Onion Peeling for CDMA- Symmetric Rate Under RMS-Bandwidth Constraints

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Abstract

Multiuser coding is a relatively new field of research, whereas single-user coding is well established. For this reason there has been an interest in methods which allow the users to choose single-user codes independently of each other. This paper considers the synchronous, code-division multiple-access (CDMA) channel, where the users are root-mean-square (RMS) bandlimited. If the signature waveforms are all identical, the discrete-time model reduces to the conventional, multiple-access Gaussian channel. The other extreme is when the signature waveforms are all orthogonal (e.g., time-division multiple-access (TDMA)) in which case the discrete-time model is a bank of single-user channels. The latter clearly allows single-user coding at the expense of bandwidth, while the former has been discussed in [1] and [2] where information-theoretic arguments have shown that sequential decoding (onion peeling) can be used in such a way as to achieve maximum rate-sum capacities. The mediate ground lends itself to sequential decoding based on soft outputs from a multiuser detector. The result is a sequential decoder for CDMA with a degree of freedom that determines to what extent the signature waveforms are correlated. In other words, there is a give-and-take relationship between coding and spreading, and this can be exploited using the results on signal design in [5] to achieve significant gains over the extremes of linearly-dependent and orthogonal signaling.

1 The System

Consider a PAM, synchronous, Gaussian CDMA (PSG-CDMA) channel where users use signature waveforms that are time-limited to $[0,T]$. The bandwidth of a set of such signals is defined to be the energy-weighted RMS (EW-RMS) bandwidth which is equal to the RMS bandwidth of the power-spectral density of the transmitted signal in a PSG-CDMA channel. The signal that is received at some central location is modeled as the sum of M transmitting users and white Gaussian noise; it is given by

$$r(t) = \sum_{k=1}^{N} \sum_{i=1}^{M} b_i(k) u_i(t-kT) + n(t), \tag{1}$$

where $\{b_i(k)\}_k$ is the set of information symbols sent by the i^{th} user, $u_i(t)$ is the signature waveform of i^{th} user, and $n(t)$ is an additive, white, Gaussian noise process with a power-spectral density of $N_0/2$. If each user's input waveform is power constrained to be w_i, then without loss it can be assumed that $\frac{1}{NT}\sum_{k=1}^{N} b_i^2(k) = w_i$ for each i, and that energy of the i^{th} user's waveform is $1 = \int_0^T u_i^2(t)dt$. Define the elements of the correlation matrix, $\mathbf{R}$, associated with a given set of users as

$$R_{mn} = \int_0^T u_m(t)u_n(t)dt, \tag{2}$$

so that $\mathbf{R}$ has unit diagonal elements, and let $\mathbf{W}$ be a diagonal matrix with $\{w_1, w_2, \ldots, w_M\}$ as its diagonal. Assume without loss that the users' powers are ordered so that $w_1 \geq w_2 \geq \cdots \geq w_M$.

The square of the EW-RMS bandwidth of a set of signature waveforms is given by

$$B^2 = \frac{1}{\text{tr}(\mathbf{W})} \sum_{i=1}^{M} w_i B_i^2, \tag{3}$$

where $B_i^2 = \int_{-\infty}^{\infty} f^2 |U_i(f)|^2 df$ is the square of the RMS bandwidth of $u_i(t)$, and $U_i(f)$ is the Fourier transform of $u_i(t)$. In [5] it was shown that for a given positive-definite matrix $\tilde{\mathbf{R}} \triangleq \mathbf{W}^{1/2}\mathbf{R}\mathbf{W}^{1/2}$, there is a set of signature waveforms that have $\tilde{\mathbf{R}}$ as their correlation matrix, and the EW-RMS bandwidth of these waveforms is less than or equal to that of any other set of signature waveforms that also have $\tilde{\mathbf{R}}$ as their correlation matrix. This minimum bandwidth is given by

$$B_0^2 = \frac{1}{(2T)^2} \frac{\text{tr}(\mathbf{\Lambda}\mathbf{\Pi})}{\text{tr}(\mathbf{W})}, \tag{4}$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\tilde{\mathbf{R}}$ in decreasing order, and $\mathbf{\Pi}$ is a diagonal matrix whose i^{th} diagonal element is given by i^2. The capacity region for this channel is discussed in [6] and [4].

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Without loss of generality, it is assumed that the received signal is input into a bank of filters that are matched to the M signature waveforms of the users, and the outputs of these filters are sampled at integer multiples of T to yield a set of sufficient statistics for the users' symbols. The channel is memoryless so it can be considered on a per-symbol basis. The discrete-time equivalent of these sufficient statistics is given by the M-dimensional

$$\mathbf{r}(k) = \mathbf{R}\mathbf{b}(k) + \mathbf{n}(k), \tag{5}$$

where $\mathbf{b}(k)$ is an M-dimensional vector of the information symbols of the users at discrete-time k, and $\{\mathbf{n}(k)\}$ is a set of independent, identically-distributed(i.i.d.), zero-mean, M-dimensional, Gaussian random vectors with covariance $\frac{N_0}{2}\mathbf{R}$.

This paper considers the *symmetric capacity* of the users. This is the largest rate at which *all* users can transmit reliably; it is the worst-case capacity of the users. A *symmetric rate* is a rate at which all users are operating.

2 The Cholesky Sequential Decoder(CSD)

If the correlation matrix $\mathbf{R}$ is positive definite, then (following [3]) a Cholesky decomposition can be used to write $\mathbf{R} = \mathbf{L}^{\mathrm{T}}\mathbf{L}$ where $\mathbf{L}$ is a lower triangular matrix and the superscript T means transpose. If $\mathbf{r}(k)$ is processed by passing it through the linear system $(\mathbf{L}^{\mathrm{T}})^{-1}$ the result is still a sufficient statistic. Therefore let

$$\mathbf{y}(k) = \mathbf{L}\mathbf{b}(k) + \tilde{\mathbf{n}}(k), \tag{6}$$

where $\{\tilde{\mathbf{n}}(k)\}$ is a set of i.i.d., zero-mean, M-dimensional, Gaussian random vectors with covariance $\frac{N_0}{2}\mathbf{I}$.

Consider the i^{th} user:

$$y_i(k) = L_{ii}b_i(k) + \sum_{j=1}^{i-1} L_{ij}b_j(k) + \tilde{n}_i(k). \tag{7}$$

If it is assumed that symbols of users 1 to $i-1$ were decoded correctly, then their interference can be subtracted from $y_i(k)$ to leave $L_{ii}b_i(k) + \tilde{n}_i(k)$ which is readily identified as a single-user, discrete-time Gaussian channel. Thus the capacity of the effective single-user channel in nats per second is

$$\frac{1}{2T}\log\left(1 + \frac{L_{ii}^2 w_i T}{N_0/2}\right). \tag{8}$$

As long as the assumption of perfect interference cancellation (PIC) is made, then each of the M users can be encoded and decoded as for an equivalent single-user channel. This assumption will be justified by requiring $\mathbf{R}$ to be such that each of the L_{ii}^2 terms to meets some lower bound. Note that the CSD does not assume anything about the input symbol distributions, unlike onion peeling (OP) which requires that the symbols be Gaussian.

Consider the two-user case, where it is required that the symmetric capacity be $\frac{1}{2T}\log\left(1 + \frac{\gamma w_2 T}{N_0/2}\right)$, with $\gamma \in [0, 1]$ being a threshold. The effective single-user capacities of the two users are

$$C_{\text{CSD}}^1 = \frac{1}{2T}\log\left(1 + \frac{L_{11}^2 w_1 T}{N_0/2}\right) \quad \text{and}$$
$$C_{\text{CSD}}^2 = \frac{1}{2T}\log\left(1 + \frac{w_2 T}{N_0/2}\right) \tag{9}$$

since $L_{22}^2 = 1$. Now L_{11}^2 must be at least as big as $\gamma\frac{w_2}{w_1}$ for the symmetric capacity to be met. Intuitively, the smaller L_{11}^2 is required to be, the more the two signature waveforms are correlated and, hence, the smaller the required bandwidth.

So the role of γ becomes clear when it is realized that it not only reduces the symmetric capacity, but it also reduces the required bandwidth. Thus if all the users are stronger than necessary for reliable communications, choosing $\gamma < 1$ enables the CSD to use less bandwidth, but still have an acceptable symmetric capacity. Note that if all users have the same power, and $\gamma = 1$, then the correlation matrix will be diagonal and hence the signature waveforms will be orthogonal.

Now consider the problem of choosing a set of signature waveforms that minimizes the required bandwidth with a symmetric-capacity constraint. For a given set of M users with fixed powers, it is desired to determine the minimum EW-RMS bandwidth required so that the capacity of each user's effective single-user channel is $\frac{1}{2T}\log\left(1 + \frac{\gamma w_M}{N_0/2}\right)$ (of course the last user's capacity will be $\frac{1}{2T}\log\left(1 + \frac{w_M}{N_0/2}\right)$). In other words, given that each user is required to meet some worst-case capacity, find the minimum EW-RMS bandwidth necessary. The intuition behind such a problem is that CSD allows weaker users to be assisted by stronger users who can afford to make some of their power available to the weaker users. As a result, all the users will share the same worst-case capacity.

An exact analytic solution to the problem looks intractable from the outset because of the complicated relationship between the off-diagonal elements, and the eigenvalues, of the correlation matrix. The former define the signature waveforms to be used, and the latter give the EW-RMS bandwidth of these signature waveforms. However, a useful upper bound on the required EW-RMS bandwidth can be found. For M users whose powers are known, the correlation matrix, $\mathbf{R}$, as given in (2) can be built user by user, working up from the weakest user to the strongest user. The i^{th} user's capacity in the PIC sequential decoding scheme is

$$C_i = \frac{1}{2T}\log\left(1 + \frac{w_i \alpha_i T}{N_0/2}\right), \tag{10}$$

where

$$\alpha_i = 1 - \left[\begin{array}{ccc} R_{i,i+1} & \cdots & R_{i,M} \end{array} \right] \mathbf{R}^{-1}(i+1) \left[\begin{array}{c} R_{i,i+1} \\ \vdots \\ R_{i,M} \end{array} \right], \quad (11)$$

and $\mathbf{R}(i)$ denotes the principle sub-matrix of $\mathbf{R}$ formed by rows (and columns) $\{i, i+1, \cdots, M\}$. For user M, $\alpha_M = 1$. For user $M-1$ pick $R_{M-1,M}$ so that $w_{M-1}\alpha_{M-1}$ is at least as big as γw_M, where $\gamma \in [0,1]$. This process is continued so that for user i, $[R_{i,i+1} \ \cdots \ R_{i,M}]$ is chosen so that $w_i\alpha_i \geq \gamma w_M$, where $\mathbf{R}(i+1)$ has already been determined. From the second step forward, the valid choices for $[R_{i,i+1} \ \cdots \ R_{i,M}]$ are not unique. For the purposes of this paper an upper bound on the bandwidth is used which allows the correlation matrix to be specified.

Two examples are now given to illustrate the performance gains of the CSD over orthogonal signaling (e.g., TDMA). The data for plots of symmetric capacity versus the number of users are generated under the assumption that the available EW-RMS bandwidth is 1 Hz. The CSD correlation matrix is created as just discussed, and (4) can be used to find the corresponding time interval required for the CSD. Then the symmetric capacity is given by $\frac{1}{2T}\log\left(1+\frac{\gamma w_M T}{N_0/2}\right)$. For TDMA the time interval is found similarly except that the correlation matrix is diagonal, and the capacity is given by $\frac{1}{2T}\log\left(1+\frac{w_M T}{N_0/2}\right)$. When the time interval T for TDMA becomes significantly larger than that of the CSD, then the symmetric capacity of TDMA is much less than that of the CSD because the symmetric capacities are decreasing in T. The signal-to-noise ratio (SNR) is defined to be $10\log\left(\frac{w_M}{N_0}\right)$, that is the SNR of the weakest user.

Example 1 Consider M users where users $1,\ldots,M-1$ have twice the power of user M, that is, there is one weak user. Figures 1 and 2 plot the number of users that can be supported as a function of symmetric capacity. Since the last user is weaker than the others, the correlation matrix will not be orthogonal even when $\gamma = 1$. As γ decreases the required time interval decreases. Thus at lower symmetric capacities the number of users that can be supported increases. This effect is exacerbated as the SNR increases, and for low SNRs the gains of the CSD diminish except for unreasonable numbers of users. At a SNR of 20 dB and a symmetric capacity of 1 nat per second, TDMA can support 10 users, while CSD can support 17 users when $\gamma = 1$ and 19 users when $\gamma = 0.7$.

Figure 3 shows the extra bandwidth needed by TDMA to achieve the same capacity as the CSD when $\gamma = 1$. For this plot the time interval is fixed, and the result is independent of the SNR. From (4), and the fact that T and $\mathbf{W}$ are identical for both schemes, the bandwidth

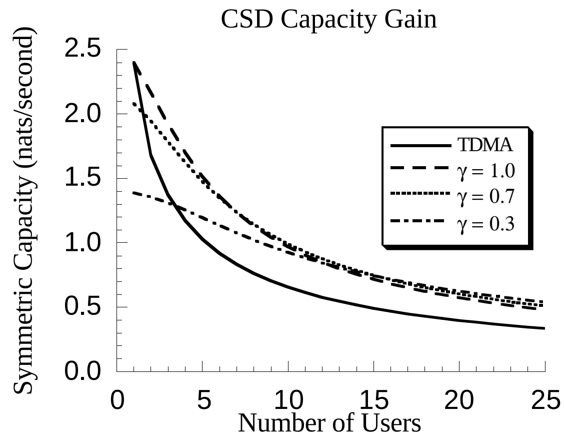

Figure 1: One Weak User, SNR = 10 dB

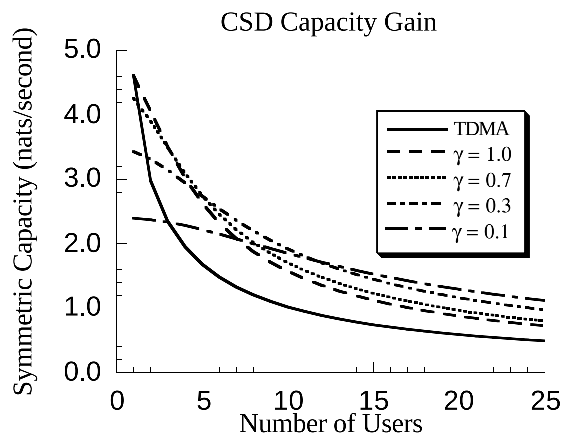

Figure 2: One Weak User, SNR = 20 dB

ratio is given as the square root of the ratio of the trace of $\mathbf{\Lambda\Pi}$ for TDMA over the trace of $\mathbf{\Lambda\Pi}$ for CSD.

Example 2 This is similar to Example 1 except that all users have the same power. This is a worst-case scenario for the CSD because when $\gamma = 1$ the CSD will have orthogonal signals just like TDMA. Still, as γ is decreased, the CSD shows considerable gains over TDMA. Results are shown in Figure 4, where at 1 nat per second TDMA can support 10 users and the CSD can support 13 when $\gamma = 0.7$, or 16 when $\gamma = 0.3$.

3 Onion Peeling(OP)

If all the users have the same signature waveforms, then the equivalent discrete-time model in (5) reduces to the conventional, multiple-access Gaussian channel,

$$r(k) = \sum_{i=1}^{M} b_i(k) + n(k), \quad (12)$$

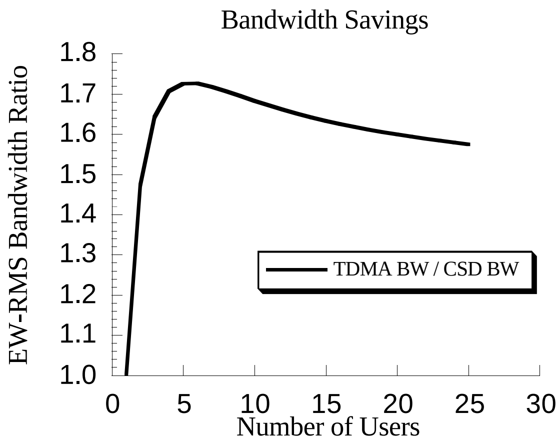

Figure 3: One Weak User, $\gamma = 1$

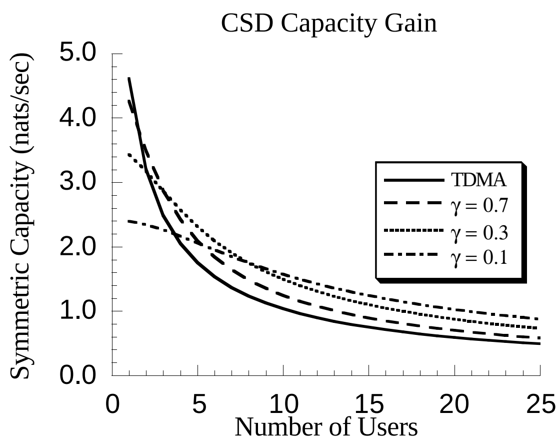

Figure 4: Equal-Weighted Users, SNR = 20 dB

where $\{b_i(k)\}_k$ is the set of information symbols of the i^{th} user and $\{n(k)\}$ is a set of i.i.d., zero-mean, Gaussian random variables with variance $\frac{N_0}{2}$. When all of the users' input symbols are Gaussian, it has been shown (see [1]) that every point in the capacity region can be achieved with single-user coding by means of OP and time sharing(TS), and more recently [2] has shown that every point in the capacity region can be achieved via OP, without resorting to TS, by decomposing some users into two users. This avoids TS because it restructures the problem so that arbitrary points on the capacity-region boundary become vertices on the capacity-region boundary of the restructured system. To facilitate tractability, the two-user case will be the primary consideration in this paper.

Consider the effective single-user channels that result from OP, where all rates and capacities are in nats per second unless stated otherwise.

Case 1: Two Channel The two-channel model is valid when the symmetric rate of the two users satisfies

$$R_{\text{sym}} \leq \frac{1}{2T} \log\left(1 + \frac{w_1 T}{N_0/2 + w_2 T}\right), \tag{13}$$

in which case user 1 can be decoded first. The capacities of the effective single-user channels are

$$C^1_{\text{OP-2A}} = \frac{1}{2T}\log\left(1+\frac{w_1T}{N_0/2+w_2T}\right) \quad \text{and}$$
$$C^2_{\text{OP-2A}} = \frac{1}{2T}\log\left(1+\frac{w_2T}{N_0/2}\right). \tag{14}$$

If, in addition,

$$R_{\text{sym}} \leq \frac{1}{2T} \log\left(1 + \frac{w_2 T}{N_0/2 + w_1 T}\right), \tag{15}$$

then user 1 can be still be decoded first and (14) holds, or user 2 can be decoded first in which case the capacities are

$$C^1_{\text{OP-2B}} = \frac{1}{2T}\log\left(1+\frac{w_1T}{N_0/2}\right) \quad \text{and}$$
$$C^2_{\text{OP-2B}} = \frac{1}{2T}\log\left(1+\frac{w_2T}{N_0/2+w_1T}\right). \tag{16}$$

Case 2: Three Channel The three-channel case is valid for any symmetric rate, and in general it is necessary when R_{sym} does not satisfy (13) and TS is not used. There are two possible choices depending on whether user 1 or user 2 is decomposed. When user 1 is decomposed, let δ be the unique number such that

$$\frac{1}{2T}\log\left(1+\frac{w_2T}{N_0/2+\delta}\right) = \min\left\{\frac{1}{2T}\log\left(1+\frac{w_2T}{N_0/2}\right),\right.$$
$$\left.\left(\frac{1}{2}\right)\frac{1}{2T}\log\left(1+\frac{(w_1+w_2)T}{N_0/2}\right)\right\}. \tag{17}$$

The two terms in the minimization brackets are the symmetric capacities of the two-user system when the capacity of user 2 is the limiting capacity, and when the rate sum is the limiting capacity, respectively. The resulting three-user channel has the following capacities for the three users, where the capacity of the second user is the symmetric capacity:

$$\begin{aligned}
C^1_{\text{OP-3A}} &= \frac{1}{2T}\log\left(1+\frac{w_1T-\delta}{N_0/2+w_2T+\delta}\right)\\
C^2_{\text{OP-3A}} &= \frac{1}{2T}\log\left(1+\frac{w_2T}{N_0/2+\delta}\right)\\
C^3_{\text{OP-3A}} &= \frac{1}{2T}\log\left(1+\frac{\delta}{N_0/2}\right).
\end{aligned} \tag{18}$$

Here user 1 is decoded first, then user 2, and finally user 3. Similarly, when user 2 is decomposed, let β be the unique number such that

$$\frac{1}{2T}\log\left(1+\frac{w_1T}{N_0/2+\beta}\right)=\frac{1}{2T}\log\left(1+\frac{w_1T}{N_0/2+w_2T}\right), \quad (19)$$

if $\frac{1}{2T}\log\left(1+\frac{w_2T}{N_0/2}\right)<\left(\frac{1}{2}\right)\frac{1}{2T}\log\left(1+\frac{(w_1+w_2)T}{N_0/2}\right)$, and

$$\frac{1}{2T}\log\left(1+\frac{w_1T}{N_0/2+\beta}\right)=\left(\frac{1}{2}\right)\frac{1}{2T}\log\left(1+\frac{(w_1+w_2)T}{N_0/2}\right) \quad (20)$$

otherwise. The resulting capacities for the three users, the capacity of the first user being the symmetric capacity, are

$$\begin{aligned} C^1_{\mathrm{OP-3B}} &= \frac{1}{2T}\log\left(1+\frac{w_1T}{N_0/2+\beta}\right)\\ C^2_{\mathrm{OP-3B}} &= \frac{1}{2T}\log\left(1+\frac{w_2T-\beta}{N_0/2+w_1T+\beta}\right)\\ C^3_{\mathrm{OP-3B}} &= \frac{1}{2T}\log\left(1+\frac{\beta}{N_0/2}\right), \end{aligned} \quad (21)$$

where user 2 is decoded first, then user 1, and finally user 3.

4 The Reliability Function

The reliability function is a powerful result that gives not only the capacity of a channel, but also indicates the exponential rate of decay of probability of error (on the average) as block lengths of the code are increased. Assuming that the input symbols are Gaussian (something already assumed to be true for OP) then a lower bound of the reliability function for the single-user, discrete-time Gaussian channel is given in Section 7.4 of [7]. In this paper it will be referred to as the Gaussian error exponent (GEE). It is not shown here because of the complexity of its description. Denote this bound by then function $E(A, R_N)$ where A is the energy per dimension and R_N is the symbol rate per dimension in a single-user, discrete-time Gaussian channel. That is $r(k) = b(k)+n(k)$ where $\frac{1}{N}\sum_{k=1}^{N} b(k)^2 \leq A$ and $\{n(k)\}$ is a set of i.i.d., zero-mean, Gaussian random variables with unit variance.

Using this function, the CSD and OP can be compared. The capacity formulas of the previous section ((14), (16), (18), (21), and (9)) are all of the form

$$\frac{1}{2T}\log\left(1+A(T)\right), \quad (22)$$

and the GEE is given by $E(A(T), RT)$ where $A(T)$ denotes a function of T, R is the symmetric rate in nats/second, and T gives the number of seconds per dimension. $E(A(T), RT)$ is a continuous function for $T \geq 0$ and it is zero for T larger than some finite number. Although the complexity of the function's description prohibits an analytical solution for the T that maximizes $E(A(T), RT)$, it is possible to find this T numerically. Thus for a given R on a channel of this form, the T that maximizes the GEE can easily be obtained. The resulting T will correspond to a minimum EW-RMS bandwidth necessary to implement the system. For OP this bandwidth will be $\frac{1}{2T}$ and for the CSD it will be $\frac{1}{2T}\frac{\mathrm{tr}(\mathbf{\Lambda\Pi})}{\mathrm{tr}(\mathbf{W})}$. The important point here is that in general the effective, single-user channel capacity is decreasing in T, but that the same is not true of $E(A(T), RT)$.

Several observations illuminate the difficulties of comparing OP to the CSD. Since the effective single-user channels are different for the two cases, the optimum values of T will in general be different. In addition, the maximum values of $E(A(T), RT)$ for the two schemes will be different. One possibility is to find the optimum values of T for OP and the CSD, and then compare the necessary bandwidths as well as the GEEs. In general the result will be that OP requires less bandwidth but also has a smaller GEE value, a vague result to say the least. Another approach is to optimize T for OP and determine the corresponding bandwidth. Then for the CSD find the value of T that corresponds to the OP bandwidth, a T that will most likely be suboptimum from a CSD standpoint in that it will not maximize its GEE. Even so, it will be seen that the CSD out performs OP significantly.

Now, for the two-user case, the GEE of the CSD is compared to that of OP.

Example 3 This compares the CSD and OP schemes for the two-user case. Plotted in Figure 5 are the GEEs. The formulas necessary for the calculations of this example can be found in the appendix. The comparisons are made for a given rate by first calculating the optimum T for OP and finding the associated bandwidth necessary. From this bandwidth the associated value of T for the CSD can be found (it will not be the optimum T from the perspective of the CSD). The error exponents of these two schemes are then plotted as a function of the rate. It is seen that the CSD dominates OP, and this domination will increase as the SNR is increased. For more disparate powers, such significant gains for the CSD are seen only at higher SNRs or when the rate is very small. In either case the effective single-user channels for OP become increasingly poor as the interfering user contributes an increasingly significant amount of "noise" energy. At very small rates this happens because less bandwidth is necessary, meaning that the value of T will become very large, thus increasing the interfering user's energy.

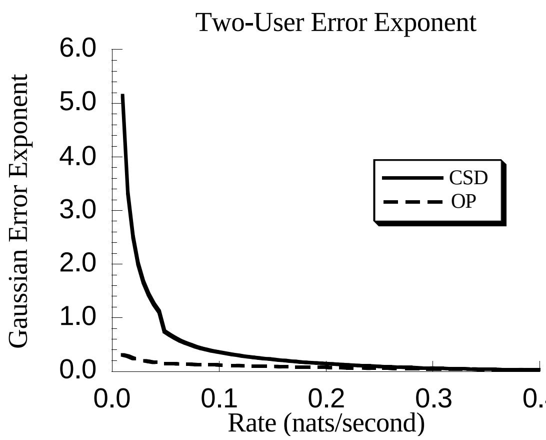

Figure 5: $w_1 = 1.5$, $w_2 = 1.0$, SNR of User 2 is 0 dB, $\gamma = 1$

Conclusion

It has been shown that the CSD yields significant gains in symmetric capacity and in the reliability function over TDMA and OP. The fact that these comparisons are favorable to the CSD, coupled with the facts that the CSD is relatively easy to implement and that it does not require the input symbols to be Gaussian, give credence to the importance of the CSD.

5 Appendix

CSD:

$$\begin{aligned} E_{\text{CSD}}(R,T) &\triangleq E\left(\frac{\gamma w_2 T}{N_0/2}, RT\right) \\ E_{\text{CSD}}(R) &\triangleq \max_T \{E_{CSD}(R,T)\} \end{aligned} \tag{23}$$

OP: When R satisfies (13), let

$$X(T) = E\left(\min\left\{\frac{w_1 T}{N_0/2 + w_2 T}, \frac{w_2 T}{N_0/2}\right\}, RT\right),$$

and when R satisfies (15), let

$$Y(T) = E\left(\min\left\{\frac{w_2 T}{N_0/2 + w_1 T}, \frac{w_1 T}{N_0/2}\right\}, RT\right).$$

Thus

$$\begin{aligned} E_{\text{OP}-2}(R,T) &\triangleq 0 \quad \text{if } R \text{ does not satisfy (13)} \\ &\triangleq X(T) \text{ if } R \text{ satisfies (13) but not (15)} \\ &\triangleq \max\{X(T), Y(T)\} \text{ if } R \text{ satisfies (15)} \end{aligned}$$

When user 1 is decomposed,

$$E_{\text{OP}-3}(R,T) \triangleq \min\left\{E\left(\frac{w_2 T}{N_0/2 + \delta}, RT\right), , \right\}$$

where

$$\begin{aligned} , &= \max_{\Delta}\left\{\min\left\{E\left(\frac{w_1 T - \delta}{N_0/2 + \delta + w_2 T}, R_1 T\right), \right.\right. \\ &\qquad \left.\left. E\left(\frac{\delta}{N_0/2}, R_3 T\right)\right\}\right\} \text{ and} \\ \Delta &= \left\{(R_1, R_3) : 0 \le R_1 \le C^1_{\text{OP}-3\text{A}}, \right. \\ &\qquad \left. 0 \le R_3 \le C^3_{\text{OP}-3\text{A}},\ R_1 + R_3 = R\right\}. \end{aligned}$$

Or when user 2 is decomposed,

$$E_{\text{OP}-3}(R,T) \triangleq \min\left\{E\left(\frac{w_1 T}{N_0/2 + \beta}, RT\right), \Phi\right\}$$

where

$$\begin{aligned} \Phi &= \max_{\Psi}\left\{\min\left\{E\left(\frac{w_2 T - \beta}{N_0/2 + \beta + w_1 T}, R_2 T\right), \right.\right. \\ &\qquad \left.\left. E\left(\frac{\beta}{N_0/2}, R_3 T\right)\right\}\right\} \text{ and} \\ \Psi &= \left\{(R_2, R_3) :\ 0 \le R_2 \le C^2_{\text{OP}-3\text{B}}, \right. \\ &\qquad \left. 0 \le R_3 \le C^3_{\text{OP}-3\text{B}},\ R_2 + R_3 = R\right\}. \end{aligned}$$

Thus,

$$E_{\text{OP}}(R) \triangleq \max_T \{\max\{E_{\text{OP}-2}(R,T), E_{\text{OP}-3}(R,T)\}\}. \tag{24}$$

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