Minimum Bandwidth Basis Functions for the Fourth-Moment Bandwidth Measure [1]

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Abstract — **We present a fourth-moment measure of the "bandwidth" of a strictly time-limited signal and obtain a minimum-bandwidth basis for $L^2(a,b)$. Such a basis consists of orthonormal waveforms with the smallest obtainable bandwidths. The primary advantage of the fourth-moment bandwidth relative to the Root Mean Square (RMS) and Fractional Out-of-Band Energy (FOBE) measures is that its basis functions have a $O(1/f^3)$ frequency roll-off compared to the $O(1/f^2)$ and $O(1/f)$ decay of the RMS and FOBE basis functions, respectively.**

I. Main Result

Every strictly time-limited pulse has a spectrum which is nonzero for an infinite range of frequencies. Hence, non-strict measures of bandwidth are used to quantify the spectral concentration of such signals. Two such measures, namely the RMS and the FOBE bandwidths, have been studied in the past. In particular, it was shown that the minimum RMS and FOBE bandwidth orthonormal basis functions for $L^2(0,T)$ are sinusoids $\sin(k\pi t/T)$ for integer k [1] and the set of time-truncated prolate-spheroidal wave functions [2], respectively. In this paper we consider the fourth-moment bandwidth and obtain the corresponding minimum bandwidth orthonormal basis.

Definition 1 (Fourth-Moment Bandwidth Measure) *For a baseband signal with energy spectrum $S_x(f)$, the fourth-moment bandwidth is defined as*

$$\mathrm{bw}(x) = \left[\frac{\int_{-\infty}^{\infty} f^4 S_x(f)\,df}{\int_{-\infty}^{\infty} S_x(f)\,df}\right]^{1/4} \tag{1}$$

Definition 2 (Minimum-Bandwidth Basis) *Let the collection of functions $B = \{\psi_i\}_{i=1}^{\infty}$ be an orthonormal basis for $L^2(-T/2,T/2)$ (the space of square-integrable functions with standard inner product), and let the bandwidth measure be defined through (1). If ψ_k has the minimum bandwidth of all L^2 functions which are orthogonal to $\{\psi_i\}_{i=1}^{k-1}$, i.e.,*

$$\psi_k = \arg \min_{\substack{x\in L^2(-\frac{T}{2},\frac{T}{2}) \\ x\perp\psi_1,\cdots,\psi_{k-1}}} \mathrm{bw}(x) \tag{2}$$

for all k, then B is a minimum-bandwidth basis *for $L^2(-T/2,T/2)$.*

The main result of this paper is that the minimum bandwidth basis functions for the fourth-moment bandwidth measure are solutions of the eigenvalue/eigenfunction equation

$$\gamma\psi(t) = \frac{1}{16\pi^4}\left(\mathbf{T}^*\left(\frac{d^4}{dt^4}(\mathbf{T}\psi)\right)\right)(t) \tag{3}$$

where $\mathbf{T}$ denotes the time-limiting operator (to the interval $[-T/2,T/2]$). The boundary conditions are imposed by requiring the solutions to lie in $H_0^2(-T/2,T/2)$ which are the time-limited elements of the Sobolev space W^2 of functions on R defined as [3],

$$W^2 = \left\{x : \left\|(1+f^2)\hat{x}(f)\right\|_2 < \infty\right\}. \tag{4}$$

It can be shown that the eigenvalues γ_k of (3), which are equal to the bandwidths of the respective basis functions, are given as $\gamma_k = (\phi_k/2\pi)^4$, where the ϕ_k's are the positive solutions to

$$\cos(\phi_k T/2)\sinh(\phi_k T/2) + \sin(\phi_k T/2)\cosh(\phi_k T/2) = 0 \tag{5}$$

and

$$\cos(\phi_k T/2)\sinh(\phi_k T/2) - \sin(\phi_k T/2)\cosh(\phi_k T/2) = 0 \tag{6}$$

The eigenfunctions are given for $t \in [-T/2,T/2]$ by

$$\psi_k(t) = \begin{cases} \sqrt{\frac{2}{T(1+\alpha_k^2)}}\left(\cos(\phi_k t) + \alpha_k\cosh(\phi_k t)\right) & k,\ \text{odd} \\ \sqrt{\frac{2}{T(1+\alpha_k^2)}}\left(\sin(\phi_k t) + \alpha_k\sinh(\phi_k t)\right) & k,\ \text{even}\end{cases}. \tag{7}$$

where $\alpha_k = -\cos(\phi_k T/2)/\cosh(\phi_k T/2)$.

A comparison of the frequency roll-off of the minimum FOBE, RMS and fourth-moment bandwidth functions can be seen in Figure 1. This figure reveals that while the minimum fourth-moment bandwidth basis function has a somewhat larger main lobe than the truncated prolate-spheroidal function and the half sinusoid, its rate of side-lobe decay is significantly better.

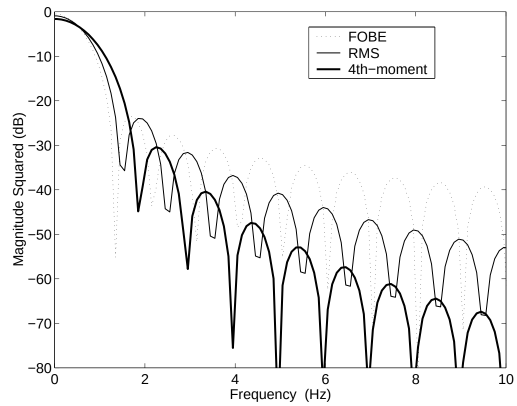

Figure 1: Magnitude spectra of $\psi_1(t)$ for the FOBE case (with $BT = 2\pi$), the RMS bandwidth and the fourth-moment measure, all for $T = 1$.

References

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