Error Exponents for the Gaussian Multiple-Access Channel

Tommy Guess
Dept. of Electrical and Computer Engineering
University of Colorado at Boulder
Boulder, CO 80309-0425 USA
Email: guess@dsp.colorado.edu

Mahesh K. Varanasi
Dept. of Electrical and Computer Engineering
University of Colorado at Boulder
Boulder, CO 80309-0425 USA
Email: varanasi@schof.colorado.edu

Abstract — **We give random-coding error-exponent bounds for the Gaussian Multiple-Access Channel (GMAC) $\underline{Y} = \mathbf{A}\underline{X} + \underline{N}$, where $\underline{X}$ is the $K \times 1$ vector input, $\mathbf{A}$ is the $M \times K$ channel matrix, and $\underline{N}$ is a zero-mean Gaussian vector with full-rank covariance matrix $\mathbf{N}$. The K users signal independently of each other, and the power of the k^{th} user is $P_k = E\{X_k^2\}$. This vector-valued channel is a generalization of the scalar-valued conventional GMAC, $Y = \sum_{k=1}^{K} X_k + N$, for which $\mathbf{A} = [1 \cdots 1]$; it arises in multiuser signaling schemes such as Code-Division and Bandwidth-Efficient Multiple Access (CDMA and BEMA) [1]. Error exponents give not only the capacity region of the channel, but also an indication of how the average probability of error decays as a function of the block length of random codes.**

I. Introduction

For a coded GMAC, the k^{th} user employs a codeword of length N to transmit a message l_k that is chosen from a set of L_k equiprobable messages. Taking all of the users together, this represents a $(N, (L_k)_{k\in\Gamma})$ multiuser code, where $\Gamma = \{1, 2, \ldots, K\}$ is the set of users. The rate of the k^{th} user is $r_k = \log(L_k)/N$, and for any subset of the users, $\mathcal{J} \subseteq \Gamma$, we let $r_{\mathcal{J}} = \sum_{k\in\mathcal{J}} r_k$. To create the multiuser codebook, it is assumed that the users' codewords are chosen independently; the codewords of the k^{th} user are chosen according to the probability density $Q_k(X_k(1), X_k(2), \ldots, X_k(N))$, where $X_k(n)$ denotes the n^{th} symbol of the k^{th} user's codeword.

The k^{th} user's message is decoded as $\hat{l}_k$ by a maximum-likelihood decoder that jointly decodes the users. There are $2^K - 1$ possible types of error that can occur, corresponding to the non-null subsets of Γ. An error event of Type $\mathcal{J}$ occurs when $\hat{l}_k \neq l_k$ for all $k \in \mathcal{J}$ and $\hat{l}_k = l_k$ for all $k \in \overline{\mathcal{J}}$. If we let $P_{e,\mathcal{J}}$ denote the probability of a Type $\mathcal{J}$ error after being averaged over the codeword ensembles, then $\sum_{\mathcal{J}\subseteq\Gamma} P_{e,\mathcal{J}}$ is an upper bound on the average probability of error. Each term in this summation can be bounded in the form of an *error exponent* to yield

$$P_{e,\mathcal{J}} \leq \exp(-N(E_{0,\mathcal{J}}(\rho, Q) - \rho r_{\mathcal{J}})) \quad \text{for all } \rho \in [0,\, 1], \qquad (1)$$

where the error exponent $E_{0,\mathcal{J}}(\rho, Q)$ depends on the probability density function of the channel, $f(\underline{Y}|\underline{X}) = (2\pi)^{-M/2}|\mathbf{N}|^{-1/2}\exp(-\frac{1}{2}(\underline{Y} - \mathbf{A}\underline{X})^{\mathrm{T}}\mathbf{N}^{-1}(\underline{Y} - \mathbf{A}\underline{X}))$, where $|\cdot|$ denotes the determinant operator. The error exponent

$$E_{r,\mathcal{J}}(r_{\mathcal{J}}, Q) = \max_{\rho\in[0,\,1]}\{E_{0,\mathcal{J}}(\rho, Q) - \rho r_{\mathcal{J}}\} \qquad (2)$$

determines the exponential rate of decay for a Type $\mathcal{J}$ error as the block length increases. The worst-case random-coding error exponent over all error types, that is

$$E_r(Q) = \min_{\mathcal{J}\subseteq\Gamma}\{E_{r,\mathcal{J}}(r_{\mathcal{J}}, Q)\}, \qquad (3)$$

dominates the average probability of error as N goes to infinity.

In the next section, we make use of the following notation. The diagonal matrix containing the users' powers is $\mathbf{P} = \mathrm{diag}(P_1, P_2, \ldots, P_K)$. Let $\mathbf{A}_{\mathcal{J}}$ be the sub-matrix of the channel matrix $\mathbf{A}$ that is formed by retaining only those columns that are in $\mathcal{J}$, and for $K \times K$ diagonal matrices such as $\mathbf{P}$, let $\mathbf{P}_{\mathcal{J}}$ be the principal sub-matrix of $\mathbf{P}$ formed by retaining only those indices in $\mathcal{J}$.

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II. The Error Exponents

First, consider the case where the users' symbol densities are Gaussian (denoted by $Q = \mathrm{G}$), i. e., $Q_k(X_k(1), \ldots, X_k(N)) = \prod_{n=1}^{N}(2\pi P_k)^{-1/2}\exp(-X_k^2(n)/(2P_k))$.

Theorem 1 (Gaussian Error Exponent)

$$E_{0,\mathcal{J}}(\rho, Q = \mathrm{G}) = \frac{\rho}{2}\log\left|\mathbf{I} + \frac{1}{1+\rho}\mathbf{P}_{\mathcal{J}}^{1/2}\mathbf{A}_{\mathcal{J}}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{A}_{\mathcal{J}}\mathbf{P}_{\mathcal{J}}^{1/2}\right|.$$

The overall Gaussian error exponent is found by substituting this into (2) and (3).

Second, consider the case where the density of the k^{th} user is Gaussian *conditioned* on the resulting codeword of this user having a power very close to P_k (denoted by $Q = \mathrm{G_S}$). That is, $Q_k(X_k(1), \ldots, X_k(N)) = \mu_k^{-1}\prod_{n=1}^{N}(2\pi P_k)^{-1/2}\exp(-X_k^2(n)/(2P_k))$ whenever $NP_k - \epsilon \leq \sum_{n=1}^{N} X_k^2(n) \leq NP_k$ for some $\epsilon > 0$, and zero otherwise; μ_k is chosen to yield a valid probability density.

Theorem 2 (Conditional Gaussian Error Exponent)

$$\begin{aligned} &E_{0,\mathcal{J}}(\rho, (d_k)_{k\in\mathcal{J}}, Q = \mathrm{G_S}) \\ &= (\rho+1)\log\left(e\,|\mathbf{K}_{\mathcal{J}}|^{-1/2}\right) - (\rho+1)(1 - \sum_{k\in\mathcal{J}} d_k P_k) + \\ &\quad \frac{\rho}{2}\log\left|\mathbf{I} + \frac{1}{(1+\rho)}\mathbf{K}_{\mathcal{J}}\mathbf{P}_{\mathcal{J}}^{1/2}\mathbf{A}_{\mathcal{J}}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{A}_{\mathcal{J}}\mathbf{P}_{\mathcal{J}}^{1/2}\right|, \end{aligned}$$

where $\mathbf{K} = (\mathbf{I} - 2\mathbf{\Delta}\mathbf{P})^{-1}$ *and* $\mathbf{\Delta} = \mathrm{diag}(d_1,\, \ldots,\, d_K)$*, with each* $d_k \in [0, 1/(2P_k))$ *being arbitrary. Here the expression in (2) is replaced by*

$$E_{r,\mathcal{J}}(r_{\mathcal{J}}, Q = \mathrm{G_S}) = \max\{E_{0,\mathcal{J}}(\rho, (d_k)_{k\in\mathcal{J}}, Q = \mathrm{G_S}) - \rho r_{\mathcal{J}}\},$$

where the maximum is over the set of $\rho \in [0,\, 1]$ *and the set of* $d_k \in [0,\, 1/(2P_k)\,)$*. The overall error exponent is again given by (3).*

For the special case of the conventional GMAC, these two theorems reduce to the results given in [2].

References

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[2] R. G. Gallager, "A Perspective on Multiaccess Channels", *IEEE Trans. Inform. Theory*, IT-31, no. 2, pp. 124-142, Mar. 1985.