This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the ICC 2007 proceedings.

The Effect of Ordered Detection and Antenna Selection on Diversity Gain of Decision Feedback Detector

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***Abstract*— The decision feedback detector (DFD) can achieve the high spectral efficiency of a MIMO channel in that it converts the MIMO channel into multiple parallel *layers*, through which independently coded data substreams may be spatially multiplexed and be transmitted over the same time and frequency slot. Because of independent coding/decoding, the DFD may apply arbitrarily ordered detection. In this paper, we analyze the effect of detection ordering on the diversity gain per layer of the DFD in a MIMO Rayleigh-fading channel. For a MIMO channel with M_t transmit and M_r ($M_r \geq M_t$) receive antennas, we derive an upper bound to the diversity gain per layer for *any* detection ordering, i.e., $\mathsf{D}_i \leq (M_r - i + 1)(M_t - i + 1)$ for $1 \leq i \leq M_t$. We show that the DFD using the so-called *greedy ordering rule* can achieve the diversity gain upper bound. We further study the diversity-multiplexing (D-M) gain tradeoff of DFD in a *pruned* MIMO channel where L_t ($L_t \leq M_t$) transmit and L_r ($L_t \leq L_r \leq M_r$) receive antennas are selected out of the full system. It is shown that the D-M tradeoff of DFD in an optimally pruned channel is $d_{\mathrm{dfd,opt}}(r) = (M_r - L_t + 1)(M_t - L_t + 1)(1 - \frac{r}{L_t})$. Such a tradeoff-optimally pruned system can be obtained by a fast antenna selection algorithm. This result has interesting implications to the multi-access communications with user selection. The theoretical analysis is validated by the numerical examples.**

I. Introduction

The decision feedback detector (DFD) [1], which is also known as the V-BLAST architecture [2], can achieve the high spectral efficiency of a multi-input multi-output (MIMO) channel in that it can convert via successive interference cancellation (SIC) the MIMO channel into multiple parallel *layers*, through which the independently coded data substreams can be spatially multiplexed and be transmitted over the same time and frequency slot. Because of the independent coding/decoding, the DFD can apply arbitrarily ordered detection, which influences the system performance, including the diversity gain performance. In this paper, we analyze the effect of ordered detection on the diversity gain per layer of the DFD in a MIMO Rayleigh-fading channel with M_t transmit antennas and M_r ($M_r \geq M_t$) receive antennas. Although this problem is important for understanding the performance of DFD (V-BLAST), only limited results are available in the literature [3][4][5]. By relating the layer gains to the singular values of the channel matrix, we derive an upper bound to the diversity gain per layer for *any* detection ordering, which is $\mathsf{D}_i \leq (M_r - i + 1)(M_t - i + 1)$ for $1 \leq i \leq M_t$. We further prove that the DFD using the so-called *greedy ordering rule* can achieve the diversity gain upper bound. It is known that the DFD with fixed detection ordering yields layers with diversity gain $\mathsf{D}_i = M_r - i + 1$ for $1 \leq i \leq M_t$. We see that applying ordered detection for DFD can dramatically improve the diversity gain per layer except for the M_tth; the first detected layer has diversity gain only $\mathsf{D}_{M_t} = M_r - M_t + 1$ even with optimal detection ordering [4][5].

Based on the above results on ordered detection, we further study the diversity-multiplexing (D-M) gain tradeoff of DFD in a *pruned* MIMO channel where L_t ($L_t \leq M_t$) transmit and L_r ($L_t \leq L_r \leq M_r$) receive antennas are selected out of the full M_r-by-M_t system. Antenna selection for MIMO systems has been extensively studied as it can significantly reduce the hardware complexity of the system while keeping the benefit of MIMO (see, e.g., [6]). We show that the optimal D-M tradeoff of DFD in the pruned channel is $d_{\mathrm{dfd,opt}}(r) = (M_r - L_t + 1)(M_t - L_t + 1)(1 - \frac{r}{L_t})$. Hence for DFD, applying antenna selection provides the extra benefit of improving its D-M tradeoff in the low multiplexing gain regime. Such a tradeoff-optimally pruned system can be obtained by a fast antenna selection algorithm rather than exhaustive search. This result represents a significant improvement over [7], in which the authors obtain (loose) upper and lower bounds to the D-M tradeoff of DFD with transmit antenna selection only.

The remainder of this paper is organized as follows. Section II introduces the channel model and the QR representation of DFD. Section III derives the upper bounds of the diversity gain per layer yielded by any ordered DFD, which is shown to be achievable. Leveraging the results of Section III, we derive the optimal D-M tradeoff of DFD with antenna selection in Section IV. Section V presents numerical examples to verify our theoretical analysis. Section VI gives the conclusion of this paper and discusses the implications of our results to multi-access communication (MAC) with user selection.

II. Channel Model and Preliminaries

A. Channel Model

Consider a communication system with M_t transmit and M_r receive antennas in a frequency flat fading channel. The sampled baseband signal is given by

$$\mathbf{y} = \mathbf{H}\mathbf{\Pi}\mathbf{x} + \mathbf{z}, \tag{1}$$

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where $\mathbf{x} \in \mathbb{C}^{M_t \times 1}$ is the information symbols, $\mathbf{\Pi} \in \mathbb{R}^{M_t \times M_t}$ is a permutation matrix corresponding to the detection ordering, $\mathbf{y} \in \mathbb{C}^{M_r \times 1}$ is the received signal, and $\mathbf{H} \in \mathbb{C}^{M_r \times M_t}$ is the iid Rayleigh flat fading channel matrix. We assume that $\mathbf{z} \sim N(0, \sigma_z^2 \mathbf{I}_{M_r})$ is the circularly symmetric complex Gaussian noise where $\mathbf{I}_{M_r}$ denotes an identity matrix with dimension M_r. Denote $P_x \triangleq \mathbb{E}[\mathbf{x}^* \mathbf{x}]$ as the total input power. Here $\mathbb{E}[\cdot]$ stands for the expectation, and $(\cdot)^*$ is the conjugate transpose. The input SNR is defined as

$$\rho = \frac{P_x}{\sigma_z^2}. \tag{2}$$

B. Representation of DFD with QR Decomposition

We note that the DFD may suppress the interference by either zero-forcing (ZF) or minimum mean squared error (MMSE) criteria. In this paper, we constrain our discussion to the ZF case. It is well-known that the DFD can be concisely represented by the QR decomposition $\mathbf{H} = \mathbf{QR}$, where $\mathbf{Q}$ is an $M_r \times M_t$ matrix with its orthonormal columns being the interference suppression vectors, and $\mathbf{R}$ is an $M_t \times M_t$ upper triangular matrix with positive diagonal. Correspondingly, the ordered DFD can be represented by applying the QR decomposition to $\mathbf{H}$ with its columns permuted, i.e., $\mathbf{H\Pi} = \mathbf{QR}$ where $\mathbf{\Pi}$ is a permutation matrix.

Now we can rewrite (1) as

$$\mathbf{y} = \mathbf{QRx} + \mathbf{z}. \tag{3}$$

Multiplying $\mathbf{Q}^*$ to both sides of (3) yields

$$\tilde{\mathbf{y}} = \mathbf{Rx} + \tilde{\mathbf{z}}, \tag{4}$$

where $\tilde{\mathbf{y}} = \mathbf{Q}^* \mathbf{y}$ and $\tilde{\mathbf{z}} = \mathbf{Q}^* \mathbf{z}$. The sequential signal detection, which involves the *decision feedback*, is as follows:

for $i = M_t : -1 : 1$
$\quad \hat{s}_i = \mathcal{Q}\left[\left(\tilde{y}_i - \sum_{j=i+1}^{M_t} r_{ij}\sqrt{w_j}\hat{s}_j\right)/r_{ii}\right]$
end

where r_{ij} is the (i,j)th entry of $\mathbf{R}$ and $\mathcal{Q}[\cdot]$ stands for mapping to the nearest point in the symbol constellation. Ignoring the error-propagation effect, we see that the MIMO channel is decomposed into M_t parallel layers

$$\tilde{y}_i = r_{ii} x_i + \tilde{z}_i, \quad i = 1, 2, \cdots, M_t. \tag{5}$$

Because $\mathbb{E}[\tilde{\mathbf{z}}\tilde{\mathbf{z}}^*] = \sigma_z^2 \mathbf{I}$, the output SNR of the ith layer is $r_{ii}^2 P_x / (M_t \sigma_z^2) = r_{ii}^2 \rho / M_t$. Hence given the input SNR, the output SNRs of the substreams are completely determined by the diagonal entries of the upper triangular matrix $\mathbf{R}$ which in turn depend on the permutation matrix $\mathbf{\Pi}$.

With fixed $\mathbf{\Pi}$ the diagonal elements of $\mathbf{R}$ are statistically independent with $\chi^2_{2(M_r - i+1)}$ distribution [8]. The diversity gain of a SISO channel only depends on the distribution of the channel gain around zero [8]. Using this fact, one can readily show that the diversity gain of the ith layer is

$$\mathsf{D}_i \triangleq \lim_{\epsilon \to 0_+} \frac{\log \mathbb{P}(r_{ii}^2 < \epsilon)}{\log \epsilon} = M_r - i + 1, \quad \text{for } 1 \le i \le M_t. \tag{6}$$

To conclude this section, we recall the following useful theorem implied in [9].

Theorem 2.1: Consider the iid Rayleigh fading channel $\mathbf{H}$ given in (1) with ordered singular values $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{M_t} > 0$. Then

$$\lim_{\epsilon \to 0_+} \frac{\log \mathbb{P}(\lambda_i^2 \le \epsilon)}{\log \epsilon} = (M_t - i + 1)(M_r - i + 1). \tag{7}$$

III. On Ordered Detection

In this section, we study the diversity gain per layer of DFD with ordered detection. With channel-dependent permutation matrix $\mathbf{\Pi}$, the distributions of r_{ii}^2's (the diagonal of $\mathbf{R}$ in the QR decomposition $\mathbf{H\Pi} = \mathbf{QR}$) are usually intractable. Therefore the diversity gain analysis is considerably complicated. We focus on computing the maximal diversity gains per layer of DFD using any detection ordering rule. We first derive an upper bound which is then proved to be achieved by a so-called *greedy detection ordering*.

A. Upper Bound of Diversity Gain per Layer

Let us write a permuted channel matrix in its column form:

$$\mathbf{H\Pi} = \left[\mathbf{h}_{\pi(1)}, \cdots, \mathbf{h}_{\pi(i-1)}, \mathbf{h}_{\pi(i)}, \cdots, \mathbf{h}_{\pi(M_t)}\right] = \mathbf{QR}.$$

Denote $\mathbf{H}_i \triangleq \left[\mathbf{h}_{\pi(1)}, \cdots, \mathbf{h}_{\pi(i-1)}\right]$. Then

$$r_{ii}^2 = \mathbf{h}_{\pi(i)}^* \mathbf{P}_{\mathbf{H}_i}^{\perp} \mathbf{h}_{\pi(i)},$$

where $\mathbf{P}_{\mathbf{H}_i}^{\perp} = \mathbf{I} - \mathbf{H}_i(\mathbf{H}_i^*\mathbf{H}_i)^{-1}\mathbf{H}_i^*$. Thus r_{ii}^2 is a function of $\mathbf{H}_i$ and $\mathbf{h}_{\pi(i)}$, and it is invariant to the column permutation of $\mathbf{H}_i$. Hence out of the $M_t!$ detection ordering, one may have up to $\begin{pmatrix} M_t \\ i-1 \end{pmatrix} \cdot (M_t - i + 1) = \frac{M_t!}{(i-1)!(M_t-i)!}$ different values of r_{ii}^2 $(1 \le i \le M_t)$, for which we have established the following theorem.

Theorem 3.1: Consider the ordered QR decomposition $\mathbf{H\Pi} = \mathbf{QR}$ where $\mathbf{\Pi}$ is a permutation matrix which is a function of $\mathbf{H}$. Let r_{ii} be the ith diagonal of $\mathbf{R}$. The inequality

$$\lim_{\epsilon \to 0_+} \frac{\log \mathbb{P}(r_{ii}^2 < \epsilon)}{\log \epsilon} \le (M_t - i + 1)(M_r - i + 1), \quad 1 \le i \le M_t, \tag{8}$$

holds for any ordering rule. In other words, the diversity gain of the ith layer

$$\mathsf{D}_i \le (M_t - i + 1)(M_r - i + 1), \quad 1 \le i \le M_t. \tag{9}$$

Proof: Let $\mathbf{H\Pi} = \mathbf{U\Lambda V}^*$ be the singular value decomposition (SVD) of the permuted channel matrix, where the diagonal entries of $\mathbf{\Lambda}$ are in non-increasing order. An ordered QR decomposition is denoted by $\mathbf{H\Pi} = \mathbf{QR}$. Let $\mathbf{H}_1 \in \mathbb{C}^{M_r \times i}$ and $\mathbf{V}_1 \in \mathbb{C}^{M_t \times i}$ be the submatrices consisting of the first i columns of $\mathbf{H\Pi}$ and $\mathbf{V}^*$, respectively. The ith diagonal entry of $\mathbf{R}$ is (see, e.g., [10])

$$r_{ii}^2 = \frac{1}{[(\mathbf{H}_1^*\mathbf{H}_1)^{-1}]_{ii}} = \frac{1}{[(\mathbf{V}_1^*\mathbf{\Lambda}^2\mathbf{V}_1)^{-1}]_{ii}}, \quad 1 \le i \le M_t. \tag{10}$$

Let us partition the matrices:

$$\mathbf{V}_1 = \begin{pmatrix} \mathbf{V}_{11} \\ \mathbf{V}_{12} \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2 \end{pmatrix}, \tag{11}$$

where $\mathbf{V}_{11} \in \mathbb{C}^{i\times i}$, $\mathbf{V}_{12} \in \mathbb{C}^{(M_t-i)\times i}$, $\mathbf{\Lambda}_1 \in \mathbb{C}^{i\times i}$, and $\mathbf{\Lambda}_2 \in \mathbb{C}^{(M_t-i)\times(M_t-i)}$. Then

$$\mathbf{V}_1^*\mathbf{\Lambda}^2\mathbf{V}_1 = \mathbf{V}_{11}^*\mathbf{\Lambda}_1^2\mathbf{V}_{11} + \mathbf{V}_{12}^*\mathbf{\Lambda}_2^2\mathbf{V}_{12}. \tag{12}$$

Let α be the minimal number such that[1] $\alpha\mathbf{V}_{11}^*\mathbf{V}_{11} \succeq \mathbf{V}_{12}^*\mathbf{V}_{12}$. α is a function of $\mathbf{V}_1$ and hence is independent of $\mathbf{\Lambda}$ since for an iid Rayleigh fading channel, the singular vector matrix $\mathbf{V}$ and the singular value matrix $\mathbf{\Lambda}$ are independent [11]. Because the diagonal of $\mathbf{\Lambda}$ is in non-increasing order,

$$\mathbf{V}_{12}^*\mathbf{\Lambda}_2^2\mathbf{V}_{12} \preceq \lambda_i^2\mathbf{V}_{12}^*\mathbf{V}_{12} \preceq \alpha\lambda_i^2\mathbf{V}_{11}^*\mathbf{V}_{11} \preceq \alpha\mathbf{V}_{11}^*\mathbf{\Lambda}_1^2\mathbf{V}_{11}. \tag{13}$$

It follows from (12) and (13) that $\mathbf{V}_1^*\mathbf{\Lambda}^2\mathbf{V}_1 \preceq (1+\alpha)\mathbf{V}_{11}^*\mathbf{\Lambda}_1^2\mathbf{V}_{11}$. Invoking the fact that $\mathbf{A}^{-1} \succeq \mathbf{B}^{-1}$ if $\mathbf{A} \preceq \mathbf{B}$, we have

$$(\mathbf{V}_1^*\mathbf{\Lambda}^2\mathbf{V}_1)^{-1} \succeq \frac{1}{1+\alpha}(\mathbf{V}_{11}^*\mathbf{\Lambda}_1^2\mathbf{V}_{11})^{-1} = \frac{1}{1+\alpha}\mathbf{V}_{11}^{-1}\mathbf{\Lambda}_1^{-2}\mathbf{V}_{11}^{-*}. \tag{14}$$

Here we have assumed that $\mathbf{V}_{11}$ is nonsingular, which is true with probability one. In special case where $i = M_t$, we have $\mathbf{V}_{11} = \mathbf{V}_1$ and hence $\alpha = 0$. Hence it follows from (10) and (14) that

$$r_{ii}^2 \le \frac{1+\alpha}{[\mathbf{V}_{11}^{-1}\mathbf{\Lambda}_1^{-2}\mathbf{V}_{11}^{-*}]_{ii}} = \frac{1+\alpha}{\sum_{j=1}^{i}|v_{ij}|^2\lambda_j^{-2}} \le \frac{(1+\alpha)\lambda_i^2}{|v_{ii}|^2}, \quad 1 \le i \le M_t \tag{15}$$

where v_{ij} is the (i,j)th entry of $\mathbf{V}_{11}^{-1}$. As both v_{ii} and α are independent of λ_i, so is $\zeta \triangleq \frac{1+\alpha}{|v_{ii}|^2}$. Out of the $M_t!$ detection orderings, we have $\frac{M_t!}{(i-1)!(M_t-i)!}$ different r_{ii}^2's whose associated ζ's are indexed as $\zeta_k,\ k = 1, 2, \cdots, \frac{M_t!}{(i-1)!(M_t-i)!}$. Denote $r_{ii,\max}^2$ the maximal among the $\frac{M_t!}{(i-1)!(M_t-i)!}$ different r_{ii}^2's, and $\zeta_{\max} = \max_{1\le k\le \frac{M_t!}{(i-1)!(M_t-i)!}}\{\zeta_k\}$. Then $r_{ii,\max}^2 \le \zeta_{\max}\lambda_i^2$ with $\zeta_{\max}$ and λ_i^2 independent of each other. Using this property, we have

$$\mathbb{P}\left(r_{ii,\max}^2 < \epsilon\right) \ge \mathbb{P}\left(\zeta_{\max}\lambda_i^2 < \epsilon\right) \ge \mathbb{P}(\zeta_{\max} < c)\mathbb{P}(\lambda_i^2 < \epsilon/c) \tag{16}$$

For any positive c, we can find some finite constant c such that $\mathbb{P}(\zeta_{\max} < c)$ is a strictly positive number. Hence

$$\begin{aligned}\lim_{\epsilon\to 0_+}\frac{\log\mathbb{P}(r_{ii,\max}^2 < \epsilon)}{\log\epsilon} &\le \lim_{\epsilon\to 0_+}\frac{\log\mathbb{P}(\lambda_i^2 < \epsilon/c)}{\log\epsilon} \\ &= (M_t-i+1)(M_r-i+1) \quad \text{for} \quad 1 \le i \le M_t.\end{aligned}$$

Theorem 3.1 is proven. ■

This theorem is a significant improvement over [5], where the authors derived a loose upper bound $\lim_{\epsilon\to 0_+}\frac{\log\mathbb{P}(r_{ii,\max}^2<\epsilon)}{\log\epsilon} \le (M_r-1)(M_t-i+1)$. It follows from (9) that $\mathrm{D}_{M_t} \le M_r - M_t + 1$, i.e., the first detected layer has diversity gain no higher than $M_r - M_t + 1$. It is well-known that if equal rates are allocated across the layers, the overall system performance of DFD (V-BLAST) is limited by the first detected layer. Hence an interesting corollary of Theorem 3.1 is that the DFD (V-BLAST) system with any detection ordering has diversity gain no more than $M_r - M_t + 1$, which agrees with the result in [4].

[1]We write $\mathbf{A} \succeq 0$ if $\mathbf{A}$ is a positive semi-definite matrix, and $\mathbf{A} \succeq \mathbf{B}$ or $\mathbf{B} \preceq \mathbf{A}$ if $\mathbf{A} - \mathbf{B} \succeq 0$.

B. Greedy Ordering Achieves Maximal Diversity Gain

We now introduce the greedy ordering rule. It is shown that the DFD with the greedy ordering achieves the upper bound of the diversity gain given in Theorem 3.1.

Associated with the greedy ordering is the Greedy QR decomposition which plays a key role in the GRT-SMA scheme proposed in [12]. The Greedy QR decomposition consists of M_t recursive steps. We elaborate the first step. The subsequent steps are easily inferred.

In the first step, we go through the following procedures.

(i) Calculate Euclidean norms $\{\|\mathbf{h}_i\|\}_{i=1}^{M_t}$.
(ii) Permute $\mathbf{h}_1$ and $\mathbf{h}_j$ where $j = \arg\max_{1\le i\le M_t}\{\|\mathbf{h}_i\|\}$. This operation can be represented by $\mathbf{H}_1 = \mathbf{H\Pi}_1$ with $\mathbf{\Pi}_1$ being the permutation matrix. (If $j = 1$, $\mathbf{\Pi}_1$ degrades to be $\mathbf{I}_{M_t}$)
(iii) Apply a Householder matrix $\mathbf{Q}_1$ to transform the first column of $\mathbf{H}_1$ to a scaled $\mathbf{e}_1$, where $\mathbf{e}_1$ is the first column of $\mathbf{I}_{M_r}$.

The procedure (i–iii) can be illustrated by

$$\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \xrightarrow{\mathbf{Q}_1^*\mathbf{H\Pi}_1} \begin{pmatrix} r_{11} & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{pmatrix}. \tag{17}$$

In the next step, the same procedures are applied to the trailing $(M_r-1)\times(M_t-1)$ submatrix on the right hand side of (17), which yields a permutation matrix $\mathbf{\Pi}_2$ and a Householder matrix $\mathbf{Q}_2$. After M_t recursive steps, we obtain the desired QR decomposition:

$$\mathbf{R} = \mathbf{Q}^*\mathbf{H\Pi},$$

or equivalently,

$$\mathbf{H\Pi} = \mathbf{QR} \tag{18}$$

where $\mathbf{\Pi} = \mathbf{\Pi}_1\mathbf{\Pi}_2\cdots\mathbf{\Pi}_{M_t-1}$ is permutation matrix and $\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2\cdots\mathbf{Q}_{M_t}$ is unitary matrix ($\mathbf{Q}_{M_t} = \mathbf{I}$ if $M_t = M_r$). In summary, at the ith step this ordering algorithm "greedily" attempts to make the ith diagonal element of $\mathbf{R}$ as large as possible. [2]

The probability density functions of r_{ii}^2's of the Greedy QR decomposition are difficult to obtain if not impossible. However, we have informative bounds on $\{r_{ii}^2\}_{i=1}^{M_t}$ which enable us to obtain the diversity gains of the M_t layers.

Theorem 3.2: Consider a matrix $\mathbf{H} \in \mathbb{C}^{M_r\times M_t}$ with nonzero singular values $\lambda_1 \ge \lambda_2 \cdots \ge \lambda_{M_t} > 0$. Let

$$\mathbf{H\Pi} = \mathbf{QR} \tag{19}$$

be the Greedy QR decomposition. Then

$$\frac{\sum_{j=i}^{M_t}\lambda_j^2}{M_t-i+1} \le r_{ii}^2 \le \lambda_i^2\prod_{j=1}^{i-1}(M_t-j+1), \quad i = 1, 2, \cdots, M_t. \tag{20}$$

Proof: See [12]. ■

[2]The Greedy QR decomposition is not new. The built-in Matlab function is $[\mathbf{Q}, \mathbf{R}, \mathbf{\Pi}]$ = QR($\mathbf{H}$).

It follows from the lower bound in (20) that

$$\begin{aligned}\lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(r_{ii}^2<\epsilon)}{\log\epsilon} &\geq \lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(\lambda_i^2<(M_t-i+1)\epsilon)}{\log\epsilon}\\ &= \lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(\lambda_i^2<\epsilon)}{\log\frac{\epsilon}{M_t-i+1}}\\ &= \lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(\lambda_i^2<\epsilon)}{\log\epsilon}\\ \text{(see Theorem 2.1)} &= (M_t-i+1)(M_r-i+1). \quad (21)\end{aligned}$$

On the other hand, it follows from the upper bound in (20) that

$$\lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(r_{ii}^2<\epsilon)}{\log\epsilon}\leq (M_t-i+1)(M_r-i+1). \quad (22)$$

Therefore

$$\lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(r_{ii}^2<\epsilon)}{\log\epsilon}= (M_t-i+1)(M_r-i+1). \quad (23)$$

Now we have proven the following theorem.

Theorem 3.3: The ith layer of DFD based on the greedy ordering rule, has diversity gain

$$\mathsf{D}_i=(M_t-i+1)(M_r-i+1), \quad 1\leq i\leq M_t. \quad (24)$$

We see from Theorems 3.1 and 3.3 is that the upper bound given in Theorem 3.1 is sharp, i.e., it can be achieved by the DFD with greedy ordering. Moreover, *the greedy ordering rule is diversity gain-optimal among all ordering rules*.

IV. On Antenna Selection

The result established in Section III has immediate implications to the diversity gain performance of DFD in the MIMO channel with antenna selection. We consider the general case where only $L_t\leq M_t$ transmit antennas and $L_r\leq M_r$ receive antennas are selected for data transmission. To make DFD work, we also constrain $L_r\geq L_t$. Denote $\mathcal{S}_t\subset\{1,2,\cdots,M_t\}$ and $\mathcal{S}_r\subset\{1,2,\cdots,M_r\}$ the sets of the indices of antennas selected at transmitter and receiver sides, respectively. The cardinality of the sets $|\mathcal{S}_t|=L_t$ and $|\mathcal{S}_r|=L_r$. We first propose a fast algorithm to determine $\mathcal{S}_t$ and $\mathcal{S}_r$.

A. Fast Antenna Selection Algorithm

The fast antenna selection algorithm applies the same antenna selection routine to the transmit antennas and receive antennas separately. To select the L_t transmit antennas, we apply the iterative Greedy QR decomposition procedure introduced in Section III-B. After L_t recursive steps, we obtain

$$\mathbf{H\Pi}=\mathbf{QR} \quad (25)$$

where $\mathbf{\Pi}=\mathbf{\Pi}_1\mathbf{\Pi}_2\cdots\mathbf{\Pi}_{L_t}$ is a permutation matrix, $\mathbf{Q}=\mathbf{Q}_1\mathbf{Q}_2\cdots\mathbf{Q}_{L_t}$, and $\mathbf{R}$ whose first L_t columns form an upper triangular matrix with positive diagonal elements $\{r_{ii}\}_{i=1}^{L_t}$. (If the procedure is repeated for M_t steps, we obtain the Greedy QR decomposition.) Denoting $\tilde{\mathbf{\Pi}}$ and $\tilde{\mathbf{R}}$ the submatrices consisting of the first L_t columns of $\mathbf{\Pi}$ and $\mathbf{R}$, respectively, we have $\mathbf{H}\tilde{\mathbf{\Pi}}=\mathbf{Q}\tilde{\mathbf{R}}$. We select the transmit antennas whose indices correspond to the nonzero rows of $\tilde{\mathbf{\Pi}}$, and denote their indices as $\mathcal{S}_t$. We denote the channel matrix after transmit antenna selection as $\mathbf{H}_{:,\mathcal{S}_t}\triangleq\mathbf{H}\tilde{\mathbf{\Pi}}\in\mathbb{C}^{M_r\times L_t}$.

To select the $L_r<M_r$ receive antennas, we apply the same procedure to $\mathbf{H}^T_{:,\tilde{\mathcal{S}}_t}\in\mathbb{C}^{L_t\times M_r}$. In this case L_r recursive steps are involved. We denote $\mathcal{S}_r$ as the set of indexes of selected receive antennas. Hence, the pruned channel matrix can be denoted by $\mathbf{H}_{\mathcal{S}_r,\mathcal{S}_t}\in\mathbb{C}^{L_r\times L_t}$.

The following theorem reveals the relationship between the singular values of $\mathbf{H}$ and $\mathbf{H}_{\mathcal{S}_r,\mathcal{S}_t}$.

Theorem 4.1: Let $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{M_t}$ be the singular values of $\mathbf{H}\in\mathbb{C}^{M_r\times M_t}$. Let $\breve{\lambda}_1\geq\cdots\geq\breve{\lambda}_{L_t}$ be the singular values of the pruned channel matrix $\mathbf{H}_{\mathcal{S}_r,\mathcal{S}_t}$ obtained using the proposed fast antenna selection algorithm. Then

$$\lambda_n^2\prod_{i=1}^n\frac{1}{(M_r-i+1)(M_t-i+1)}\leq\breve{\lambda}_n^2\leq\lambda_n^2 \quad \text{for } n=1,\cdots,L_t.$$

Proof: See [13]. ■

B. D-M Tradeoff of DFD with Antenna Selection

Denote $\mathbf{H}_{\mathcal{S}_r,\mathcal{S}_t}\in\mathbb{C}^{L_r\times L_t}$ as the pruned channel matrix. Let $\mathbf{H}_{\mathcal{S}_r,\mathcal{S}_t}=\breve{\mathbf{Q}}\breve{\mathbf{R}}$ be the QR decomposition. For DFD, the ith data substream experiences a fading channel whose channel gain is $\breve{r}_{ii}$ which is the ith diagonal element of $\breve{\mathbf{R}}$. To study the D-M gain tradeoff DFD with antenna selection, we need to analyze the distributions of $\breve{r}_{ii}^2$ around origin, for which we have the following theorem.

Theorem 4.2: Consider the iid Rayleigh channel given in (1). For any pruned channel matrix $\mathbf{H}_{\mathcal{S}_r,\mathcal{S}_t}\in\mathbb{C}^{L_r\times L_t}$, the following inequality holds

$$\lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(\breve{r}_{ii}^2<\epsilon)}{\log\epsilon}\leq (M_t-i+1)(M_r-i+1), \quad 1\leq i\leq L_t. \quad (26)$$

Moreover, if the pruned channel is obtained through the proposed fast antenna selection algorithm, then

$$\lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(\breve{r}_{ii}^2<\epsilon)}{\log\epsilon}= (M_t-i+1)(M_r-i+1), \quad 1\leq i\leq L_t. \quad (27)$$

Proof: Denote $\mathbf{H}_{:,\mathcal{S}_t}\in\mathbb{C}^{M_r\times L_t}$ as the channel matrix after transmit antenna selection, and $\mathbf{H}_{:,\mathcal{S}_t}=\tilde{\mathbf{Q}}\tilde{\mathbf{R}}$ as its QR decomposition. Then according to Theorem 3.1, for any $\mathcal{S}_t$ the diagonal elements of $\tilde{\mathbf{R}}$ satisfy

$$\lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(\tilde{r}_{ii}^2<\epsilon)}{\log\epsilon}\leq (M_t-i+1)(M_r-i+1) \quad \text{for } 1\leq i\leq L_t(=M_t). \quad (28)$$

For any submatrix of $\mathbf{H}_{:,\mathcal{S}_t}$, which we denote as $\mathbf{H}_{\mathcal{S}_t,\mathcal{S}_t}$ whose QR decomposition is $\mathbf{H}_{\mathcal{S}_r,\mathcal{S}_t}=\breve{\mathbf{Q}}\breve{\mathbf{R}}$, it is routine to show that $\breve{r}_{ii}^2\leq\tilde{r}_{ii}^2$ for $1\leq i\leq M_t$. Therefore, it follows from (28) that

$$\lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(\breve{r}_{ii}^2<\epsilon)}{\log\epsilon}\leq (M_t-i+1)(M_r-i+1). \quad (29)$$

Now the first part of Theorem 4.2 is proven.

According to Theorem 4.1, $\xi_i\lambda_i\leq\breve{\lambda}_i\leq\lambda_i$ for some positive constant ξ_i. Then it follows from Theorem 2.1 that

$$\lim_{\epsilon\to 0_+}\frac{\log \mathbb{P}(\breve{\lambda}_i^2<\epsilon)}{\log\epsilon}= (M_t-i+1)(M_r-i+1), \quad 1\leq i\leq M_t. \quad (30)$$

Denote $\mathbf{H}_{\mathcal{S}_r,\mathcal{S}_t}\breve{\mathbf{\Pi}} = \breve{\mathbf{Q}}\breve{\mathbf{R}}$ the greedy QR decomposition. It follows from Theorem 3.2 that $\breve{r}_{ii}^2 \geq \frac{\breve{\lambda}_i^2}{L_t-i+1}$. Hence

$$\begin{aligned}\lim_{\epsilon\to 0_+}\frac{\log\mathbb{P}(\breve{r}_{ii}^2<\epsilon)}{\log\epsilon} &\geq \lim_{\epsilon\to 0_+}\frac{\log\mathbb{P}(\breve{\lambda}_i^2<\epsilon)}{\log\epsilon}\\ &= (M_t-i+1)(M_r-i+1) \quad (31)\\ &\quad \text{for } 1\leq i\leq M_t.\end{aligned}$$

Combining (31) and (29), we have proven the second part of Theorem 4.2. ■

Following [9], we adopt the symbol $\doteq$ to denote *exponential equality*, i.e., we write $f(\rho)\doteq\rho^b$ if $\lim_{\rho\to\infty}\frac{\log f(\rho)}{\log\rho}=b$. Denote the overall multiplexing gain as r ($r\leq L_t$). Each substream has multiplexing gain $\frac{r}{L_t}$. The ith substream is in outage if $\breve{r}_{ii}^2<\rho^{\frac{r}{L_t}-1}$, which, according to Theorem 4.2, has probability

$$P_{\text{outage,dfd}}^{(i)} \doteq \rho^{(M_r-i+1)(M_t-i+1)(\frac{r}{L_t}-1)}, \quad 1\leq i\leq L_t$$

if the fast antenna selection algorithm is used. Hence the overall outage probability of the DFD is dominated by that of the L_tth substream, i.e.,

$$P_{\text{outage,dfd}} \doteq \rho^{(M_r-L_t+1)(M_t-L_t+1)(\frac{r}{L_t}-1)}.$$

Therefore the D-M gain tradeoff of the DFD combined with the fast antenna selection algorithm is

$$d_{\text{dfd,p}}(r) = (M_r-L_t+1)(M_t-L_t+1)\left(1-\frac{r}{L_t}\right). \quad (32)$$

Here the subscript "p" stands for "pruned" system. Moreover, as clearly implied in Theorem 4.2, this tradeoff is also the upper bound to the DFD with any antenna selection approach. As a special case, when $L_t=M_t$,

$$d_{\text{dfd,full}}(r) = (M_r-M_t+1)\left(1-\frac{r}{M_t}\right), \quad (33)$$

which is the D-M tradeoff of DFD even with optimal detection ordering.

Figure 1 shows the D-M gain tradeoff of the DFD in the optimally pruned MIMO channel. Note that antenna selection can improve the D-M tradeoff of DFD at low multiplexing gain regime.

V. Numerical Examples

We present two numerical examples to validate the preceding theoretical analysis.

In the first example, we compare the outage probabilities $\mathbb{P}(r_{ii}^2<\epsilon)$ and $\mathbb{P}(r_{ii,\max}^2<\epsilon)$ in a 3-by-3 system. Here r_{ii} is the gain of the ith layer obtained via DFD using the greedy ordering rule, and $r_{ii,\max}$ is the maximal layer gain over all the $M_t!$ permutations, for $i=1,\dots,M_t$. We run 10^5 Monte Carlo trials to obtain Figure 2. The probabilities $\mathbb{P}(r_{ii,\max}^2<\epsilon)$, $i=2,3$ are the marked solid lines while $\mathbb{P}(r_{ii}^2<\epsilon), i=2,3$ are represented by the marked dot lines. It is easy to see from Section III-B that the Greedy QR yields $r_{11}=r_{11,\max}$. Hence $\mathbb{P}(r_{11,\max}^2<\epsilon)=\mathbb{P}(r_{11}^2<\epsilon)$ and they are represented by the leftmost unmarked solid line. We may observe that

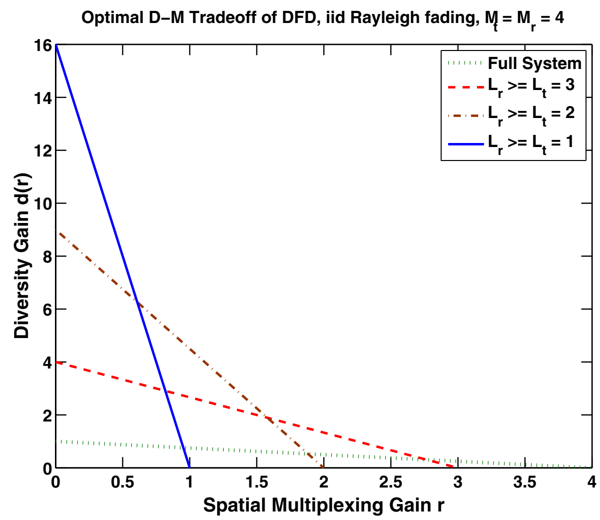

Fig. 1. D-M tradeoffs of the full and pruned DFD systems

the greedy ordering achieves the maximal diversity gains, which agrees with the analysis in Section III-B. From the theoretical analysis, the diversity gains of the three layers are $\mathsf{D}_1=9, \mathsf{D}_2=4$ and $\mathsf{D}_3=1$. At first sight, one may see through comparing the two lines $-$ and $-\circ-$ that the diversity gain difference of r_{11} and $r_{22,\max}$ is seemingly smaller than the theoretical analysis: $\mathsf{D}_1=9, \mathsf{D}_2=4$. Indeed, with a large diversity gain, the outage probability curve approaches a vertical line and increasing the diversity gain further yields only marginal performance gain. We may infer that in the high diversity gain regime, coding gain is more relevant to the system performance. It is important to note that there does *not* exist an ordering which can yield $r_{ii,\max}$ for each i simultaneously.

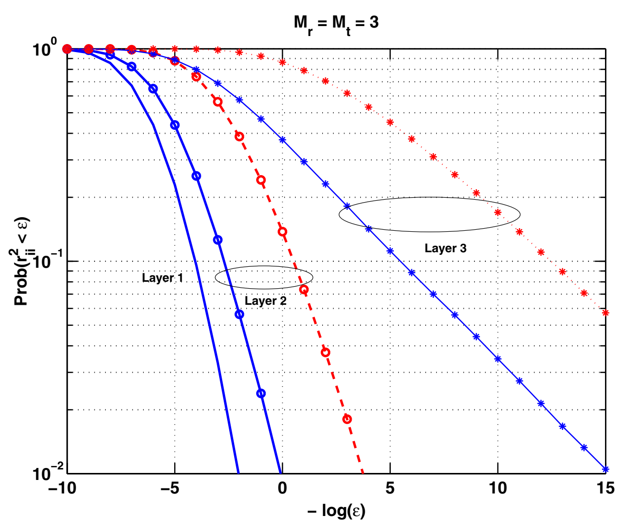

Fig. 2. The outage probabilities of the layers. The solid lines stand for the outage probability $\mathbb{P}(r_{ii,\max}^2<\epsilon)$ and the marked dot lines represent $\mathbb{P}(r_{ii}^2<\epsilon)$.

In the second example, a system with $M_r=4$ and $M_t=3$

is considered. We compare the pruned system where only $L_r = 3$ receive antennas are used by the fast antenna selection algorithm against the full system. Figure 3 shows the diversity gains of the layers obtained via the DFD with the greedy ordering in the full system (the solid lines) and the pruned system (the dashed lines). The figure is obtained by averaging over 10^5 Monte Carlo trials. Comparing the solid lines and the dashed lines, we observe that the layers of the DFD in the pruned system have *no* diversity gain loss compared to the DFD in the full system, although antenna selection does cost some coding gain. The diversity gains of the three layers are $\mathsf{D}_1 = 12, \mathsf{D}_2 = 6, \mathsf{D}_3 = 2$. This simulation result verifies Theorem 4.2.

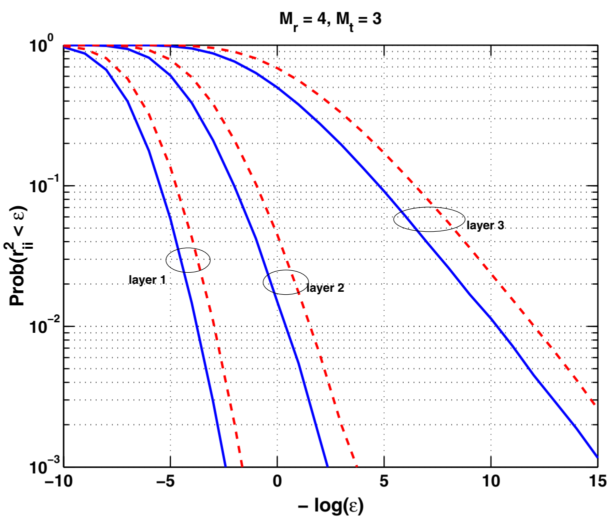

Fig. 3. The outage probabilities of the layers of the full system (the solid lines) and the pruned system where only $L_r = 3$ antennas are used (the dot lines).

VI. Conclusion and Discussion

In this paper, we have analyzed the effect of ordered detection on the diversity gain performance of the DFD/V-BLAST in a MIMO Rayleigh-fading channel. We obtain a sharp upper bound to the diversity gain per layer for *any* detection ordering, which is $\mathsf{D}_i \leq (M_r - i + 1)(M_t - i + 1)$ for $1 \leq i \leq M_t$. Using the so-called *greedy ordering rule*, the DFD can achieve this upper bound. As a corollary, we see that optimal ordering rule does *not* improve the diversity gain of the weakest layer. We also studied the diversity-multiplexing (D-M) gain tradeoff of DFD in a *pruned* MIMO channel where L_t ($L_t \leq M_t$) transmit and L_r ($L_t \leq L_r \leq M_r$) receive antennas are selected out of the full M_r-by-M_t system. We show that the optimal D-M tradeoff of DFD in the pruned channel is $d_{\text{dfd,opt}}(r) = (M_r - L_t + 1)(M_t - L_t + 1)(1 - \frac{r}{L_t})$. Such a tradeoff-optimally pruned system can be obtained by a fast antenna selection algorithm. Two numerical examples are provided to validate the theoretical analysis.

The DFD is also applicable to multi-access communication (MAC) where multi-users communicate with the multi-antenna base station (BS). Consider a multi-access channel with M_t users and the BS which has M_r antennas. In practice, M_t is usually far greater than M_r. Hence to make the DFD work, only $L_t \leq M_r < M_t$ users are selected for simultaneous transmission. The user selection is made possible by $\log_2 \binom{M_t}{L_t}$ bits feedback. Such user selection is tantamount to the transmit antenna selection that we discussed in Section IV. Through this opportunistic user selection, the MAC even with the *suboptimal* DFD has the D-M tradeoff $d_{\text{dfd,opt}}(r)$ given in (32), which may outperform the MAC with the *optimum* ML receiver but with no user selection. The latter is studied in [14], which shows that the maximal diversity gain is no greater than M_r. Such contrast is not surprising though. The fundamental D-M tradeoff given in [14] is derived under the assumption that the users have no CSI and apply equal rate. Our result indicates the huge gain yielded by the finite rate feedback which facilitates the collaboration between the multi-users and BS for opportunistic data transmission. If different rates may be applied to the L_t selected users, even a better D-M tradeoff than (32) can be achieved (see [12]).

References

[1] M. Varanasi and T. Guess, “Optimum decision feedback multiuser equalization with successive decoding achieves the total capacity of the Gaussian multiple-access channel,” *Proceedings of the Thirty-First Asilomar Conference on Signals, Systems and Computers*, vol. 2, pp. 1405 – 1409, Nov 2-5 1997.

[2] G. J. Foschini, G. D. Golden, R. A. Valenzuela, and P. W. Wolniansky, “Simplified processing for high spectral efficiency wireless communication employing multiple-element arrays,” *Wireless Personal Communications*, vol. 6, pp. 311–335, March 1999.

[3] S. Loyka and F. Gagon, “Performance analysis of the V-BLAST algorithm: an analytical approach,” *IEEE Transactions on Wireless Communications*, vol. 3, pp. 1326–1337, July 2004.

[4] Y. Jiang, X. Zheng, and J. Li, “Asymptotic analysis of V-BLAST,” *Proceedings of IEEE GlobeCom*, St. Louis, MO, November 2005.

[5] H. Zhang, H. Dai, Q. Zhou, and B. L. Hughes, “On the diversity-multiplexing tradeoff for ordered SIC receivers over MIMO channels,” *IEEE International Conference on Communications (ICC), Istanbul, Turkey*, June 2006.

[6] A. Gorokhov, D. A. Gore, and A. J. Paulraj, “Receive antenna selection for MIMO spatial multiplexing: Theory and algorithms,” *IEEE Transactions on Signal Processing*, vol. 51, pp. 2796 – 2807, November 2003.

[7] H. Zhang, H. Dai, Q. Zhou, and B. L. Hughes, “Transmit antenna selection for spatial multiplexing systems: A geometrical approach,” *IEEE Transactions on Information Theory*, vol. 52, pp. 5297–5311, Dec. 2006.

[8] D. Tse and P. Viswanath, *Fundamentals of Wireless Communications*. Cambridge Press, 2005.

[9] L. Zheng and D. Tse, “Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels,” *IEEE Transactions on Information Theory*, vol. 49, pp. 1073–1096, May 2003.

[10] J. H. Winters, J. Salz, and R. D. Gitlin, “The impact of antenna diversity on the capacity of wireless communication systems,” *IEEE Transactions on Communications*, vol. 42, pp. 1740–1750, February/March/April 1994.

[11] A. M. Tulino and S. Verdu, *Random Matrix Theory and Wireless Communications*. Hanover, MA 02339, USA: Now Publishers Inc., 2004.

[12] Y. Jiang and M. Varanasi, “A class of spatial multiplexing architecture combining rate-tailored transmission and ordered BLAST detection Part I: On ordered detection,” *IEEE Transactions on Wireless Communications*, submitted.

[13] Y. Jiang and M. Varanasi, “Diversity-multiplexing tradeoff of MIMO systems with antenna selection: Theory and case study,” *IEEE Transactions on Information Theory*, submitted.

[14] D. Tse, P. Viswanath, and L. Zheng, “Diversity-multiplexing tradeoff in multiple access channel,” *IEEE Transactions on Information Theory*, vol. 50, pp. 1859–1874, September 2004.