Diversity-multiplexing tradeoff for QAM based Multiuser Cooperation Strategies

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Abstract — **For multiple users cooperating to communicate to a destination, two practical cooperation schemes based on QAM symbols were recently proposed by the authors. The first scheme was a two-phase cooperation strategy based on QAM space-time codes. The second scheme, known as the Self-information Canceling Linear (SCL) scheme was specifically designed for a two-user system for higher coding gain but at a higher complexity. In this work, we perform a diversity-multiplexing tradeoff (DMT) analysis of the error probability with these two schemes. For multiplexing gains higher than zero, the size of the inherent QAM constellation is increased with the signal-to-noise ratio. The DMT analysis is performed accounting for the decoding errors in the inter-user communication. It is found that the two-phase scheme achieves the diversity order of the outage probability of this strategy for all multiplexing gains. However, the diversity order with the SCL scheme is uniformly greater than that with the two-phase scheme and the SCL scheme also achieves the maximum multiplexing gain inherent in the system. Such achievability results had not been obtained before for any practical scheme that also accounted for the inter-user communication errors.**

I. Introduction

In a multiuser wireless system, cooperative communication can improve the reliability of each user's information at the receiver. Practical cooperation strategies based on QAM information symbols were proposed and analyzed by the authors in [1]. Two practical schemes, namely, a two-phase scheme and a Self-information Canceling Linear (SCL) scheme were proposed in [1]. The performance analysis of these schemes was presented accounting for the users not always detecting a decoding error during the inter-user communication. It was shown that for m users, the two-phase scheme achieves the full diversity order of m at a fixed rate. It was also shown that the SCL scheme achieves the maximum diversity order of 2 in a two user system.

In this paper, we analyze these two schemes from a diversity-multiplexing tradeoff (DMT) point of view. Hence, this work is a generalization of the performance analysis of the schemes of [1] to the case when the rate is allowed to increase with the signal-to-noise ratio. For the DMT analysis presented in this paper, we scale the QAM constellation size with the signal-to-noise ratio and compute the resulting diversity order.

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The concept of diversity-multiplexing tradeoff (DMT) was proposed originally for the multi-input multi-output (MIMO) fading channel in [2]. It was shown therein that the diversity order of the outage probability is an upper bound on the best achievable diversity order on the MIMO channel for all multiplexing gains. To prove the achievability of the DMT curve of the outage probability, the diversity order of the error probability of a Gaussian ensemble of codebooks was computed. The optimum DMT tradeoff curve was shown to be achieved in some cases using the Gaussian codebook method but no new code design rules were provided. More recently, a new set of techniques is presented in [3] for the DMT analysis of error probabilities on the multi-input multi-output (MIMO) channel. It was shown in [3] that space-time codes based on QAM information symbols can also achieve the optimal DMT curve, namely, the DMT curve of the MIMO outage probability. The property required for the space-time codes based on QAM symbols to meet the optimal DMT curve thereby serves as a useful design criterion for high performance practical space-time codes.

In this work, we utilize the techniques presented in [3] to perform the DMT analysis of the practical cooperation schemes based on QAM information symbols. This analysis reinforces the two schemes proposed in [1] as practical multiuser cooperation strategies for good performance. Also, the choice of certain parameters in the description of these schemes can be inferred from the diversity order results of this paper. Moreover, the cooperation rule employed to decide which users cooperate is modified from the one in [1] to account for the multiplexing gains being greater than zero. However, similar to the analysis in [1], we do not assume that the cooperating users know whether the inter-user communication is error-free.

In the first part of this paper, we perform the DMT analysis for the error probability with the two-phase scheme. In the second part of the paper, we perform the DMT analysis for the error probability with the SCL scheme. In the cooperation schemes considered in this paper, each user and the destination is equipped with only one antenna. The fading coefficient between users i and j is denoted by $\mathbf{h}_{i,j}$ and that between user i and the destination is denoted by $\mathbf{g}_i$. All fading and noise coefficients are i.i.d zero-mean unit variance complex normal random variables.

II. Two-phase scheme

The two-phase scheme considered in this section is based on the system model proposed in [4] for m cooperating users.

This scheme, originally proposed by the authors in [1], was shown to be a practical cooperation strategy to achieve the maximum diversity order at a fixed rate. In this paper, we allow the rate of the scheme to increase with the signal-to-noise ratio. Therefore, the description of the scheme, as presented next, is slightly different compared to the one in [1]. However, as in [1], we still account for the fact that the users may not detect an error during the inter-user communication.

The entire bandwidth in the two-phase scheme is divided into m non-overlapping bands. In the time domain, communication occurs in two phases, the first being an inter-user communications phase and the second a cooperative communications phase. Each user employs a standard QAM constellation $\mathcal{I}$ consisting of 2^{2R} points on a complex integer grid. The average energy of $\mathcal{I}$ is P but the actual symbols from this QAM constellation are multiplied by a factor θ before being sent by the user. The spectral efficiency is R bits/channel use. We require the coherence time T of the channel to be of the form $T = 2 \cdot 2^{\nu}, \nu \geq 1$, with $m-1 \leq \frac{T}{2}$ if $m > 2$. If $m = 2$, then no constraint on T is needed. We set $M = \frac{T}{2}$ and define a frame as consisting of $T = 2M$ time slots.

In the first phase, each user s transmits M information symbols in the s-th band and simultaneously receives the information from the other users in the remaining bands. At the end of this phase, for each user s, there exists a subset $\mathcal{D}(s)$ of the remaining users that are allowed to cooperatively relay the information of s. The set $\mathcal{D}(s)$ is chosen according to a certain cooperation rule to be described later. The users in $\mathcal{D}(s)$ perform ML decoding to obtain estimates of the QAM symbols of user s using the signal they received in the s-th band. In the second phase, the cooperating users in $\mathcal{D}(s)$, for each s, create a pre-determined row of a space-time code using these estimates and transmit it in the s-th band. The space-time code employed is the rate one LPST code construction proposed in [5] and consists of codewords of the form

$$\theta \begin{bmatrix} z_1 & z_2 & z_3 & \dots & z_M \\ iz_M & z_1 & z_2 & \dots & z_{M-1} \\ \dots & \dots & \dots & \dots & \dots \\ iz_2 & iz_3 & \dots & iz_M & z_1 \end{bmatrix}, \tag{1}$$

where the information symbols $z_1, \dots, z_M$ are drawn from $\mathcal{I}$. The LPST code given above exhibits the full transmit diversity of M [5] on the MIMO channel. Therefore, the space-time code of size $\mathcal{D}(s) \times M$ obtained by using any $\mathcal{D}(s)$ fixed rows of the LPST code in (1) will exhibit a transmit diversity of $\mathcal{D}(s)$ on the MIMO channel.

The destination processes the received statistics from both phases with the assumption that the estimates obtained by the cooperating users in $\mathcal{D}(s)$ are the actual information symbols of user s, for each s, and applies the sphere-decoding algorithm [6] with a rearranged channel matrix.

The cooperation rule used to select the set $\mathcal{D}(s)$ is slightly different from the one used in [1]. We choose a particular cooperation rule so that for two users q and s, user q belongs to $\mathcal{D}(s)$ only when the following condition is met,

$$\log(1 + |\mathbf{h}_{q,s}|^2\theta^2 P) \geq 2Rc, \tag{2}$$

where c is a constant that we specify later. Due to the presence of the cooperation rule, any given user does not transmit for all time. However, the average signal-to-noise ratio SNR satisfies

$$\theta^2 P/m \leq SNR \leq \theta^2 P, \tag{3}$$

which implies that [1]

$$SNR \doteq \theta^2 P. \tag{4}$$

In the diversity-multiplexing tradeoff (DMT) formulation, the spectral efficiency R is allowed to grow with SNR as $\log(SNR^r)$, where r is known as the multiplexing gain. Thus, we have that $2R = \log(SNR^{2r})$, which means that the average energy of the QAM constellation scales as $P \doteq SNR^{2r}$. The constant θ should then be chosen to satisfy $\theta^2 \doteq SNR^{1-2r}$ so that the SNR constraints imposed by (4) are met.

In [1], the constant c in the cooperation rule (2) was replaced by a function $c(P)$ of P. A careful choice of this function was necessary to perform an analysis for fixed rate schemes ($i.\ e.$, $r = 0$). In this paper, however, we consider multiplexing gains greater than 0 and for this case choosing c to be a constant suffices. The DMT analysis of the two phase scheme described here is summarized in the following proposition.

Proposition 1: For $m = 2$ and multiplexing gains $r > 0$, the error probability of the two-phase scheme achieves the diversity order of $2(1-2r)$.

Proof: Let us first restrict ourselves to a multiplexing gain $r \in (0, \frac{1-\delta}{2})$, for some $\delta \in (0,1)$. Let us also choose the constant c such that

$$1 < c < \frac{1}{1-\delta} \tag{5}$$

With this choice of c, we would have that $2cr < 1$ and

$$\frac{1-\delta}{2} < \frac{1}{2c} < \frac{1}{c+1}, \tag{6}$$

which would be made use of later in the proof. Let $\theta\mathbf{x}_1$ be the symbol transmitted by user 1, where $\mathbf{x}_1$ is a symbol from the QAM constellation. Let $\mathcal{C}$ denote the event that the users cooperate and $\bar{\mathcal{C}}$ that they do not. Let $\mathcal{E}$ denote the event that the destination makes an error in decoding when $\mathbf{x}_1$ is sent by user 1. The probability of error can be written as

$$P(\mathcal{E}) = P(\mathcal{E}/\mathcal{C})P(\mathcal{C}) + P(\mathcal{E}/\bar{\mathcal{C}})P(\bar{\mathcal{C}}). \tag{7}$$

The probabilities of $\mathcal{C}$ and $\bar{\mathcal{C}}$ are easily found as

$$\begin{aligned} P(\mathcal{C}) &= P(|\mathbf{h}_{1,2}|^2 \geq \frac{2^{2Rc}-1}{\theta^2 P}) = \exp\left(-\frac{SNR^{2rc}-1}{\theta^2 P}\right) \\ P(\bar{\mathcal{C}}) &= 1 - P(\mathcal{C}) \leq \left(\frac{SNR^{2rc}-1}{\theta^2 P}\right). \end{aligned}$$

To evaluate $P(\mathcal{E}/\bar{\mathcal{C}})$, note that conditioned on $\bar{\mathcal{C}}$, the destination only needs to process the signal it receives in the first phase to decode $\mathbf{x}_1$. This received signal is given by

$$\mathbf{r} = \theta\mathbf{g}_1\mathbf{x}_1 + \mathbf{n}, \tag{8}$$

[1] $f \doteq g$ means that $\lim_{SNR\to\infty} \frac{\log(f)}{\log(SNR)} = \lim_{SNR\to\infty} \frac{\log(g)}{\log(SNR)}$, $\dot{\geq}$ and $\dot{\leq}$ defined similarly

where $\mathbf{n}$ is the noise variable. We now use a *sphere-bounding* technique [3] to upper bound $P(\mathcal{E}/\bar{\mathcal{C}})$. For a given realization of $\mathbf{g}_1$, the destination does not make an error if the noise $\mathbf{n}$ lies within a sphere of radius $|\mathbf{g}_1\theta|$ centered at $\mathbf{x}_1$. The probability that the noise lies outside this sphere is therefore an upper bound on the decoding error probability. Averaging this upper bound over the distribution of $\mathbf{g}_1$, we obtain

$$P(\mathcal{E}/\bar{\mathcal{C}}) \quad \leq \quad \int e^{-\theta^2|\mathbf{g}_1|^2}e^{-|\mathbf{g}_1|^2}d|\mathbf{g}_1|^2 = \frac{1}{1+\theta^2}.$$

We thus get that

$$P(\mathcal{E}/\bar{\mathcal{C}})P(\bar{\mathcal{C}}) \leq \frac{SNR^{2rc}-1}{\theta^2(1+\theta^2)P} \doteq SNR^{-2(1-(c+1)r)}.$$

As a function of r, the diversity order of the term $P(\mathcal{E}/\bar{\mathcal{C}})P(\bar{\mathcal{C}})$ is therefore the part of the straight line segment joining $(r=0, d=2)$ and $(r=1/(c+1), d=0)$ between $r=0$ and $r=\frac{1-\delta}{2} < 1/(c+1)$. This is the bold curve shown in Figure 1.

To evaluate $P(\mathcal{E}/\mathcal{C})$, we further condition on the event that user 2 makes an error in decoding user 1's information before taking part in the cooperation phase. If $\hat{\mathbf{x}}_1$ denotes the estimate of user 1's information made by user 2, we can write

$$P(\mathcal{E}/\mathcal{C}) \leq P(\mathcal{E}/\mathcal{C}, \hat{\mathbf{x}}_1 = \mathbf{x}_1) + P(\hat{\mathbf{x}}_1 \neq \mathbf{x}_1/\mathcal{C}). \tag{9}$$

Let $\mathbf{h} = \mathbf{h}_{1,2}$. The signal received by user 2 in the first phase is

$$\mathbf{y} = \mathbf{h}\theta\mathbf{x}_1 + \mathbf{w}, \tag{10}$$

where $\mathbf{w}$ is the noise variable. The conditional density of $|\mathbf{h}|^2$ conditioned on $\mathcal{C}$ is

$$\frac{e^{-|\mathbf{h}|^2}}{\exp(-\frac{SNR^{2rc}-1}{\theta^2 P})} \cdot \chi_{[\frac{SNR^{2rc}-1}{\theta^2 P},\infty)}(|\mathbf{h}|^2), \tag{11}$$

where $\chi_I(.)$ is the indicator function for the set I. Using the sphere-bounding approach again, we can write

$$\begin{aligned} P(\hat{\mathbf{x}}_1 \neq \mathbf{x}_1/\mathcal{C}) \quad &\leq \quad \frac{\int_{|\mathbf{h}|^2 \in [\frac{SNR^{2rc}-1}{\theta^2 P},\infty)} e^{-\theta^2|\mathbf{h}|^2}e^{-|\mathbf{h}|^2}d|\mathbf{h}|^2}{\exp(\frac{SNR^{2rc}-1}{\theta^2 P})} \\ &= \quad \frac{\exp(-\frac{SNR^{2rc}-1}{P})}{1+\theta^2} \doteq SNR^{-\infty}, \end{aligned}$$

where the last step follows because $c > 1$. Hence, the term $P(\hat{\mathbf{x}}_1 \neq \mathbf{x}_1/\mathcal{C})$ can be neglected when computing the diversity order of $P(\mathcal{E}/\mathcal{C})$ in (9). The diversity order of $P(\mathcal{E}/\mathcal{C})$ is equal to the diversity order of the other term $P(\mathcal{E}/\mathcal{C}, \hat{\mathbf{x}}_1 = \mathbf{x}_1)$ in (9). This corresponds to the case that user 2's estimate of user 1's data is error free. The statistic that the destination employs under the event $\mathcal{C}$ can be written as

$$\mathbf{s} = \sqrt{|\mathbf{g}_1|^2 + |\mathbf{g}_2|^2}\theta\mathbf{x}_1 + \mathbf{n}. \tag{12}$$

Using the sphere-bounding approach again, we get that

$$\begin{aligned} P(\mathcal{E}/\mathcal{C}, \hat{\mathbf{x}}_1 = \mathbf{x}_1) \quad &\leq \quad E_{\mathbf{g}_1,\mathbf{g}_2}[e^{-(|\mathbf{g}_1|^2+|\mathbf{g}_2|^2)\theta^2}] \quad (13) \\ = \frac{1}{(1+\theta^2)^2} \quad &\doteq \quad SNR^{-2(1-2r)}. \quad (14) \end{aligned}$$

The curve corresponding to the diversity order as a function of r for $P(\mathcal{E}/\mathcal{C})$ therefore lies completely above the curve obtained for $P(\mathcal{E}/\bar{\mathcal{C}})P(\bar{\mathcal{C}})$ above. The diversity order of $P(\mathcal{E}/\mathcal{C})P(\mathcal{C})$ is even higher than that of $P(\mathcal{E}/\mathcal{C})$. Thus, the diversity order of $P(\mathcal{E})$ is at least the diversity order of the term $P(\mathcal{E}/\bar{\mathcal{C}})P(\bar{\mathcal{C}})$ which is at least

$$d \geq 2(1-(c+1)r), \quad r \in (0, \frac{1-\delta}{2}). \tag{15}$$

Now, since δ was arbitrary, we choose $\delta \to 0$ to get that the diversity order achievable by the proposed scheme is given by $d \geq 2(1-2r), \quad r \in (0, 1/2)$. ■

It is possible to generalize the techniques used in the proof above to the case of $m > 2$ using a certain sphere-bounding technique presented in [3]. For $m > 2$, we first write the error probability $P(\mathcal{E})$ for user 1 as the sum

$$\sum_{\mathcal{O} \subset \{2,\ldots,m-1\}} P(\mathcal{E}/\mathcal{O})P(\mathcal{O}),$$

where $\mathcal{O}$ is the set of users cooperating for the first user's information in the second phase. Then, $P(\mathcal{E}/\mathcal{O})$ is upper bounded by conditioning on the event $\mathcal{I}$ that any of the estimates of user 1's data made by the users in $\mathcal{O}$ in phase one is incorrect. Thus,

$$P(\mathcal{E}/\mathcal{O}) \leq P(\mathcal{E}/\mathcal{O}, \overline{\mathcal{I}}) + P(\mathcal{I}/\mathcal{O})$$

The probability $P(\mathcal{I}/\mathcal{O})$ decays exponentially with SNR due to the nature of our cooperation rule and hence does not contribute to the diversity order. To compute the diversity order of $P(\mathcal{E}/\mathcal{O}, \overline{\mathcal{I}})$, we note that the estimates made by users in $\mathcal{O}$ are all correct and then apply the sphere-bound approach of [3]. The resulting diversity order is summarized next.

Proposition 2: For any m and multiplexing gains $r > 0$, the error probability of the two-phase scheme achieves the diversity order of $m(1-2r)$.

The results of Propositions 1 and 2 imply that the proposed practical cooperation scheme achieves the same diversity order as that of the outage probability for this model [4] for any number of users. The achievability of the diversity order promised by the outage probability had not been shown before using a practical cooperation scheme that also accounted for the inter-user communication errors.

III. Two-user SCL scheme

In this section, we consider the two-user Self-Information Canceling Linear (SCL) scheme proposed in [1]. In this scheme, which is meant for two users, there is no division of bandwidth into separate bands or division of time into separate phases. It was shown in [1] that the SCL scheme achieves the maximum diversity order of 2 at a fixed rate ($r = 0$). The performance of the SCL scheme was also shown to be better than the two-phase scheme due to the smaller size of the QAM constellation used. In this paper, we consider the diversity order of the SCL scheme for multiplexing gains higher than 0 and, therefore, modify the description of SCL scheme appropriately. The description of the SCL scheme and the associated DMT analysis of the error probability are presented now.

Let the coherence time $T = n$ time slots with $n >> 1$. Each of the two users in the SCL scheme are active in alternate time slots. The information symbol for each user is drawn from a 2^R-QAM constellation of average energy P, where R is the spectral efficiency of the scheme. The information symbols are scaled by a factor θ. The two users cooperate if

$$\log(1 + \theta^2 P|\mathbf{h}_{1,2}|^2) \geq cR, \quad (16)$$

where $c > 1$ is a constant. When (16) is met, the scheme is described as follows. In the first time slot, the first user transmits its information symbol. Subsequently, for time slot $t \in [2, n-1]$, the signal sent by a user is a linear combination of its own information symbol and an estimate of the other users information of time slot $t-1$. This estimate of the other users data is obtained using the statistic obtained in time slot $t-1$, by first subtracting its own information symbol from time slot $t-2$ and then performing an ML decoding. Let (a, b) be the coefficients of the linear combination with $|a|^2 + |b|^2 = 1$. In the last time slot, no new information is transmitted and the active user transmits the estimate of the information of the other user for time slot $n-1$. The destination processes the received signal assuming that all the estimates are error free and performs an equivalent ML decoding under that assumption. When (16) is not met, there is no cooperation and in each time slot the active user only transmits its own information symbol.

For a multiplexing gain r, we have that $R = r\log(SNR)$ which means that the average energy P of the QAM constellation satisfies $P \doteq SNR^r$. The constant θ is now chosen as $\theta^2 \doteq SNR^{1-r}$ so that $\theta^2 P \doteq SNR$.

The coefficients (a, b) play an important role in the performance of the SCL scheme. These coefficients have to be chosen from any appropriately scaled row of a two dimensional Full Modulation Diversity (FMD) generator matrix $\mathbf{G}$ (for e.g. from [7]). The unnormalized minimum product distance of the lattice generated by $\mathbf{G}$ is defined as

$$\min_{\substack{\mathbf{u} \in \mathcal{Z}[i]^2 \setminus 0 \\ \mathbf{x} = \mathbf{G}\mathbf{u}}} |\mathbf{x}_1 \mathbf{x}_2|^2$$

The DMT curve of the error probability with the SCL scheme is provided in the following proposition.

Proposition 3: For the SCL scheme, let the row vector $[\ a \quad b\ ]$ correspond to any appropriately scaled row of a two dimensional FMD generator matrix which generates a lattice with non-zero unnormalized minimum product distance. Then, for multiplexing gains $r > 0$, the error probability of the two-user SCL scheme achieves a diversity order given by

$$\begin{aligned} 2 - 3r, &\quad 0 < r \leq 0.5, \\ 1 - r, &\quad 0.5 \leq r < 1. \end{aligned}$$

Proof: Once again, we begin by restricting $r \in (0, 1-\delta)$ for some $\delta \in (0, 1)$ and by choosing a c such that

$$1 < c < \frac{1}{1-\delta}. \quad (17)$$

With this choice of c, we have $cr < 1$ and

$$1 - \delta < \frac{1}{c} < \frac{2}{c+1}. \quad (18)$$

For time slot i, let $\mathbf{x}_i$ be the new QAM information symbol for user 1 if i is odd and for user 2 if i is even. We also assume that n is even because the proof is analogous for n being odd. Let $\hat{\mathbf{x}}_i$ denote the estimate of $\mathbf{x}_i$ obtained by the cooperating user. Thus, if i is odd, then $\hat{\mathbf{x}}_i$ is the estimate made by user 2 in time slot i. If i is even, then $\hat{\mathbf{x}}_i$ is the estimate made by user 1 in time slot i. Let $\mathcal{C}$ denote the event that the users cooperate and $\bar{\mathcal{C}}$ the event that they do not. Let $\mathcal{E}$ denote the event that the destination makes an error in decoding the information stream of user 1. Using the same principle as before, we can write

$$P(\mathcal{E}) = P(\mathcal{E}/\mathcal{C})P(\mathcal{C}) + P(\mathcal{E}/\bar{\mathcal{C}})P(\bar{\mathcal{C}}). \quad (19)$$

Consider the second term first. If we let $\mathbf{h} = \mathbf{h}_{1,2}$, then

$$P(\bar{\mathcal{C}}) = P(|\mathbf{h}|^2 < \frac{SNR^{rc} - 1}{\theta^2 P}) \leq \frac{SNR^{rc} - 1}{\theta^2 P}. \quad (20)$$

Using the sphere-bounding approach again, one can write

$$P(\mathcal{E}/\bar{\mathcal{C}}) \dot{\leq} \frac{n/2}{1+\theta^2} \doteq SNR^{-(1-r)}, \quad (21)$$

which leads to

$$P(\mathcal{E}/\bar{\mathcal{C}})P(\bar{\mathcal{C}}) \dot{\leq} SNR^{(c+1)r-2}. \quad (22)$$

Thus, the diversity order of $P(\mathcal{E}/\bar{\mathcal{C}})P(\bar{\mathcal{C}})$ is at least the part of the straight line joining $(r = 0, d = 2)$ and $(r = \frac{2}{c+1}, d = 0)$ between $r = 0$ and $r = 1 - \delta < \frac{2}{c+1}$. This is the upper bold curve shown in Figure 2

In order to find the diversity order of $P(\mathcal{E}/\mathcal{C})$, we further condition on the event that any of the estimates made by the two cooperating users are in error. As before, we can write

$$P(\mathcal{E}/\mathcal{C}) \leq P(\mathcal{E}/\mathcal{C}, \hat{\mathbf{x}}_i = \mathbf{x}_i \forall i) + P(\hat{\mathbf{x}}_i \neq \mathbf{x}_i, \text{ for any } i/\mathcal{C}).$$

As shown next, the second term above decays exponentially with SNR which would mean that the diversity order of $P(\mathcal{E}/\mathcal{C})$ is at least that of the first term above.

In the i-th time slot, the signal received by the cooperating user may be written as

$$\mathbf{y}_i = \mathbf{h}(\theta a \mathbf{x}_i + \theta b \hat{\mathbf{x}}_{i-1}) + \mathbf{w}_i, \quad (23)$$

where $\mathbf{w}_i$ are the noise variables. The probability that any of the estimates made by the cooperating users is incorrect for any given inter-user channel realization $\mathbf{h}$ is

$$P_{\mathbf{h}}(\hat{\mathbf{x}}_i \neq \mathbf{x}_i, \text{ for any } i) = P_{\mathbf{h}}(\cup_{i=1}^{n-1}(\hat{\mathbf{x}}_i \neq \mathbf{x}_i)). \quad (24)$$

We write this event as a disjoint union of events obtained from the self-information cancellation being error free until time slot $i-1$ for each i. When self-information cancellation is error free until time slot $i-1$, then the estimate in time slot i being incorrect only depends on the noise variable $\mathbf{w}_i$. Thus,

$$\begin{aligned} P_{\mathbf{h}}(\cup_{i=1}^{n-1}(\hat{\mathbf{x}}_i \neq \mathbf{x}_i)) &= P_{\mathbf{h}}(\cup_{i=1}^{n-1}(\hat{\mathbf{x}}_i \neq \mathbf{x}_i \setminus \cup_{j=1}^{i-1} (\hat{\mathbf{x}}_j \neq \mathbf{x}_j))) \\ &= \sum_{i=1}^{n-1} P_{\mathbf{h}}(\hat{\mathbf{x}}_i \neq \mathbf{x}_i \setminus \cup_{j=1}^{i-1} (\hat{\mathbf{x}}_j \neq \mathbf{x}_j)) \quad (25) \\ &\leq \sum_{i=1}^{n-1} P_{\mathbf{h}}(|\mathbf{w}_i| > \theta|\mathbf{h}a|) \leq (n-1)e^{-\theta^2|\mathbf{h}a|^2}. \quad (26) \end{aligned}$$

The probability of any estimate being incorrect conditioned on the users cooperating can therefore be upper bounded by taking the expectation of (26) with respect to the conditional density of $\mathbf{h}$ given by

$$\frac{e^{-|\mathbf{h}|^2}}{\exp(-\frac{SNR^{rc}-1}{\theta^2 P})} \cdot \chi_{[\frac{SNR^{rc}-1}{\theta^2 P},\infty)}(|\mathbf{h}|^2), \quad (27)$$

Thus, we have that

$$P(\hat{\mathbf{x}}_i \neq \mathbf{x}_i, \text{ for any } i/\mathcal{C}) \leq \frac{n-1}{1+\theta^2|a|^2} e^{-|a|^2 \frac{SNR^{cr}-1}{P}}, \quad (28)$$

which decays exponentially with SNR because $c > 1$.

To find the diversity order of $P(\mathcal{E}/\mathcal{C}, \hat{\mathbf{x}}_i = \mathbf{x}_i, \forall i)$, we note that conditioned on all the estimates being error-free, the effective signal received at the destination is similar to the case wherein there is only one user but with two transmit antennas and which employs a space-time code of the form shown below.

$$\mathbf{S}(\{\mathbf{x}_i\}) = \theta \begin{bmatrix} a\mathbf{x}_1 & 0 & a\mathbf{x}_3 + b\mathbf{x}_2 & \dots & 0 \\ 0 & a\mathbf{x}_2 + b\mathbf{x}_1 & 0 & \dots & b\mathbf{x}_{n-1} \end{bmatrix}. \quad (29)$$

For two valid codewords $\mathbf{S}(\{\mathbf{x}_i\})$ and $\mathbf{S}(\{\mathbf{x}'_i\})$ and a particular pair of channel realizations $\mathbf{g}_1$ and $\mathbf{g}_2$, the Euclidean distance between the two codewords in the equivalent faded constellation is

$$\theta^2 \|[\mathbf{g}_1\mathbf{g}_2]\mathbf{S}(\{\Delta\mathbf{x}_i\})\|^2 \quad (30)$$

where $\Delta\mathbf{x}_i = \mathbf{x}_i - \mathbf{x}'_i, 1 \leq i \leq n-1$. Define d_1 and d_2 as

$$d_1(\{\Delta\mathbf{x}_i\}) = |a\Delta\mathbf{x}_1|^2 + (\sum_{l=1}^{n/2-1} |a\Delta\mathbf{x}_{2l+1} + b\Delta\mathbf{x}_{2l}|^2)$$

$$d_2(\{\Delta\mathbf{x}_i\}) = (\sum_{l=1}^{n/2-1} |a\Delta\mathbf{x}_{2l} + b\Delta\mathbf{x}_{2l-1}|^2) + |b\Delta\mathbf{x}_{n-1}|^2,$$

so that

$$\|[\mathbf{g}_1\mathbf{g}_2]\mathbf{S}(\{\Delta\mathbf{x}_i\})\|^2 = |\mathbf{g}_1|^2 d_1(\{\Delta\mathbf{x}_i\}) + |\mathbf{g}_2|^2 d_2(\{\Delta\mathbf{x}_i\}).$$

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the two-dimensional FMD matrix from which the coefficients a and b are derived. Then, from the assumption in the proposition, there exists a constant $d_p > 0$ such that

$$(a\Delta\mathbf{z}_1 + b\Delta\mathbf{z}_2)(c\Delta\mathbf{z}_1 + d\Delta\mathbf{z}_2) \geq d_p \quad (31)$$

for any $\Delta\mathbf{z}_1, \Delta\mathbf{z}_2 \in Z[i]$, with at least one of $\Delta\mathbf{z}_1$ or $\Delta\mathbf{z}_2$ being non-zero. Now, let $\Delta\mathbf{z}_1$ and $\mathbf{z}_2$ correspond to the differences of two pairs of points in the 2^R-QAM constellation with at least one of $\Delta\mathbf{z}_1$ or $\Delta\mathbf{z}_2$ being non-zero. We then have that

$$|(a\Delta\mathbf{z}_1 + b\Delta\mathbf{z}_2)(c\Delta\mathbf{z}_1 + d\Delta\mathbf{z}_2)|^2 \dot{\geq} SNR^0. \quad (32)$$

Due to the energy constraints imposed on the chosen constellation, we have that

$$|c\Delta\mathbf{z}_1 + d\Delta\mathbf{z}_2|^2 \dot{\leq} SNR^r, \quad (33)$$

which implies that

$$|a\Delta\mathbf{z}_1 + b\Delta\mathbf{z}_2|^2 \dot{\geq} SNR^{-r}, \quad (34)$$

Using this fact, we can infer the following lower bounds.

$$\min_{\{\Delta\mathbf{x}_i\}} d_1(\{\Delta\mathbf{x}_i\}) \quad \dot{\geq} \quad SNR^{-r} \quad (35)$$

$$\min_{\{\Delta\mathbf{x}_i\}} d_2(\{\Delta\mathbf{x}_i\}) \quad \dot{\geq} \quad SNR^{-r}. \quad (36)$$

Furthermore, it is easy to see by inspection that for a given symbol difference sequence $\{\Delta\mathbf{x}_i\}$, at least one of the terms in the expression for either $d_1(\{\Delta\mathbf{x}_i\})$ or $d_2(\{\Delta\mathbf{x}_i\})$ does not involve a linear combination of two non-zero symbols $\Delta\mathbf{x}_l$. This means that we have one of the following two cases,

$$\min_{\{\Delta\mathbf{x}_i\}} \|[\mathbf{g}_1\mathbf{g}_2]\mathbf{S}(\{\Delta\mathbf{x}_i\})\|^2 \dot{\geq} |\mathbf{g}_1|^2 SNR^0 + |\mathbf{g}_2|^2 SNR^{-r}$$

$$\text{or } \min_{\{\Delta\mathbf{x}_i\}} \|[\mathbf{g}_1\mathbf{g}_2]\mathbf{S}(\{\Delta\mathbf{x}_i\})\|^2 \dot{\geq} |\mathbf{g}_1|^2 SNR^{-r} + |\mathbf{g}_2|^2 SNR^0$$

which therefore leads to

$$\min_{\{\Delta\mathbf{x}_i\}} \|[\mathbf{g}_1\mathbf{g}_2]\mathbf{S}(\{\Delta\mathbf{x}_i\})\|^2 \dot{\geq} \min\left(|\mathbf{g}_1|^2 SNR^0 + |\mathbf{g}_2|^2 SNR^{-r}, \; |\mathbf{g}_1|^2 SNR^{-r} + |\mathbf{g}_2|^2 SNR^0\right).$$

Following [2], we make the substitutions

$$|\mathbf{g}_1|^2 = SNR^{-\alpha_1}, \quad |\mathbf{g}_2|^2 = SNR^{-\alpha_2},$$

so that the minimum Euclidean distance between the codewords in the faded constellation is

$$d^2_{min}(\alpha_1,\alpha_2) = \theta^2 \min_{\{\Delta\mathbf{x}_i\}} \|[\mathbf{g}_1\mathbf{g}_2]\mathbf{S}(\{\Delta\mathbf{x}_i\})\|^2$$
$$\dot{\geq} \min(SNR^{1-r-\alpha_1} + SNR^{1-2r-\alpha_2}, SNR^{1-2r-\alpha_1} + SNR^{1-r-\alpha_2})$$
$$\doteq SNR^{\min(\max(1-r-\alpha_1,1-2r-\alpha_2),\max(1-2r-\alpha_1,1-r-\alpha_2))}.$$

Following [3], the sphere-bound for a particular channel realization pair (α_1, α_2) implies that

$$P(\mathcal{E}/\mathcal{C}, \hat{\mathbf{x}}_i = \mathbf{x}_i, \forall i, \alpha_1, \alpha_2) \leq e^{-d^2_{min}(\alpha_1,\alpha_2)/4} \sum_{k=0}^{n-1} \frac{[d^2_{min}(\alpha_1,\alpha_2)/4]^k}{k!}. \quad (37)$$

We must now average this bound over the density of (α_1, α_2). This can be done by applying the technique of diversity order computation of [2]. Neglecting the regions for (α_1, α_2) where the density of (α_1, α_2) decays exponentially and also where the upper bound of (37) decays exponentially, we get that the diversity order of $P(\mathcal{E}/\mathcal{C}, \hat{\mathbf{x}} = \mathbf{x}_i, \forall i)$ is the minimum of $\alpha_1 + \alpha_2$ over the region defined by the relations

$$\alpha_1 > 0, \alpha_2 > 0,$$
$$\min(\max(1-r-\alpha_1, 1-2r-\alpha_2), \max(1-2r-\alpha_1, 1-r-\alpha_2)) < 0.$$

When $1 - 2r > 0$, this region is shown in Figure 3. The minimum of $\alpha_1 + \alpha_2$ for this case is $(1-r) + (1-2r) = 2 - 3r$.

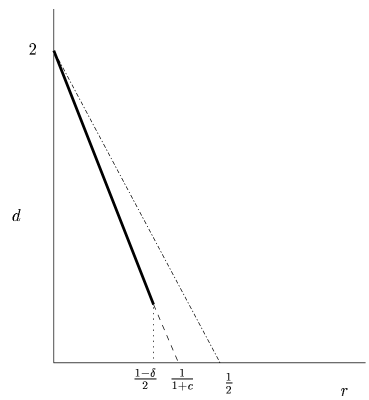

Fig. 1. Two-phase DMT

When $1 - 2r \leq 0$, the required region is shown in Figure 4. The minimum of $\alpha_1 + \alpha_2$ for this case is $1 - r$. Therefore, the diversity order of $P(\mathcal{E}/\mathcal{C})$ is the lower bold curve shown in Figure 2. This diversity order of $P(\mathcal{E})$ is at least the minimum of the two tradeoff curves corresponding to that of $P(\mathcal{E}/\bar{\mathcal{C}})P(\bar{\mathcal{C}})$ and $P(\mathcal{E}/\mathcal{C})$. By taking $\delta \to 0$, we see that the diversity order of $P(\mathcal{E})$ is at least the curve corresponding to $P(\mathcal{E}/\mathcal{C})$ which is given in the proposition. ■

Thus, Proposition 3 implies that the SCL scheme is a practical cooperation scheme to achieve the maximum multiplexing gain of $r = 1$ inherent in the system. Also, the SCL scheme provides a better DMT curve than with the two-phase strategy for all multiplexing gains $r \in (0, 1)$.

The complexity of the SCL scheme, however, is much higher than that of the two-phase scheme. A naive implementation of the SCL scheme for a framelength of T would require a sphere decoder of size T complex dimensions, thereby leading to a complexity that is a non-linear polynomial in T. However, under the assumption of the estimates being error free, the signal at the receiver is similar to that for an inter-symbol interference (ISI) channel. Thus, a variant of the Viterbi decoding algorithm can be employed in the SCL scheme to limit the complexity to be linear in T and the size of the inherent QAM constellation.

IV. CONCLUSIONS

A diversity-multiplexing tradeoff analysis of practical cooperation strategies based on QAM symbols was performed. The diversity order of error probability was computed for multiplexing gains greater than zero for the specific coding scheme obtained by increasing the QAM constellation size with the signal-to-noise ratio. For a two-phase cooperation strategy for multiple users, it was shown that the diversity order of the error probability matches that of the outage probability for this strategy for all multiplexing gains. However, this diversity order is not the best possible among all cooperation strategies. For a two user cooperation strategy known as the SCL scheme, achievability of both maximum diversity order and the maximum possible multiplexing gain inherent in the system was shown. The diversity order of the error probability with the SCL scheme is also uniformly better than that with the two-phase scheme.

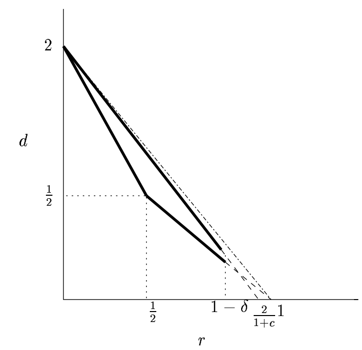

Fig. 2. SCL DMT

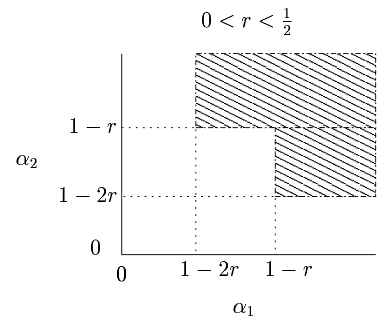

Fig. 3. Feasible region, small r

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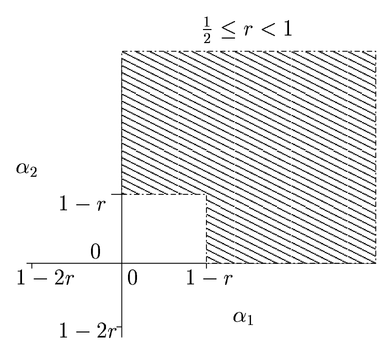

Fig. 4. Feasible region, large r