Algebraic Space-Time Codes with Full Diversity and Low Peak-To-Mean Power Ratio

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Abstract— **A new class of full diversity space-time codes is proposed that leads to a significantly smaller Peak-to-Mean Envelope Power Ratio (PMEPR) compared to the recently invented Diagonal Algebraic Space-Time (DAST) and Threaded Algebraic Space-Time (TAST) codes. Moreover, the proposed "Low PMEPR Space-Time" (LPST) codes exhibit identical performance and decoding complexity compared to the TAST codes. Additionally, unlike the TAST codes, the LPST codes meet an upper bound known as the Singleton bound on the maximum achievable rate by a space-time code. The key to the construction of the LPST codes is an improved spreading scheme that outperforms the originally suggested Hadamard spreading scheme for the PMEPR reduction of DAST codes. Several properties of the LPST codes in relation to PMEPR are studied. Numerical results are presented to support the significant advantage of the proposed codes in terms of reduced PMEPR for high rate wireless communications.**

I. Introduction

The aim of this paper is to design high rate space-time codes that simultaneously achieve both good performance and low PMEPR. The PMEPR is an important parameter to be considered during hardware design because large values of PMEPR pose difficulty in the design of amplifiers and their operating characteristics at the transmitter. The system under consideration is an M transmit and N receive antenna quasi-static fading channel.

With the recently proposed DAST scheme [1] , performance improvement over the orthogonal designs [2] can be obtained for $M > 2$ at high spectral efficiencies. The DAST codes were obtained by rotating a multi-dimensional QAM constellation and placing the co-ordinates of the resulting lattice point along the diagonal of the codeword. This structure of the DAST codes leads to an undesirably large PMEPR due to two reasons. One reason is the expansion of the final transmitted signal constellation and the other is the absence of a signal on the off-diagonal positions of the codewords. The DAST codes also suffer from a low rate of 1 symbol per channel use and do not meet the Singleton bound [3].

The TAST codes [4] are an extension of DAST in the sense that L layers of rotated and scaled information symbols are threaded in a matrix in a non-overlapping fashion and the scaling constants are chosen to ensure full transmit diversity. The TAST codes result in a considerable performance improvement over the DAST codes although at a significant increase in decoding complexity. Moreover, for $L < M$, the PMEPR of the TAST codes is large due to the same reasons as for the DAST code. For $L < M$, the TAST codes do not meet the Singleton bound either even though they have a higher rate of L symbols per channel use compared to the DAST codes. The condition $L < M$ may be required for reducing the decoding complexity of the TAST codes. For instance, when $N < M$, it was suggested in [4] to use $L = N < M$ layers.

In this paper, we show that for certain values of M, the algebraic structure of the complex rotation used in the TAST code can be utilized to obtain a spreading matrix that significantly reduces the PMEPR of the TAST code. The proposed spreading mechanism does not affect either the performance or the decoding complexity of the original TAST code. Moreover, the proposed space-time codes meet the Singleton bound for every value of L. A special case of the proposed spreading mechanism leads to a full diversity space-time code that transmits only QAM information symbols.

This paper is organized as follows. Definitions of the relevant quantities are presented in Section II. The optimum spread of the DAST code is described in Section III-A. The extension of the proposed spreading scheme to TAST and the properties of the new LPST codes are described in Section III-B. Numerical comparisons of the PMEPR of the LPST and TAST codes are presented in Section IV. The conclusions are summarized in Section V.

II. Preliminaries

Let $\mathcal{C}$ be a space-time codebook of $M \times T$ sized complex matrices. Each row of a codeword $\mathbf{X} \in \mathcal{C}$ corresponds to the signal for a particular antenna. The PMEPR for the r-th transmit antenna, denoted by ϵ_r, is defined as

$$\epsilon_r = \frac{\max\limits_{\substack{\mathbf{X}\in\mathcal{C} \\ 1\le t\le T}} |\mathbf{X}_{r,t}|^2}{\mathrm{E}_{\mathbf{X}}\left[\frac{1}{T}\sum_{t=1}^{T}|\mathbf{X}_{r,t}|^2\right]}, \quad 1 \le r \le M. \tag{1}$$

where $\mathrm{E}_{\mathbf{X}}[.]$ denotes the expectation over all codewords $\mathbf{X} \in \mathcal{C}$. All the codewords are assumed to be equally likely. The definition of ϵ_r remains the same irrespective of the fading model and the extent of channel state information known at the receiver. Note that ϵ_r, as defined above, is invariant to a scaling of the codebook $\mathcal{C}$ by a constant. Hence, the PMEPR is independent of the signal-to-noise ratio.

It will also be useful to define the PMEPR of any multi-dimensional complex vector valued constellation. For an M-dimensional constellation $\mathcal{S}$, the PMEPR of the r-th co-ordinate is defined as

$$\zeta_r = \frac{\max\limits_{\mathbf{s}\in\mathcal{S}} |\mathbf{s}_r|^2}{\mathrm{E}_{\mathbf{s}}\left[|\mathbf{s}_r|^2\right]}, \quad 1 \le r \le M, \tag{2}$$

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where $E_{\mathbf{s}}[.]$ denotes the expectation over all constellation points $\mathbf{s} \in \mathcal{S}$. The symbol $\mathcal{I}$ denotes a finite constellation of q points in the complex plane and will be one of two types, $\mathcal{I}_i$ or $\mathcal{I}_j$, in this paper. Let $i = \sqrt{-1}$ and $j = e^{i\frac{2\pi}{3}}$. The constellations $\mathcal{I}_i$ and $\mathcal{I}_j$ are obtained as

$$\mathcal{I}_i = \{a + ib : a, b \text{ odd integers}, -\sqrt{q}+1 \leq a, b \leq \sqrt{q}-1\},$$
$$\mathcal{I}_j = \{a + jb : a, b \text{ odd integers}, -\sqrt{q}+1 \leq a, b \leq \sqrt{q}-1\},$$

where the size q is an even power of 2. The constellation $\mathcal{I}_i$ is the standard q-QAM constellation and $\mathcal{I}_j$ is carved out of the well known A_2 lattice [5]. The average energy of $\mathcal{I}$ is denoted by E_{av} and a constellation point in $\mathcal{I}$ with the maximum envelope by z_{max}. The PMEPR of $\mathcal{I}$ is given by $\epsilon_{\mathcal{I}} = \frac{|z_{max}|^2}{E_{av}}$. In particular, $\epsilon_{\mathcal{I}_i} = 3\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)$ and $\epsilon_{\mathcal{I}_j} = \frac{9}{2}\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)$.

If A is the number of possible values for each position of a codeword in $\mathcal{C}$ and $d(\mathcal{C})$ is the transmit diversity of $\mathcal{C}$, then the quantity $r_m(\mathcal{C}) = \frac{1}{T}\log_A |\mathcal{C}|$, called the modulation rate, is upper bounded as $r_m(\mathcal{C}) \leq M - d(\mathcal{C}) + 1$ for $T \geq M$. This bound is referred to as the Singleton bound for the maximum rate in [3].

Notation: the operator $\otimes$ refers to the Kronecker product of two matrices [6]. The K dimensional identity matrix is denoted by $\mathbf{I}_K$. The symbol $\lfloor\cdot\rfloor$ represents the floor function and the symbol $\lfloor\cdot\rfloor_M$ represents modulo operation of an integer with respect to the integer M. Let Q, R and C denote the fields of rational, real and complex numbers.

III. Low PMEPR Space-Time Codes

The space-time codes $\mathcal{C}$ considered in this section are square designs ($T = M$). The DAST and TAST codes are known to achieve the full transmit diversity of M [1,4]. In the following, the construction of the DAST and TAST codes is described with a more insightful representation. Such a representation reveals a structure that aids in solving the problem of PMEPR reduction for both DAST and TAST. The relevant results of algebraic number theory applied in this paper can be found in [7,8].

Consider the following two disjoint sets of integers

$$\begin{aligned}\mathcal{M}_1 &= \{m | m = 2^{\kappa}, \kappa \in Z^+\}\\ \mathcal{M}_2 &= \{m | m = 3^{\kappa_1}2^{\kappa_2}, \kappa_1 \in Z^+, \kappa_2 \in Z^+ \cup \{0\}\}.\end{aligned}$$

The algebraic number fields needed in this section will depend on whether $M \in \mathcal{M}_1$ or $M \in \mathcal{M}_2$. However, to simplify the presentation, a common notation will be followed for the two cases described next.

1. $M \in \mathcal{M}_1$: Define the element $\theta = e^{\frac{i2\pi}{4M}}$ so that the algebraic number field $Q(\theta)$ contains $Q(i)$ as a subfield. The degree $[Q(\theta) : Q(i)] = M$ and the minimal polynomial of θ over $Q(i)$ is $\mu_{Q(i),\theta}(x) = x^M - e$, where the constant $e = i$. Moreover, $Q(\theta)$ is a Galois extension of $Q(i)$. Set $F = Q(i), O_F = Z[i]$ and the finite constellation $\mathcal{I} = \mathcal{I}_i$.
2. $M \in \mathcal{M}_2$: In this case, define the element $\theta = e^{\frac{i2\pi}{6M}}$ so that the algebraic number field $Q(\theta)$ contains $Q(j)$ as a subfield. The degree $[Q(\theta) : Q(j)] = M$ and the minimal polynomial of θ over $Q(j)$ is $\mu_{Q(j),\theta}(x) = x^M - e$, where the constant $e = -j^2 = 1 + j$. Once again, $Q(\theta)$ is a Galois extension of $Q(j)$. Set $F = Q(j), O_F = Z[j]$ and the finite constellation $\mathcal{I} = \mathcal{I}_j$.

The M conjugates of θ in both cases above are given by $\theta^{(k)} = \omega_M^{k-1}\theta, 1 \leq k \leq M$, where $\omega_M = \exp(\frac{2i\pi}{M})$. The set $(1, \theta, \theta^2, \ldots, \theta^{M-1})$ forms a basis of the extension $Q(\theta)/F$. For a set of M numbers $(z_1, \ldots, z_M) \in O_F^M$, the element $\alpha = \sum_{l=1}^{M} z_l\theta^{l-1}$ is an algebraic integer in $Q(\theta)$. Let each information symbol z_l be chosen from the finite subset $\mathcal{I}$ of O_F. Let $(\alpha^{(1)}, \ldots, \alpha^{(M)})$ be the conjugates of α obtained by the application of the M automorphisms of $\text{Gal}(Q(\theta)/F)$ to α. We then have that

$$\alpha^{(k)} = \sum_{l=1}^{M} z_l\theta^{l-1}\omega_M^{(k-1)(l-1)}, \quad 1 \leq k \leq M. \quad (3)$$

One can view the conjugates of α as the co-ordinates of an M-dimensional complex constellation $\mathcal{S}$. This constellation is, in fact, a rotation of the input $\mathcal{I}^M$ constellation and exhibits full modulation diversity [8]. The generator matrix of the rotated constellation is $\mathbf{G} = \mathbf{S}_M\mathbf{D}_\theta$, where $\mathbf{D}_\theta = \text{diag}(1, \theta, \ldots, \theta^{M-1})$ and $\mathbf{S}_M$ is the $M \times M$ IDFT matrix given by $[\mathbf{S}_M]_{kl} = \omega_M^{(k-1)(l-1)}$. Thus, $\mathcal{S} = \{\mathbf{Gu} | \mathbf{u} \in \mathcal{I}^M\}$. Let ζ_s be the PMEPR for the s-th co-ordinate of the rotated constellation $\mathcal{S}$.

A. *Optimum Spread for DAST*

The general construction of the DAST code is given by [1,9]

$$\mathbf{\Xi}_M(z_1, \ldots, z_M) = \mathbf{U}\text{diag}(\alpha^{(1)}, \ldots, \alpha^{(M)})\mathbf{V}, \quad (4)$$

where $\mathbf{U}$ and $\mathbf{V}$ are fixed full rank matrices normalized so that the transmitted power is the same as that with $\mathbf{U} = \mathbf{I}_M$ and $\mathbf{V} = \mathbf{I}_M$. If the matrices $\mathbf{U}$ and $\mathbf{V}$ are unitary then the performance of the code is exactly the same as that with $\mathbf{U} = \mathbf{I}_M$ and $\mathbf{V} = \mathbf{I}_M$ [9, 10]. When $\mathbf{U}$ and $\mathbf{V}$ are identity matrices, the PMEPR of the transmit antennas is found using (1) and (2) to be $\epsilon_r = M\zeta_r, 1 \leq r \leq M$. The factor of M in ϵ_r appears due to the absence of signal on the off-diagonal positions of the codeword. This factor can be avoided if M is a Hadamard dimension, i. e., 2 or a multiple of 4, by choosing $\mathbf{V} = \mathbf{I}_M$ and $\mathbf{U}$ to be the normalized Hadamard matrix $\frac{1}{\sqrt{M}}\mathcal{H}_M$ [1]. An $M \times M$ matrix $\mathcal{H}_M$ is a Hadamard matrix if it consists of ± 1 only and if $\mathcal{H}_M^{\mathsf{T}}\mathcal{H}_M = M\mathbf{I}_M$. The effect of the Hadamard transform is to repeat a scaled version of $\alpha^{(j)}$ in the j-th column. The PMEPR of the DAST code with the Hadamard transform is $\epsilon_r = \max_{1 \leq s \leq M} \zeta_s, 1 \leq r \leq M$.

A closed form expression for ζ_s, was obtained in [11] for $M \in \mathcal{M}_1$ and is given by

$$\zeta_s = \epsilon_{\mathcal{I}_i} \times \frac{1}{2M\sin^2(\frac{\pi}{4M})} \triangleq \zeta, \quad 1 \leq s \leq M. \quad (5)$$

Now, $\epsilon_r = \max_{1 \leq s \leq M} \zeta_s = \zeta, 1 \leq r \leq M$. Thus, even with the Hadamard transform, the PMEPR of all transmit antennas is $w = \frac{1}{2M\sin^2(\frac{\pi}{4M})}$ times more than the PMEPR of the QAM constellation $\mathcal{I}_i$, a factor that grows almost linearly for large M. For $M \in \mathcal{M}_2$, the ζ_s are not even the same for each s but it can be shown that $\max_{1\le s\le M} \zeta_s \ge w\epsilon_{\mathcal{I}_j}$, where $w = \frac{1}{4M\sin^2(\frac{\pi}{6M})}$. Therefore, if $M \in \mathcal{M}_2$ and M is a multiple of 4, then the PMEPR ϵ_r with the Hadamard transform is at least a factor of w times the PMEPR of the input constellation $\mathcal{I}_j$. If $M \in \mathcal{M}_2$ and M is not a multiple of 4, then a Hadamard spread is not even possible for the original DAST code.

In this subsection, a unitary spreading matrix $\mathbf{\Theta}_M$ is presented for $M \in \mathcal{M}_1 \cup \mathcal{M}_2$ such that by setting $\mathbf{U} = \mathbf{\Theta}_M$ and $\mathbf{V} = \mathbf{\Theta}_M^{-1}$, the PMEPR for each transmit antenna becomes exactly equal to the PMEPR of the input constellation $\mathcal{I}$, thereby outperforming the Hadamard transform in this objective by at least the factor w mentioned above.

It is first noted that multiplication of $\alpha^{(1)} = \alpha$ by θ^j, $1 \le j \le M-1$, leads to an algebraic integer whose representation in the basis $(1, \theta, \ldots, \theta^{M-1})$ is given by $(ez_{M-j+1}, ez_{M-j+2}, \ldots, ez_M, z_1, \ldots, z_{M-j})$. Such a cyclic shift of the representation of α and multiplication by e in the first j positions is due to the special structure of $\mu_{F,\theta}(x)$ so that $\theta^{M-1+j} = e\theta^{j-1}$. Thus,

$$\begin{bmatrix} 1 \\ \theta \\ \vdots \\ \theta^{M-1} \end{bmatrix} \alpha^{(1)} = \mathbf{Z}(z_1, z_2, \ldots, z_M) \begin{bmatrix} 1 \\ \theta \\ \vdots \\ \theta^{M-1} \end{bmatrix}, \tag{6}$$

where the matrix formatting function $\mathbf{Z}$ is given by

$$\mathbf{Z}(z_1, \ldots, z_M) = \begin{bmatrix} z_1 & z_2 & z_3 & \ldots & z_M \\ ez_M & z_1 & z_2 & \ldots & z_{M-1} \\ ez_{M-1} & ez_M & z_1 & \ldots & z_{M-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ez_2 & ez_3 & \ldots & ez_M & z_1 \end{bmatrix}. \tag{7}$$

The matrix representation of α, as in (6), is an instance of a well-known linear map of a number field (see [7, Chapter 2, Problem 17]). Applying the M automorphisms of $Q(\theta)$ to (6) and compiling all the M equations into a matrix notation, we get that

$$\mathbf{D}_\theta \mathbf{S}_M \mathrm{diag}(\alpha^{(1)}, \ldots, \alpha^{(M)}) = \mathbf{Z}(z_1, z_2, \ldots, z_M)\mathbf{D}_\theta \mathbf{S}_M. \tag{8}$$

If $\mathbf{\Theta}_M \triangleq \frac{1}{\sqrt{M}}\mathbf{D}_\theta \mathbf{S}_M$, then $\mathbf{\Theta}_M^\dagger \mathbf{\Theta}_M = \mathbf{I}_M$ and we get from (8) that

$$\mathbf{Z}(z_1, z_2, \ldots, z_M) = \mathbf{\Theta}_M \mathrm{diag}(\alpha^{(1)}, \ldots, \alpha^{(M)})\mathbf{\Theta}_M^{-1}. \tag{9}$$

Thus, the conjugates $(\alpha^{(1)}, \ldots, \alpha^{(M)})$ are, in fact, the eigenvalues of the matrix $\mathbf{Z}(z_1, \ldots, z_M)$. It can now be seen from (9) and (4) that setting $\mathbf{U} = \mathbf{\Theta}_M$ and $\mathbf{V} = \mathbf{\Theta}_M^{-1}$ leads to $\mathbf{\Xi}_M(z_1, \ldots, z_M) = \mathbf{Z}(z_1, \ldots, z_M)$.

For $M \in \mathcal{M}_1$, all the entries of $\mathbf{Z}(z_1, \ldots, z_M)$ are elements of the input q-QAM constellation as multiplication of any point in the square QAM constellation $\mathcal{I}_i$ by $e = i$ leads to another point in the same constellation $\mathcal{I}_i$. Hence, the proposed spread for $M \in \mathcal{M}_1$ results in a space-time code that transmits only QAM information symbols at the rate of 1 symbol per channel use and the PMEPR for each antenna becomes equal to the PMEPR of $\mathcal{I}_i$. For $M \in \mathcal{M}_2$, multiplication of a constellation point in $\mathcal{I}_j$ by $e = -j^2$ does not necessarily lead to another constellation point in $\mathcal{I}_j$. Nevertheless, since $|e| = 1$, the PMEPR for each antenna with the proposed spread is equal to the PMEPR of $\mathcal{I}_j$. Therefore, there is no increase in PMEPR of the new space-time code with respect to the input constellation $\mathcal{I}$ for any $M \in \mathcal{M}_1 \cup \mathcal{M}_2$. Moreover, this has been achieved without any change in the performance of the code since the matrix $\mathbf{\Theta}_M$ is unitary.

It is shown in [12] that a space-time code that transmits only information symbols from $\mathcal{I}$, as in $\mathbf{Z}(z_1, \ldots, z_M)$, has the smallest PMEPR for each transmit antenna among all codes that transmit a linear combination of independent information symbols from $\mathcal{I}$. It is in this sense that the spreading matrix for the DAST codes proposed in this section is optimal.

B. TAST Code spread

Let us consider next the more general problem of decreasing the PMEPR of the TAST codes. The original TAST construction is summarized first. Let $\mathbf{z} = [z_{1,1}, \ldots, z_{1,M}, \ldots, z_{L,1}, \ldots, z_{L,M}]^T$ be a vector of LM information symbols from the input constellation $\mathcal{I}$, where L is the number of layers in the TAST code $(1 \le L \le M)$ and $M \in \mathcal{M}_1 \cup \mathcal{M}_2$. The constellation $\mathcal{S}$ obtained from the rotation of $\mathcal{I}^M$ is employed for all the layers as explained next. The algebraic integer α_l corresponding to the information symbols of the l-th layer is given by

$$\alpha_l = \sum_{k=1}^{M} z_{l,k}\theta^{k-1}, \quad 1 \le l \le L. \tag{10}$$

The L-layer TAST code, denoted by $\mathcal{T}_{M,L,N}$, is constructed as

$$\mathcal{T}_{M,L,N}(\mathbf{z}) = \sum_{l=1}^{L} \phi_l \mathbf{P}^{l-1}\mathrm{diag}(\alpha_l^{(1)}, \ldots, \alpha_l^{(M)}), \tag{11}$$

where the M-dimensional permutation matrix $\mathbf{P}$ is given by $\mathbf{P} = \begin{bmatrix} \mathbf{0} & 1 \\ \mathbf{I}_{M-1} & \mathbf{0} \end{bmatrix}$ and $\mathbf{P}^0 = \mathbf{I}_M$. The constants $(\phi_1, \ldots, \phi_L) \in C^L$, known as the Diophantine numbers for the L layers, are critical in ensuring full transmit diversity of the TAST code [4] and their choice is discussed later. However, for the sake of PMEPR computation in general, no restrictions are imposed on the ϕ_l.

The PMEPR of the r-th antenna with the L-layer TAST code is given by

$$\epsilon_r = \frac{\max_{1\le l\le L}\left(|\phi_l|^2 \zeta_{\lfloor r-l \rfloor_M + 1}\right)}{\frac{1}{M}\sum_{l=1}^{L}|\phi_l|^2}, \quad 1 \le r \le M, \tag{12}$$

where we have used the fact that $E_{\mathbf{x}\in\mathcal{S}}[|\mathbf{x}_s|^2]$ is the same for all $1 \le s \le M$. Consider first the case $M \in \mathcal{M}_1$ so that the expression for ϵ_r simplifies to

$$\epsilon_r = \zeta \left(\frac{\max_{1\le l\le L} |\phi_l|^2}{\frac{1}{M}\sum_{l=1}^{L}|\phi_l|^2} \right), \quad 1 \le r \le M, \tag{13}$$

where the constant ζ is the PMEPR of the rotated constellation $\mathcal{S}$, as defined in (5). The quantity in brackets in (13) is always greater than $\frac{M}{L}$ thus providing a lower bound on ϵ_r. If

$L = M$ and $|\phi_l| = 1, 1 \le l \le M$, then $\epsilon_r = \zeta, 1 \le r \le M$. Since M is a multiple of a Hadamard dimension for $M \in \mathcal{M}_1$, it is possible to spread the TAST code for $L < M$ layers by Hadamard sequences, while preserving the performance, so that the PMEPR is reduced. Specifically, this is achieved by using the modified codewords

$$\mathcal{T}'_{M,L,N}(\mathbf{z}) = \sqrt{\frac{L'}{M}}(\mathcal{H}_{\frac{M}{L'}} \otimes \mathbf{I}_{L'})\mathcal{T}_{M,L,N}(\mathbf{z}).$$

where L' is the smallest integer greater than or equal to L such that M/L' is a Hadamard dimension. The PMEPR of the TAST code spread in the Hadamard fashion is smaller than the expression in (13) by a factor of M/L'. Clearly, the Hadamard spread is not possible if $L > M/2$, thereby exposing the limitation of this spreading scheme. The least value of PMEPR is obtained if M/L itself is equal to 2 or a multiple of 4. A particular example of such a spread was provided in [4] wherein for $M = 4, L = 2$, the $\mathcal{T}'_{4,2,2}$ code results in $\epsilon_r = \zeta$. In any case, we have the following lower bounds on the PMEPR of the TAST code with the Hadamard spread for $M \in \mathcal{M}_1$:

$$\epsilon_r \ge \zeta \frac{L'}{L} \ge \zeta, \quad 1 \le r \le M. \tag{14}$$

When $M \in \mathcal{M}_2$, knowledge of the magnitudes of ϕ_l is required to separate the terms in the maximization in (12) because ζ_l varies with l. Consequently, we will resort to numerical evaluation of the ϵ_r for $M \in \mathcal{M}_2$ later on. Again, the spread of the TAST code by Hadamard sequences is possible for $L < M, M \in \mathcal{M}_2$, if $\lfloor M \rfloor_4 = 0$.

Now, it can be shown that the modulation rate $r_m(\mathcal{T}_{M,L,N}) = r_m(\mathcal{T}'_{M,L,N}) = \frac{L}{M}$, which is smaller than the maximum possible modulation rate of 1 given by the Singleton bound if $L < M$. In this section, we obtain a new space-time code with the same performance as $\mathcal{T}_{M,L,N}$ but that satisfies the Singleton bound. The higher modulation rate of the proposed code makes the PMEPR smaller than ζ when $L < M$ and $M \in \mathcal{M}_1 \cup \mathcal{M}_2$. Moreover, this reduction in PMEPR is possible for more values of L than $M/2$, as in the Hadamard spread.

The main idea is to apply the spreading $\mathbf{\Theta}_M$ as in (9) to the TAST code in (11) so that the performance remains unaffected. First, the product $\mathbf{\Theta}_M \mathbf{P}^{l-1}$ is expressed in a convenient form using the structure of the IDFT matrix $\mathbf{S}_M$. If we set $\mathbf{\Psi}_l = \text{diag}(1, \omega_M^{l-1}, \omega_M^{2(l-1)}, \ldots, \omega_M^{(M-1)(l-1)})$, then

$$\begin{aligned} \mathbf{S}_M \mathbf{P}^{l-1} &= \mathbf{\Psi}_l \mathbf{S}_M && (15)\\ \Rightarrow \mathbf{\Theta}_M \mathbf{P}^{l-1} &= \mathbf{\Psi}_l \mathbf{\Theta}_M. && (16) \end{aligned}$$

Hence, applying the spreading $\mathbf{\Theta}_M$ to the TAST codeword in (11), we get from (16) and (9) that

$$\begin{aligned} \mathbf{\Theta}_M \mathcal{T}_{M,L,N}(\mathbf{z}) \mathbf{\Theta}_M^{-1} &= \sum_{l=1}^{L} \phi_l \mathbf{\Psi}_l \mathbf{\Theta}_M \text{diag}(\alpha_l^{(1)}, \ldots, \alpha_l^{(M)}) \mathbf{\Theta}_M^{-1} \\ &= \sum_{l=1}^{L} \phi_l \mathbf{\Psi}_l \mathbf{Z}(z_{l,1}, z_{l,2}, \ldots, z_{l,M}) \qquad (17)\\ &\triangleq \mathcal{L}_{M,L,N}(\mathbf{z}), \end{aligned}$$

where the formatter $\mathbf{Z}$ is the same as that in (7). The new code $\mathcal{L}_{M,L,N}$ given by (17) will be referred to as the Low PMEPR Space-Time (LPST) code. The justification for this term will be provided shortly. As an example, the LPST code for $M = 4$ and $L = 2$ is shown below. The information symbols are $(z_1, \ldots, z_4)$ for the first layer and $(z_5, \ldots, z_8)$ for the second layer.

$$\begin{bmatrix} \phi_1 z_1 + \phi_2 z_5 & \phi_1 z_2 + \phi_2 z_6 & \phi_1 z_3 + \phi_2 z_7 & \phi_1 z_4 + \phi_2 z_8 \\ i\phi_1 z_4 - \phi_2 z_8 & \phi_1 z_1 + \phi_2 i z_5 & \phi_1 z_2 + \phi_2 i z_6 & \phi_1 z_3 + \phi_2 i z_7 \\ i\phi_1 z_3 - i\phi_2 z_7 & i\phi_1 z_4 - i\phi_2 z_8 & \phi_1 z_1 - \phi_2 z_5 & \phi_1 z_2 - \phi_2 z_6 \\ i\phi_1 z_2 + \phi_2 z_6 & i\phi_1 z_3 + \phi_2 z_7 & i\phi_1 z_4 + \phi_2 z_8 & \phi_1 z_1 - \phi_2 i z_5 \end{bmatrix}$$

In general, the (r, c) element of $\mathcal{L}_{M,L,N}(\mathbf{z})$ is given by

$$\mathcal{L}_{M,L,N}(r, c) = \sum_{l=1}^{L} \phi_l \omega_M^{(r-1)(l-1)} \gamma_{r,c} z_{l, M - \lfloor r-c-1 \rfloor_M}, \tag{18}$$

where the term $\gamma_{r,c} = e$ if $c < r$ and $\gamma_{r,c} = 1$ if $c \ge r$. Thus, with the $\mathcal{L}_{M,L,N}$ code, each antenna transmits a linear combination of L independent information symbols from $\mathcal{I}$ in each time slot. Due to this linearity, the low complexity sphere decoder [13] can be applied at the receiver for optimum decoding of the LPST codes.

The improved PMEPR properties of the $\mathcal{L}_{M,L,N}$ code are shown next. The average power of the signal on any transmit antenna is $P_{av} = (\sum_{l=1}^{L} |\phi_l|^2) E_{av}$. The PMEPR for the r-th antenna with the $\mathcal{L}_{M,L,N}$ code is obtained from (18) as

$$\begin{aligned} \epsilon_r &= \frac{1}{P_{av}} \max_{\substack{1 \le c \le M \\ z_{l,k} \in \mathcal{I}, 1 \le l \le L, 1 \le k \le M}} |\mathcal{L}_{M,L,N}(r, c)|^2 && (19)\\ &= \frac{1}{P_{av}} \max_{z'_l \in \mathcal{I}} \left| \sum_{l=1}^{L} \phi_l e^{\frac{i 2\pi (r-1)(l-1)}{M}} z'_l \right|^2, && (20) \end{aligned}$$

where we have simplified with the fact that for a fixed r, the maximum of $|\mathcal{L}(r, c)|^2$ among all codewords is independent of c.

Proposition 1: The PMEPR $\epsilon_r, 1 \le r \le M$, with the $\mathcal{L}_{M,L,N}$ space-time code for $M \in \mathcal{M}_1 \cup \mathcal{M}_2$ is upper bounded by $L\epsilon_{\mathcal{I}}$.

Proof: For any (r, c) position in the code, application of the Cauchy-Schwartz inequality in (18) gives

$$\begin{aligned} |\mathcal{L}_{M,L,N}(r, c)|^2 &\le \left(\sum_{l=1}^{L} |z_{l, M - \lfloor r-c-1 \rfloor_M}|^2 \right) \left(\sum_{l=1}^{L} |\phi_l|^2 \right) \\ &\le L |z_{max}|^2 \left(\sum_{l=1}^{L} |\phi_l|^2 \right). && (21) \end{aligned}$$

Hence, we have an upper bound, independent of (r, c), for the peak envelope power for any transmit antenna. Dividing the upper bound in (21) by P_{av}, we get that $\epsilon_r \le L \frac{|z_{max}|^2}{E_{av}} = L\epsilon_{\mathcal{I}}, \quad 1 \le r \le M.$ ∎

For a given q, the upper bound of Proposition 1 on the PMEPR of all transmit antennas depends only on L. Thus, with the choice of L so that the upper bound to the PMEPR of the LPST code is less than a lower bound on the PMEPR of the TAST

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code , our construction will necessarily result in a reduction of PMEPR for all the transmit antennas. Such a guarantee on the number of layers is obtained for $M \in \mathcal{M}_1$ from the lower bound on the PMEPR of the TAST code in (14) and is presented in the next proposition. A proof of the following result can be found in [12].

Proposition 2: For $M \in \mathcal{M}_1$ and a fixed q, the L layer LPST code $\mathcal{L}_{M,L,N}$ necessarily has a smaller PMEPR than that obtained with a Hadamard spread of the TAST code $\mathcal{T}_{M,L,N}$ if

$$1 \leq L \leq \left\lfloor \frac{1}{\sqrt{2}\sin(\frac{\pi}{4M})} \right\rfloor \triangleq L_M^g. \tag{22}$$

The values of L_M^g for a few values of $M \in \mathcal{M}_1$ are shown in Table I. It is seen from Table I that the reduction of PMEPR can be obtained for large values of L. The new code $\mathcal{L}_{M,L,N}$ exploits the absence of some layers to make the PMEPR smaller than that of the rotation $\mathcal{S}$. This is because the LPST code transmits a linear combination of only L information symbols as opposed to M symbols in the original TAST code. This is unlike the spreading by Hadamard sequences, as in $\mathcal{T}'_{4,2,2}$, where the spreaded code continues to transmit a linear combination of M information symbols even if $L < M$. For $M = 4$, it can be seen from Table I that our construction leads to PMEPR reduction for $L = 3$ also, whereas a spread of $\mathcal{T}_{4,3,N}$ by Hadamard sequences is not possible.

Note that for $L = 1$, the $\mathcal{L}_{M,1,N}$ code is the same as the optimum spreaded DAST code of Section III-A. If $L > 1$, the PMEPR of the transmit antennas depend on the Diophantine numbers and can also vary across the transmit antennas. Nevertheless, for a fixed set of Diophantine numbers and $M \in \mathcal{M}_1$, the following proposition shows that some of the antennas exhibit the same PMEPR with the LPST code.

Proposition 3: For any $M \in \mathcal{M}_1 \backslash \{2\}$ and $L > 1$, the transmit antennas can partitioned into $M/4$ disjoint groups so that every antenna in the same group exhibits the same PMEPR with the LPST code. For $M = 2$, both the transmit antennas have the same PMEPR.

Proof: If the index of a particular antenna is r, then the antenna with the index $\lfloor r + (M/4) \rfloor_M$ for $M \geq 4$ and with the index $\lfloor r + 1 \rfloor_M$ for $M = 2$ also gives the same maximum absolute value of $\mathcal{L}_{M,L,N}(r, c)$ in (18). This is because multiplication of any point in $\mathcal{I}_i$ by i gives rise to another valid point in $\mathcal{I}_i$. The average power for all transmit antennas is the same. Thus, for $M \geq 4$ there are 4 antennas in the same group with the same PMEPR and there are $M/4$ such groups. Similarly, for $M = 2$, both the antennas have the same PMEPR. ■

The selection of ϕ_l to guarantee full transmit diversity of the TAST codes is non-trivial in general with no imposed constraints on the Diophantine numbers. Hence, it was suggested in [4] to parameterize all the Diophantine numbers with a single Diophantine number as $(\phi_1, \ldots, \phi_L) = (1, \phi^{\frac{1}{M}}, \ldots, \phi^{\frac{L-1}{M}})$. With this structure, full transmit diversity is obtained by choosing ϕ such that the set $(1, \phi, \ldots, \phi^{L-1})$ is algebraically independent over $Q(\theta)$. If $|\phi| \neq 1$ and $M \in \mathcal{M}_1$, then (13) implies that the PMEPR of the TAST code is necessarily greater than the PMEPR of the inherent rotation. For the purpose of illustration, we shall constrain $\phi = e^{i\lambda}$, for $\lambda \in \mathbb{R}$. In this case, the PMEPR ϵ_r of the original TAST code is independent of λ. However, with the LPST code, ϵ_r depends on λ and this warrants an optimum selection of λ that leads to the least PMEPR, while preserving the diversity advantage of the code. Henceforth, we use $\epsilon_r(\lambda)$ to denote the PMEPR of the r-th antenna for a particular choice of λ. The behavior of the PMEPR with respect to λ is generalized in the following proposition.

Proposition 4: For any $M \in \mathcal{M}_1$, $\epsilon_r(\lambda) = \epsilon_r(\lambda + M\frac{\pi}{2})$.

Proof: Replacing λ by $\lambda + M\frac{\pi}{2}$ leads to a multiplication by i for the coefficients of the linear combination in (18). Hence, the maximum of $|\mathcal{L}_{M,L,L}(r, c)|^2$ among all codewords remains unchanged. Since $|i| = 1$, the average power for any transmitter also remains unchanged and thus the result of the proposition follows. ■

The performance of the TAST codes, in terms of coding gain, depends on the Diophantine numbers ϕ_l. The advantage of our construction is that for $1 \leq L \leq L_M^g, M \in \mathcal{M}_1$ and the choice of $\{\phi_l\}_{l=1}^L$ that optimizes the coding gain of $\mathcal{T}_{M,L,N}$, the corresponding LPST code $\mathcal{L}_{M,L,N}$ also enjoys the same performance but at a lower PMEPR for the transmit antennas. Numerical computation for $M = 3 \in \mathcal{M}_2$ in Section IV also leads to a similar conclusion for the PMEPR with the LPST code. Moreover, the following result states that the $\mathcal{L}_{M,L,N}$ code meets the Singleton bound.

Proposition 5: For $M \in \mathcal{M}_1 \cup \mathcal{M}_2, 1 \leq L \leq M$ and the choice of $\{\phi_l\}_{l=1}^L$ such that $\mathcal{T}_{M,L,N}$ exhibits full diversity, the corresponding LPST code $\mathcal{L}_{M,L,N}$ satisfies $r_m(\mathcal{L}_{M,L,N}) = 1$, thereby satisfying the Singleton bound with equality.

IV. NUMERICAL RESULTS

In this section, specific examples of the LPST codes are discussed. The Diophantine numbers in all the examples are chosen to be $\phi_l = \phi^{\frac{l-1}{M}}, 1 \leq l \leq L$, with $\phi = e^{i\lambda}, \lambda \in R$. A plot of the variability of the true PMEPR with λ for $M = 4$ and $1 \leq L \leq 4$ is shown in Figure 1 for 16-QAM information symbols. In this case, $\epsilon_r(\lambda)$ is the same for all $1 \leq r \leq 4$ and the LPST code results in significant reduction in PMEPR for $1 \leq L \leq 3$ and for every value of λ. A similar plot for $M = 8$, $1 \leq L \leq 7$ and 4-QAM is shown in Figure 2. In this case, there are two disjoint groups of antennas, corresponding to the odd and even indices, so that each antenna in the same group exhibits the same PMEPR. The set of increasing solid lines correspond to one of the two groups for increasing L as shown. The set of increasing dotted lines correspond to the other group for increasing $L(> 1)$. For a given L, let $\epsilon_L^{max}(\lambda) = \max_{1 \leq r \leq M} \epsilon_r(\lambda)$. Using $\epsilon_L^{max}(\lambda)$, one can determine the range for λ so that the PMEPR for all the transmit antennas is less than a required threshold. The minimum of $\epsilon_L^{max}(\lambda)$ occurring at a certain $\lambda_{L,min}$ reflects the potential savings in PMEPR achievable with the LPST code compared to the TAST code with the Hadamard spread. The corresponding PMEPR savings of the LPST code at $\lambda_{L,min}$ are tabulated in Table II for $1 \leq L \leq 7$. The $\epsilon_L^{max}(\lambda)$ curve for $M = 3$ is shown in Figure 3. It is clear from the figure that for all values of λ, the PMEPR with the LPST code is smaller than that of TAST for $L = 1$ and $L = 2$. Even for $L = 3$, there exists a range for λ wherein the PMEPR of the LPST code is smaller than that of the corresponding TAST code.

TABLE I

VALUES OF L_M^g

M	L_M^g
2	1
4	3
8	7
16	14
32	28

TABLE II

POTENTIAL PMEPR SAVINGS WITH $\mathcal{L}_{8,L,L}$

L	$Gain(dB)$
1	8.2
2	5.2
3	5.2
4	2.9
5	3.4
6	2.4
7	0.97

V. CONCLUSIONS

The problem of designing high rate and full diversity space-time codes with low PMEPR was addressed. The principal means of obtaining the Low PMEPR Space Time codes proposed in this paper is pre-multiplication and post-multiplication of a high performance space-time code by suitable unitary matrices. Without sacrificing the performance of the original code, the PMEPR was shown to be necessarily reduced in most cases. The full diversity LPST codes also meet the Singleton bound with equality irrespective of the rate (in symbols per channel use). For the DAST code with M transmit antennas, our construction leads to a decrease in the PMEPR by approximately a factor of M relative to the Hadamard spreading scheme. For rates beyond 1 symbol per channel use and up to a guaranteed amount, the LPST codes have significantly smaller PMEPR than the corresponding TAST codes.

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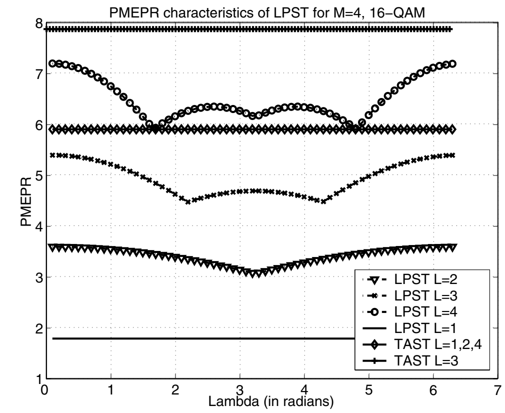

Fig. 1. PMEPR reduction for $M = 4$

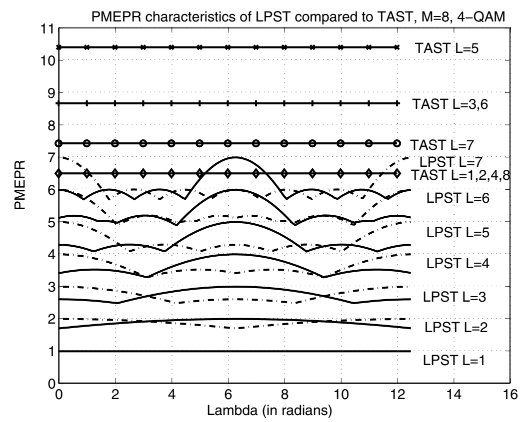

Fig. 2. PMEPR reduction for $M = 8$

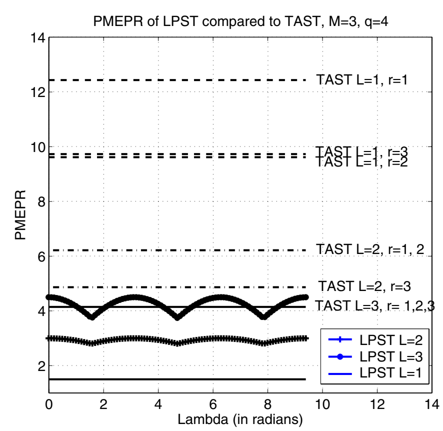

Fig. 3. PMEPR reduction for $M = 3$