our claim would be proved. The rest of the proof shows that $\sqrt{\boldsymbol{Q}_k}$ and $\sqrt{\boldsymbol{D}_k}$ are indeed positive definite.

By factoring $\det(\boldsymbol{Q}_k)$ (which is positive), we have

$$\sqrt{\boldsymbol{Q}_k(1,1)\boldsymbol{Q}_k(2,2)} - |\boldsymbol{Q}_k(1,2)| > 0$$

which means that $\det(\sqrt{\boldsymbol{Q}_k}) > 0$. Using this fact and from an inspection of $\sqrt{\boldsymbol{Q}_k}$, we have that both its eigenvalues are positive which implies $\sqrt{\boldsymbol{Q}_k}$ is positive definite. The same argument applies for $\sqrt{\boldsymbol{D}_k}$. □

We are now ready to state the result that we were aiming to prove.

Proposition 8: The symmetric energy of the optimum decision feedback detector $\phi^{\text{O-NDF}}$ is greater than that of the optimum decorrelating detector $\phi^{\text{O-NPD}}$ (and, hence, *a fortiori* greater than that of $\phi^{\text{G-NPD}}$) for *any* order of detection.

Proof: The result is a simple consequence of Lemma 1 and Proposition 7. □

The preceding result is the noncoherent counterpart of the result on the robustness of symmetric energy performance of the coherent decorrelating decision feedback detector to the order of detection (see Proposition 1 of [1]). Somewhat surprisingly, such a result is not true of the relative performances of the $\phi^{\text{G-NPD}}$ and the $\phi^{\text{G-NDF}}$. In fact, it is possible to find an example where the symmetric energy of $\phi^{\text{G-NDF}}$ is less than that of $\phi^{\text{G-NPD}}$ for every permutation of the users.

Proposition 7 is also the basis for obtaining algorithms that determine the order of detection for the optimum NDF detector with certain performance guarantees. Let the users be numbered $1, \ldots, K$ to start with and let the desired ordering be $o_1, \ldots, o_K$.

Proposition 9 [Ordering Rule I]: $\phi^{\text{O-NDF}}$ outperforms $\phi^{\text{O-NPD}}$ in terms of asymptotic effective energy for *every* user, if they are detected in the order of nonincreasing asymptotic effective energies for the optimum decorrelating detector, i.e., $o_1, \ldots, o_K$ are such that

$$\mathcal{E}_{(o_1)}^{\text{O-NPD}} \geq \cdots \geq \mathcal{E}_{(o_K)}^{\text{O-NPD}}. \tag{60}$$

Proof: The ordering rule in (60) implies that

$$\mathcal{E}_{(o_k)}(\phi^{\text{O-NPD}}) \leq \min_{m \in \{1,\ldots,k-1\}} \left\{\mathcal{E}_{(o_m)}(\phi^{\text{O-NPD}})\right\}. \tag{61}$$

Using Proposition 7, we know that

$$\mathcal{E}_{(o_i)}(\phi_g^{\text{O-NDF}}) \geq \mathcal{E}_{(o_i)}(\phi^{\text{O-NPD}})$$

for any user i. Therefore, it follows that

$$\mathcal{E}_{(o_k)}(\phi^{\text{O-NPD}}) \leq \min_{m \in \{1,\ldots,k-1\}} \left\{\mathcal{E}_{(o_m)}(\phi_g^{\text{O-NDF}})\right\}. \tag{62}$$

Using Proposition 7 for user k, it is also true that

$$\mathcal{E}_{(o_k)}(\phi_g^{\text{O-NDF}}) \geq \mathcal{E}_{(o_k)}(\phi^{\text{O-NPD}})$$

which combined with (62) yields

$$\mathcal{E}_{(o_k)}(\phi^{\text{O-NPD}}) \leq \min_{m \in \{1,\ldots,k\}} \left\{\mathcal{E}_{(o_m)}(\phi_g^{\text{O-NDF}})\right\}. \tag{63}$$

It now only remains to invoke Proposition 6. □

Although a simple way to ensure that decision feedback guarantees an enhancement in user-wise performance over the optimum decorrelating detector with the optimum NDF detector, Ordering Rule I is by no means the best way to achieve such an improvement. Moreover, the margins of improvements in performance with decision feedback can be made significantly higher with a more sophisticated ordering rule. Before we state our improved ordering rule, we need the following extension of Proposition 7, the proof of which is similar. The details are left to the reader.

Lemma 8: Given any order of detection $o_1, o_2, \ldots, o_K$, swapping users o_j and o_k, $k > j$, with other users fixed, leads to an increase in the genie-aided asymptotic effective energy for user o_j and a decrease in the genie-aided asymptotic effective energy for user o_k for the optimum NDF detector.

One implication of Lemma 8 is that the genie-aided asymptotic effective energy of any user attains its highest value with maximum "user-expurgation," i.e., if it were detected last. Moreover, there cannot be one ordering where the genie-aided asymptotic effective energy of every user is the highest over all possible orderings. Lemma 8 also gives an intuitive explanation of why Ordering Rule I is a good one. By ordering users in nonincreasing order of asymptotic effective energies for the optimum decorrelating detector, the worse the performance of a user is with the optimum decorrelating detector, the higher the performance gain it tries to ensure for that user in the decision feedback strategy. This intuition can be extended to obtain higher performance gain margins with decision feedback while retaining the key property of Ordering Rule I.

Proposition 10 [Ordering Rule II]: Let the new ordering of users, $o_1, \ldots, o_K$, be such that user o_1 achieves the highest asymptotic effective energy among all users when they are detected by $\phi^{\text{O-NDP}}$. Having ascertained $o_1, \ldots, o_{k-1}$ in this ordering rule, the next user o_k is chosen from the remaining $K-k+1$ users as the user that achieves the highest genie-aided asymptotic effective energy among those remaining users if each one of those users were to be detected next. This ordering rule ensures that the asymptotic effective energy of every user for the O-NDF detector is greater than the corresponding asymptotic effective energy for the optimum decorrelating detector.

Proof: Let us suppose that the statement of the theorem is not true for user o_m, the mth user in Ordering Rule II. This implies by (51) that

$$\mathcal{E}_{(o_m)}(\phi^{\text{O-NPD}}) > \min_{i \in \{1,\ldots,m\}} \left\{\mathcal{E}_{(o_i)}(\phi_g^{\text{O-NDF}})\right\}. \quad (64)$$

By Proposition 7, we know that

$$\mathcal{E}_{(o_m)}(\phi_g^{\text{O-NDF}}) \geq \mathcal{E}_{(o_m)}(\phi^{\text{O-NPD}}).$$

This implies that $\exists$ some $j < m$ such that

$$\mathcal{E}_{(o_m)}(\phi^{\text{O-NPD}}) > \mathcal{E}_{(o_j)}(\phi_g^{\text{O-NDF}}). \quad (65)$$

Now swap users o_j and o_m, the new ordering being

$$\{o_1, \ldots, o_{j-1}, \hat{o}_j, \ldots, o_{m-1}, \hat{o}_m\}.$$

This implies by (65) that $\mathcal{E}_{(\hat{o}_j)}(\phi_g^{\text{O-NDF}})$ is greater than $\mathcal{E}_{(o_j)}(\phi_g^{\text{O-NDF}})$ with $o_1, \ldots, o_{j-1}$ fixed. This violates the criterion of the ordering rule used in selecting the jth user, thereby leading to a contradiction. □

Proposition 11: Ordering Rule II for $\phi^{\text{O-NDF}}$ maximizes symmetric energy over all possible NDF detectors and over all possible $K!$ orderings.

Proof: We need only prove that the $\phi^{\text{O-NDF}}$ detector for Ordering Rule II has a symmetric energy that is no less than that of the $\phi^{\text{O-NDF}}$ detector for any other ordering rule.

An argument in [22] (which is made in the context of coherent decision feedback detection and based on the signal-to-interference-plus-noise ratio criterion under the perfect feedback assumption) can also be used in the present context to show that Ordering Rule II maximizes the worst case genie-aided asymptotic effective energy. Moreover, Lemma 1 implies that the worst case genie-aided asymptotic effective energy is equal to the worst case asymptotic effective energy (without the assumption of perfect feedback), which is the symmetric energy. Hence, the proposition. □

In the implementation of Ordering Rule I, $\mathcal{E}_k(\phi^{O-NPD})$ has to be calculated for every user k, i.e., a total of K asymptotic effective energy computations are required. In the case of Ordering Rule II, the computation that was required for Ordering Rule I has to be done just to determine user o_1. Additionally, to determine o_k (for $k \geq 2$), $\mathcal{E}_j(\phi_g^{\text{O-NDF}})$, $j \in \{1, \ldots, K\} - \{o_1, \ldots, o_{k-1}\}$, has to be calculated for each of the $(K-k+1)$ values of j. The computations for ordering users can of course be done off-line and must be updated every time the user energies change.

A. Data-Driven Ordering Rules

In this subsection, we consider the problem of ordering users when the energies are unknown. Here we are guided by the ordering rules for the known energy case but rely on the matched filter outputs $\boldsymbol{y}$ to determine the order of detection. Such randomized ordering rules that mimic Ordering Rule II are as follows.

Data-Driven Ordering I: For every received matched filter output $\boldsymbol{y}$, i.e., for every symbol period T, choose the first user to be detected, as that which has the maximum value for the decorrelator output $|z_k(j)|^2$ (over all $k \in \{1, 2, \ldots, K\}$ and $j \in \{1, 2\}$). Having made a decision for the first user, choose the second user to be that which has the maximum value for the "expurgated" decorrelation output $|z_{df}^{(2)}(j)|^2$ over the remaining $K-1$ users and $j \in \{1, 2\}$, and so on.

Data-Driven Ordering II: As a variation of the above idea, the maximum value of $|\,|z_k(1)|^2 - |z_k(2)|^2\,|$ over all k is used to choose user 1 and the maximum value of $|\,|z_{df}^{(2)}(1)|^2 - |z_{df}^{(2)}(2)|^2\,|$ over all the remaining $K-1$ users to determine user 2, and so on.

The data-driven orderings, while random, tend to often result in the same order as long as the received energies stay constant. Some changes from the most likely order will of course occasionally result. The implementation of the decision feedback detector for such randomized orderings is computationally no more complex than that for a fixed order. However, more feedforward and feedback decorrelation coefficients need to be stored to account for the possibility of having to perform decision feedback detection for more than just one ordering.

VII. Binary Signaling With Unequal Energies

In this section, we seek to extend the results of this paper to the more general case where the received energies of the two signals of a particular user can be distinct. It is assumed that while the absolute values of the energies are unknown, the ratios of the two energies are known at the receiver. This is a reasonable assumption since such ratios would most likely be equal at the receiver and the transmitter.

We show that the unequal energy problem can be transformed into an equal energy problem. For the more general case, let the kth user's signal energies be denoted by E_{k1} and E_{k2}. The linear model for the matched filter outputs in (5) will remain the same except that $\boldsymbol{A}_k$ in (10) will now be equal to $\mathrm{diag}(\sqrt{E_{k1}}e^{j\theta_{k1}}, \sqrt{E_{k2}}e^{j\theta_{k2}})$. Consequently, the decision statistic for the decision feedback strategy as given in (14) for perfect previous users' decisions is given for $j = \{1, 2\}$ as

$$\boldsymbol{z}_{df}^{(k)}(j) = \sqrt{E_{kj}}e^{j\theta_{kj}}b_{kj} + \gamma^{(k)}(j). \tag{66}$$

The decision statistic for the nondecision feedback strategy corresponding to (12) is

$$\boldsymbol{z}_k(j) = \sqrt{E_{kj}}e^{j\theta_{kj}}b_{kj} + \tau_k(j), \qquad j = \{1, 2\}. \tag{67}$$

With the knowledge of only the ratio $\frac{E_{k1}}{E_{k2}}$, we can now scale $\boldsymbol{z}_k(2)$ to obtain the following model for the decision feedback detector:

$$\begin{aligned} \boldsymbol{z}_{df}^{(k)}(1) &= \sqrt{E_{k1}}e^{j\theta_{k1}}b_{k1} + \gamma^{(k)}(1) \\ \sqrt{\frac{E_{k1}}{E_{k2}}}\boldsymbol{z}_{df}^{(k)}(2) = \bar{\boldsymbol{z}}_{df}^{(k)}(2) &= \sqrt{E_{k1}}e^{j\theta_{k2}}b_{k2} + \bar{\gamma}^{(k)}(2). \end{aligned} \tag{68}$$

For the decorrelating detector, we have the model

$$\begin{aligned} \boldsymbol{z}_k(1) &= \sqrt{E_{k1}}e^{j\theta_{k1}}b_{k1} + \tau_k(1) \\ \sqrt{\frac{E_{k1}}{E_{k2}}}\boldsymbol{z}_k(2) = \bar{\boldsymbol{z}}_k(2) &= \sqrt{E_{k1}}e^{j\theta_{k2}}b_{k2} + \bar{\tau}_k(2). \end{aligned} \tag{69}$$

The scaled decision statistics $\bar{\boldsymbol{z}}_{df}^{(k)} = (\boldsymbol{z}_{df}^{(k)}(1)\ \bar{\boldsymbol{z}}_{df}^{(k)}(2))^T$ and $\bar{\boldsymbol{z}}_k = (\boldsymbol{z}_k(1)\ \bar{\boldsymbol{z}}_k(2))^T$ for user k for the decision feedback and non-decision feedback detection strategies, respectively, can be seen to be the equal energy models corresponding to (14) and (12) with the following modifications. The noise vector $\bar{\gamma}^{(k)} = (\gamma^{(k)}(1)\ \bar{\gamma}^{(k)}(2))^T$ has a covariance matrix whose first row is

$$\left(\boldsymbol{D}_k(1,1)\ \sqrt{\frac{E_{k1}}{E_{k2}}}\boldsymbol{D}_k(1,2)\right)$$

and second row is

$$\left(\sqrt{\frac{E_{k1}}{E_{k2}}}\boldsymbol{D}_k(2,1)\ \frac{E_{k1}}{E_{k2}}\boldsymbol{D}_k(2,2)\right)$$

where $\boldsymbol{D}_k$ is defined in Section III. The noise vector $\bar{\tau}_k = (\tau_k(1)\ \bar{\tau}_k(2))^T$ has a covariance matrix whose first row is

$$\left(\boldsymbol{Q}_k(1,1)\ \sqrt{\frac{E_{k1}}{E_{k2}}}\boldsymbol{Q}_k(1,2)\right)$$

and second row is

$$\left(\sqrt{\frac{E_{k1}}{E_{k2}}}\boldsymbol{Q}_k(2,1)\ \frac{E_{k1}}{E_{k2}}\boldsymbol{Q}_k(2,2)\right)$$

where $\boldsymbol{Q}_k$ is defined in Section III.

Adapting the O-NDF detector and the optimum decorrelating detector for the equal energy case in Section IV to the scaled version of the unequal energy case, we have that the O-NDF decision rule for the more general unequal energy case to be given by

$$\hat{i}_k = \begin{cases} 1, & \text{if } |z_{df}^{(k)}(1)|^2 \geq |\bar{z}_{df}^{(k)}(2)|^2 \\ 2, & \text{else} \end{cases} \tag{70}$$

and, similarly, the optimum decorrelating rule is

$$\hat{i}_k = \begin{cases} 1, & \text{if } |z_k(1)|^2 > |\bar{z}_k(2)|^2 \\ 2, & \text{else.} \end{cases} \tag{71}$$

The genie-aided asymptotic effective energy for user k for the optimum unequal energy NDF detector is given by (72) at the bottom of the page, and the asymptotic effective energy for user k for the optimum decorrelating detector $\phi_{ue}^{\text{O-NPD}}$ is given by

$$\mathcal{E}_k(\phi_{ue}^{\text{O-NPD}}) = \min_j \left\{ \frac{2E_{kj}}{\boldsymbol{Q}_k(1,1) + \frac{E_{k1}}{E_{k2}}\boldsymbol{Q}_k(2,2) + 2\sqrt{\frac{E_{k1}}{E_{k2}}}|\boldsymbol{Q}_k(1,2)|} \right\} \tag{73}$$

where the subscript ue stands for "unequal energy" and $j \in \{1, 2\}$. The expressions for symmetric energy, $\mathcal{E}(\phi_{ue}^{\text{O-NPD}})$ and $\mathcal{E}(\phi_{ue}^{\text{O-NDF}})$, are easily derived by substituting(73) and (72), respectively, in the symmetric energy as implied by Lemma 1. We therefore have an extension of Proposition 2 to the unequal energy case.

VIII. Numerical Examples

We start with a two-user channel with signal correlation matrix

$$\boldsymbol{R} = \begin{bmatrix} 1.0000 & -0.1666 & -0.6745 & 0.0075 \\ -0.1666 & 1.0000 & 0.4251 & 0.2866 \\ -0.6745 & 0.4251 & 1.0000 & -0.3303 \\ 0.0075 & 0.2866 & -0.3303 & 1.0000 \end{bmatrix}. \tag{74}$$

Fig. 1 plots the symmetric energy of the conventional (11), the O-NPD and the GLR-based (G-NPD) decorrelating detectors as a function of the energy ratio E_2/E_1 (in decibels). The sum of the energies is fixed at $E_1 + E_2 = 1$. Both decorrelating

$$\mathcal{E}_k(\phi_{g,ue}^{\text{O-NDF}}) = \min_{\boldsymbol{i}_1^{k-1}} \min_j \left\{ \frac{2E_{kj}}{(\boldsymbol{D}_k(1,1) + \frac{E_{k1}}{E_{k2}}\boldsymbol{D}_k(2,2) + 2\sqrt{\frac{E_{k1}}{E_{k2}}}|\boldsymbol{D}_k(1,2)|)} \right\} \tag{72}$$

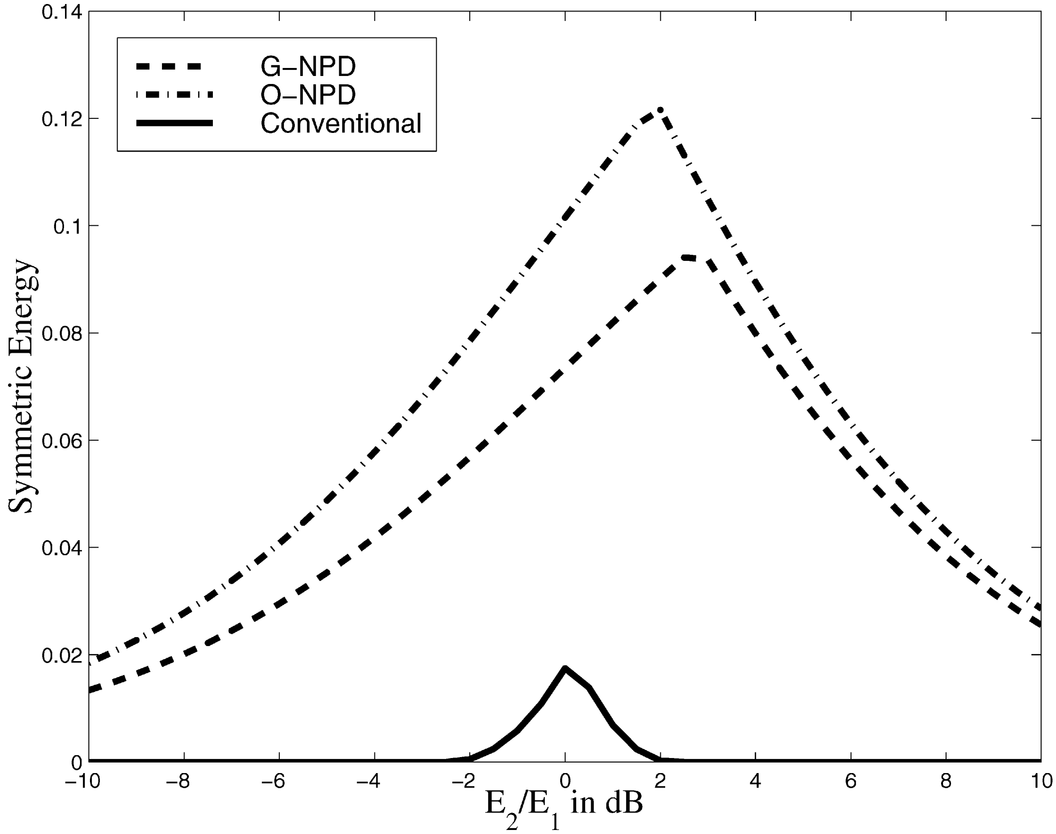

Fig. 1. Symmetric energy versus user energy ratio E_2/E_1 for a two-users system for the conventional, optimum (O-NPD) and GLR-based (G-NPD) decorrelating detectors.

detectors far outperform the conventional detector. Notice also that the O-NPD detector performs uniformly better than G-NPD detector as it must. Fig. 2 plots in addition the symmetric energy of the optimum and GLR decision feedback detectors. The symmetric energies of the decision feedback detectors are computed for the case where the users are detected in a fixed order (user 1 first and user 2 second). Both decision feedback detectors perform uniformly better than their non-decision feedback counterparts. However, recall that this is guaranteed to happen for all possible signal correlations only for the optimum decision feedback detector. When the second user is sufficiently stronger than the first, the symmetric energy performances of optimum (or GLR) detectors with and without decision feedback are indistinguishable. When the second user is more than 1 dB weaker than the first user, there is a more than 4-dB gap in symmetric energy between the O-NDF detector and the G-NPD detector of [3] and [4].

Fig. 3 compares the performance of the detectors in Fig. 2 when the decision feedback for the O-NDF and the G-NDF detectors has been implemented with reordering, i.e., users have been reordered to optimize the symmetric energy over the two possible orders of detection for each given user energy ratio. This time, the decision feedback detectors further outperform their nondecision feedback counterparts with the difference showing up for sufficiently positive values of E_2/E_1 because of the correct choice of order of detection. For $E_2/E_1 > 4$ dB, there is at least a 3-dB gap in the symmetric energies of the two decision feedback detectors.

Next, consider a two-user channel where there is low correlation between signals of each user but high correlation between signals of different users with

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0.02 & 0.2 & 0.8 \\ 0.02 & 1 & 0.7 & 0.3 \\ 0.2 & 0.7 & 1 & 0.1 \\ 0.8 & 0.3 & 0.1 & 1 \end{bmatrix}. \tag{75}$$

With the energies of users 1 and 2 being $E_1 = 5$ and $E_2 = 1$, Fig. 4 shows the BER bounds for user 2 corresponding for the optimum and GLR detectors with and without feedback. For each of the decision feedback detectors, the first user's performance would be identical to the corresponding decorrelating detector and is therefore not shown. The BER performance in this example of both the decision feedback detectors is uniformly better than that of the corresponding decorrelating detectors, and shows, as promised by the gains in asymptotic effective energies, a significant improvement in BER. The following asymptotic effective energies can be obtained: $\mathcal{E}_1(\phi^{\text{G-NPD}}) = 0.4172$, $\mathcal{E}_1(\phi^{\text{O-NPD}}) = 0.4770$, $\mathcal{E}_2(\phi^{\text{G-NPD}}) = 0.0817$, $\mathcal{E}_2(\phi^{\text{O-NPD}}) = 0.1045$, $\mathcal{E}_2(\phi_g^{\text{G-NDF}}) = 0.3233$, and $\mathcal{E}_2(\phi_g^{\text{O-NDF}}) = 0.4750$. The symmetric energies are $\mathcal{E}(\phi^{\text{G-NPD}}) = 0.0817$, $\mathcal{E}(\phi^{\text{O-NPD}}) = 0.1045$, $\mathcal{E}(\phi^{\text{G-NDF}}) = 0.3233$, and $\mathcal{E}(\phi^{\text{O-NDF}}) = 0.4750$. Thus, the symmetric energy gain due to decision feedback is 6 dB for the GLR-based detectors and 6.6 dB for the optimum detectors. The symmetric energy difference between the optimum and GLR decorrelating detectors is 1.1 dB and that between their decision feedback versions is 1.6 dB. Moreover, the upper and

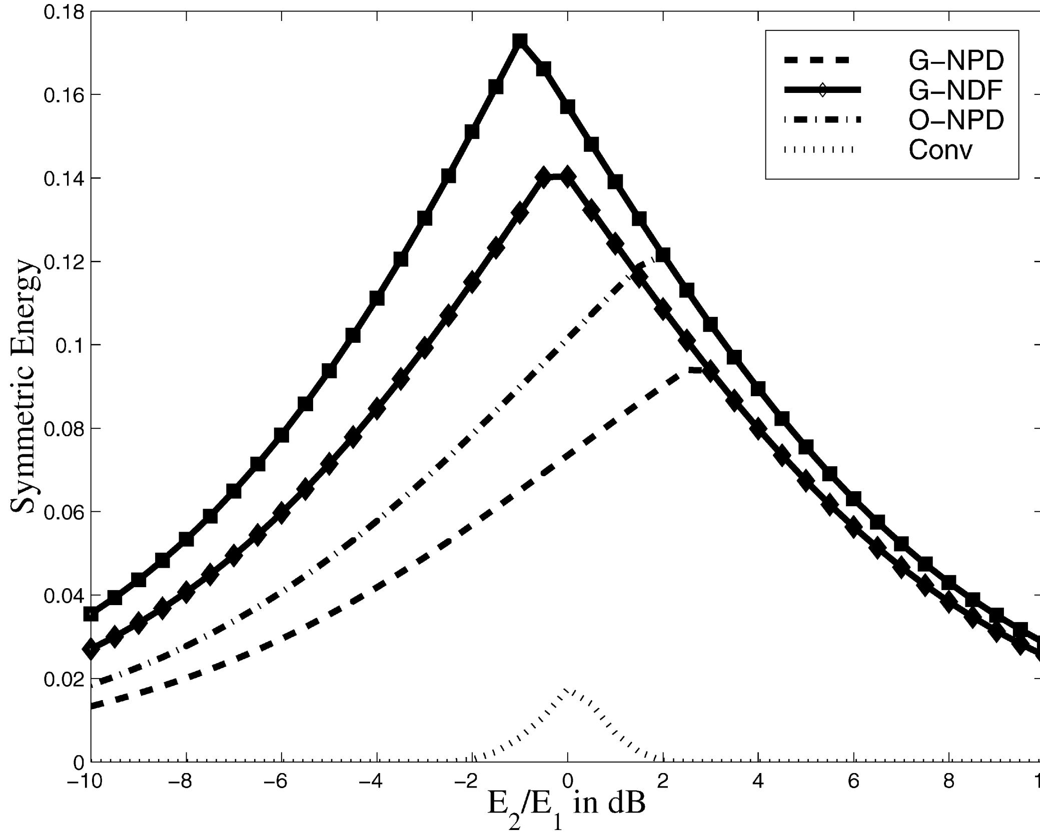

Fig. 2. Symmetric energy versus user energy ratio E_2/E_1 for a two-users system for the conventional detector, the optimum (O-NPD) and GLR-based (G-NPD) parallel detectors, and the optimum (O-NDF) and GLR-based (G-NDF) NC-DFDs for a fixed order of detection.

lower bounds on BER for the G-NDF and the O-NDF detectors converge in Fig. 4 as they must since the condition in (52) is satisfied.

Consider a three-user channel with a signal correlation matrix given by (76) at the bottom of the page, and let $E_1 = 1$, $E_2 = 1.2$ and $E_3 = 1.5$ respectively.

The asymptotic effective energies for the optimum decorrelating detector are $\mathcal{E}_1(\phi^{\text{O-NPD}}) = 0.7078$, $\mathcal{E}_2(\phi^{\text{O-NPD}}) = 0.8281$, and $\mathcal{E}_3(\phi^{\text{O-NPD}}) = 0.7185$. For the decorrelating GLR detector, we have $\mathcal{E}_1(\phi^{\text{G-NPD}}) = 0.6970$, $\mathcal{E}_2(\phi^{\text{G-NPD}}) = 0.8184$, and $\mathcal{E}_3(\phi^{\text{G-NPD}}) = 0.7074$. The genie-aided asymptotic effective energies for the two decision feedback detectors for this order of detection are $\mathcal{E}_2(\phi_g^{\text{O-NDF}}) = 0.8884$, $\mathcal{E}_3(\phi_g^{\text{O-NDF}}) = 0.8884$, and $\mathcal{E}_2(\phi_g^{\text{G-NDF}}) = 0.7985$, $\mathcal{E}_3(\phi_g^{\text{G-NDF}}) = 0.7434$. Hence, every user's asymptotic effective energy for the genie-aided O-NDF detector is at least as high as that for the optimum decorrelating detector. This ensures, by Lemma 1, a gain in symmetric energy for the O-NDF detector. Notice that the same is not true, however, for the GLR-based detectors. An application of the asymptotic effective energy bounds in (51) to this example shows that per-user performance does not necessarily get a boost with decision feedback for the second and the third users for the O-NDF detector. For the G-NDF detector, the performance of the second user, in fact, degrades due to decision feedback.

Next, we consider the same three-user channel for which the ordering in terms of decreasing user energies (which in our example is the order user $1 \rightarrow$ user $2 \rightarrow$ user 3) does not guarantee user-wise performance gain for the O-NDF detector. Ordering Rule I stipulates, based on the asymptotic effective energies for the optimum decorrelating detector, that user 2 be detected first, user 3 next, and user 1 last. In the new order of detection, the asymptotic effective energy for user 2 is 0.8281 and the genie-aided asymptotic effective energy for user 3 is 0.7231, and for user 1 is 0.7477. The reader can use Proposition 6 to check that there is a user-wise performance gain as promised by Proposition 9 for the optimum NDF detector over the optimum nondecision feedback detector.

$$\boldsymbol{R} = \begin{bmatrix} 1.0000 & -0.3585 & -0.1876 & -0.3580 & 0.0847 & -0.0055 \\ -0.3585 & 1.0000 & -0.0208 & 0.1979 & 0.3200 & 0.3000 \\ -0.1876 & -0.0208 & 1.0000 & -0.0453 & -0.0871 & -0.4488 \\ -0.3580 & 0.1979 & -0.0453 & 1.0000 & 0.3167 & 0.1103 \\ 0.0847 & 0.3200 & -0.0871 & 0.3167 & 1.0000 & 0.1645 \\ -0.0055 & 0.3000 & -0.4488 & 0.1103 & 0.1645 & 1.0000 \end{bmatrix} \tag{76}$$

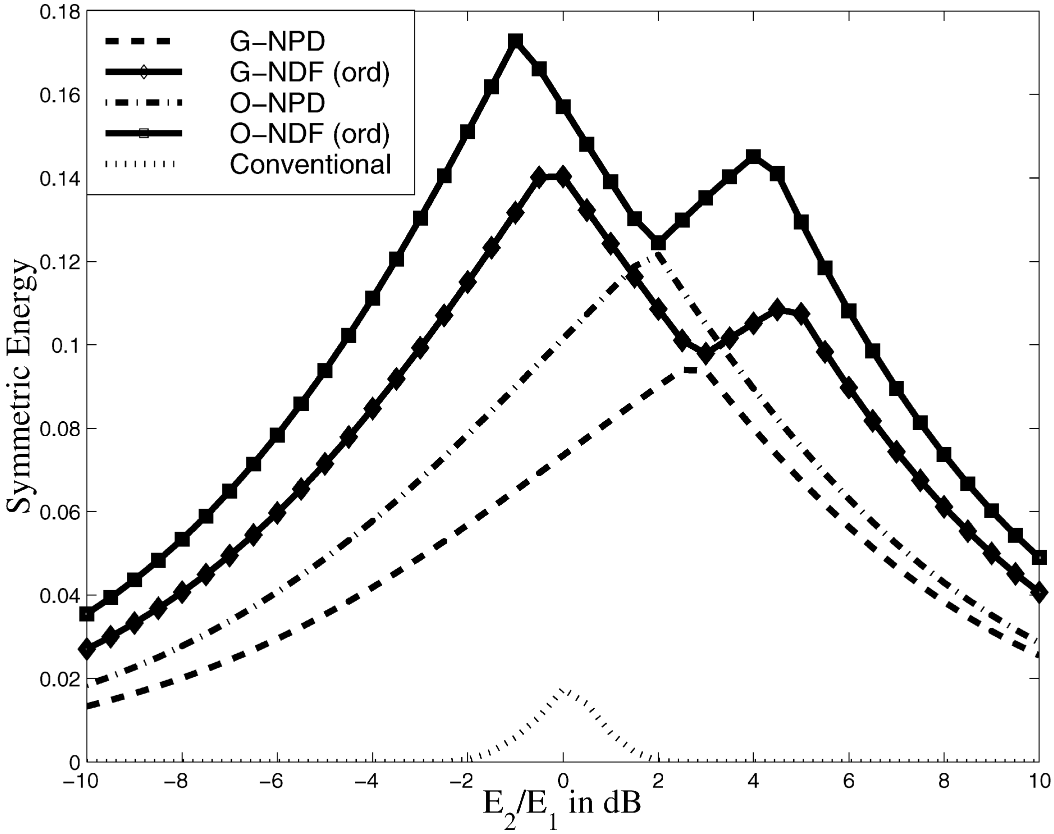

Fig. 3. Symmetric energy versus user energy ratio E_2/E_1 for a two-users system for the conventional detector, the optimum (O-NPD), and GLR-based (G-NPD) parallel detectors, and the optimum (O-NDF) and GLR-based (G-NDF) NC-DFDs with reordering to optimize symmetric energy.

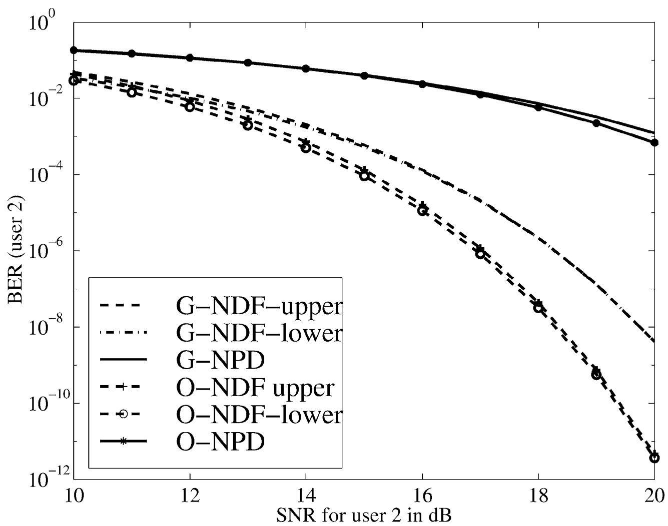

Fig. 4. BER comparison of the optimum NC-DFD (denoted O-NDF in legend), GLR based NC-DFD (G-NDF), optimum decorrelating detector (O-NPD), and GLR decorrelating detector (G-NPD) : $E_1/E_2 = 5$, BER bounds for both $\phi^{\text{G-NDF}}$ and $\phi^{\text{O-NDF}}$ converge.

Ordering Rule II, on the other hand, requires user 2 to be detected first, user 1 second, and user 3 last. For this ordering, asymptotic effective energy for user 2 is 0.8281 and the genie-aided asymptotic effective energy for user 1 is 0.7471, and for user 3 is 0.7690. Again it can be verified using Proposition 6 that there is a user-wise performance gain for the optimum NDF detector over the optimum decorrelating detector as promised by Proposition 10.

IX. Conclusion

An analytical framework for noncoherent decision feedback detection is introduced in this paper in the context of nonorthogonal binary modulation. A new K-map class of noncoherent decision feedback detectors that includes the GLR-based NDF detector of [2] is defined. The optimum NDF detector in this class that maximizes symmetric energy is derived. This detector has the desirable property of not requiring the knowledge of the re-ceived energies and the phases of the users. Even the non-decision feedback counterpart of the O-NDF detector outperforms the decorrelating GLR detector proposed in [3], [4]. Results on performance analysis of the O-NDF detector include lower and upper bounds on symbol error probability from which feasible sets of received energies are found within which it is possible, with power control, to meet QoS constraints in terms of asymptotic effective energy for each user. For fixed received energies, the performance analysis indicates that the O-NDF detector outperforms its non-decision feedback counterpart in terms of symmetric energy regardless of the order of detection. Performance bounds for decision feedback detection yield sufficient conditions under which the O-NDF detector achieves its perfect feedback, or genie-aided, performance, and conditions under which the O-NDF detector outperforms the optimum decorrelating detector. Ordering rules to ensure user-wise performance gains for the optimum NDF detector over the optimum decorrelating detector are also obtained.

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