On Multiuser Detection for Low-Rank CDMA Systems*

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Abstract

We revisit suboptimum linear and group detection for CDMA systems and focus on "over-loaded" or low-rank CDMA systems wherein the total number of signals can be larger than the dimension of the signal space. Such systems are of particular interest when bandwidth is at a premium. Specifically, we consider cases where the desired and/or the interfering signal spaces can be of reduced rank and/or have a non-trivial intersection. Because it generalizes linear detection, we consider group detection and analyze its behavior and asymptotic performance. In particular, we introduce the Group Pseudo-Inverse Detector and the Group MMSE Detector and show the asymptotic (low-noise) equivalence of the two. We also consider a signal design problem wherein an additional user must be introduced into a system which is already either "fully loaded" or "over-loaded" to achieve optimum performance of that user without increasing the system bandwidth.

1 Introduction

Suboptimum detectors for multiuser CDMA communication systems have been extensively studied. The decorrelator and its group counterpart [1] (see also [2]) are designed to project out the multiple-access interference (MAI). However, there detectors can have an unacceptable performance when the desired signals are linearly dependent on the interfering signals. For such scenarios, [3] had proposed to replace the linear decorrelator by a "pseudo-inverse" detector which has a least-squares interpretation.

As for MMSE detection, the linear MMSE detector is known to converge, for low noise power, to the decorrelator, when the desired user's signal is linearly independent of the interfering users' signals [4]. Similarly, the Group MMSE Detector proposed in [5] was shown to converge to the Group Decorrelator if the desired signals are linearly independent of the interfering signals. In either case, the asymptotic behavior of MMSE detection was not considered when the intersection of the desired and interfering signal spaces is a subspace of dimension strictly greater than zero.

In this paper, we focus on low-rank signaling systems and consider the general case of detecting a group of users so that linear detection, wherein only one user is desired, is a special case of our analysis. The linear dependences can occur within the desired or interfering signals and/or they can occur *between* the two groups. We introduce two new group detectors, namely, the Group Pseudo-Inverse Detector and a new Group MMSE Detector. We show that the former is the natural generalization of the notion of (group) decorrelation in that it is applicable to the general CDMA model including, in particular, low-rank systems, and moreover, it is the

*This work was supported in part by ARO Grant DADD19-99-1-0291 and by NSF Grant NCR-9725778.

limiting form (for low noise power) of the latter. By analyzing its asymptotic performance, we consider the problem of introducing an additional user into an already fully- or over-loaded system by maximizing a lower bound on its asymptotic efficiency without increasing the system bandwidth.

2 CDMA System Model

Consider a multiuser system wherein K users employ Quadrature Amplitude Modulation (QAM) to communicate simultaneously and synchronously over an additive white Gaussian channel (AWGN) using signature waveforms that satisfy the generalized Nyquist criterion. Given this set-up, detection on a symbol-by-symbol basis is optimal. The complex baseband equivalent representation of the received signal in a symbol interval is

$$r(t) = \sum_{k=1}^{K} \sqrt{E_k}\, e^{j\phi_k}\, b_k s_k(t) + n(t), \tag{1}$$

where E_k, ϕ_k, b_k and $s_k(t)$ represent, respectively, the received energy, phase, data and the baseband equivalent of the signature waveform for user k. The energies and phases of all users are assumed to be known at the receiver and their signature waveforms are normalized to have unit-energy. Furthermore, all users employ the same signaling alphabet $\boldsymbol{\mathcal{A}} = \{\alpha_i\}_{i=1}^{M}$ also normalized to have unit average energy. Finally, $n(t)$ is a zero-mean proper complex white Gaussian random process with power spectrum of height σ^2. An equivalent discrete-time model can be obtained by matched filtering $r(t)$ with a set of basis functions and sampling the output appropriately. This operation yields the following N-dimensional vector (N is the dimensionality of the signal space)

$$\mathbf{r} = \mathbf{SAb} + \mathbf{n}, \tag{2}$$

where $\mathbf{S} \stackrel{\triangle}{=} [\mathbf{s}_1, \ldots, \mathbf{s}_K]$ denotes the $N \times K$ signal matrix, $\mathbf{A} = \mathrm{diag}\left(\sqrt{E_1}\, e^{j\phi_1}, \ldots, \sqrt{E_K}\, e^{j\phi_K}\right)$ the diagonal matrix of complex received amplitudes, $\mathbf{b}^{\mathsf{T}} = [b_1, \ldots, b_K]$ the data of all users and $\mathbf{n}$ is a zero-mean proper complex Gaussian random vector with covariance matrix $\sigma^2 \mathbf{I}_N$.[1]

As an aside, we note that when the modulation is real (such as pulse amplitude modulation, a special case which has received considerable attention recently), an equivalent real-valued model can be derived as suggested in [1]. Linear (group) detection for this real-valued model is known to outperform linear (group) detection on the complex model [1, Corollary 2]. Furthermore, linear (MMSE and decorrelating) detectors for the equivalent real model of [1] can be shown to be equivalent to the so-called “widely linear” (or “linear-conjugate”) detectors for the complex model as derived in [6–9]. In other words, widely linear detectors are not really new and the corresponding derivations can be avoided by considering linear (group) detection for the equivalent real model as in [1].

3 Group Detection

Group detection was introduced in [1] for systems wherein a subset $G \subseteq \{1, \ldots K\}$ of the K users is to be detected (the complementary set is denoted $\bar{G}$). In this section, we describe it as a two-stage algorithm. It is convenient to introduce the subscripts $_{\mathrm{G}}$ and $_{\bar{\mathrm{G}}}$ to refer to

[1] The superscripts $^{\mathsf{T}}$, † and $^{+}$ denote the transpose, the conjugate transpose and the pseudo-inverse, respectively.

parameters pertaining to the desired and interfering users, respectively, and rearrange users so as to partition the columns of the signal matrix as $\mathbf{S} = [\mathbf{S}_{\mathrm{G}},\, \mathbf{S}_{\bar{\mathrm{G}}}]$. The model in (2) can then be written as $\mathbf{r} = \mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}} + \mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\mathbf{b}_{\bar{\mathrm{G}}} + \mathbf{n}$.

The received signal is first fed to a bank of $|G|$ filters to produce an estimate of the desired users' data under a given design criterion. This linear transformation is referred to as the Group filter and is denoted by $\mathbf{F}_{\mathrm{G}}$. Its output

$$\mathbf{y}_{\mathrm{G}} = \mathbf{F}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}} + \mathbf{F}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\mathbf{b}_{\bar{\mathrm{G}}} + \boldsymbol{\eta}_{\mathrm{G}} \tag{3}$$

is then fed to a decision rule which selects the joint hypothesis for the desired users. Since the estimate (3) is in general not MAI-free, any decision rule that ignores the residual MAI is suboptimal while the maximum likelihood (ML) rule is too complex to implement. We propose a simple decision rule which, while still suboptimal, accounts for the structure of the residual MAI: we assume that it has a Gaussian distribution thereby accounting for its first and second order moments. This rule achieves a good compromise between ease of implementation and performance as will be seen in the numerical simulations.

Since the residual MAI has zero mean and is statistically independent from the Gaussian noise $\boldsymbol{\eta}_{\mathrm{G}}$, the resulting post-filtering statistic can be written as

$$\mathbf{y}_{\mathrm{G}} = \mathbf{F}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}} + \boldsymbol{\gamma}_{\mathrm{G}}, \tag{4}$$

where $\boldsymbol{\gamma}_{\mathrm{G}}$ is the compound residual MAI + noise term. Its covariance matrix is the sum of the covariance matrices of the residual MAI and of the noise, and is denoted as $\sigma^2\mathbf{K}$, with $\mathbf{K} \triangleq \frac{1}{\sigma^2}\mathbf{F}_{\mathrm{G}}^{\dagger}\left(\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{E}_{\bar{\mathrm{G}}}\mathbf{S}_{\bar{\mathrm{G}}}^{\dagger} + \sigma^2\mathbf{I}_N\right)\mathbf{F}_{\mathrm{G}}$ and $\mathbf{E} \triangleq \mathrm{diag}(E_1, \ldots, E_K)$. The decision rule selects the joint hypothesis which yields the maximum likelihood under the Gaussian residual MAI assumption. It is easily shown to be given as

$$\hat{\mathbf{b}}_{\mathrm{G}} \in \arg\max_{\mathbf{b}_{\mathrm{G}}} \left\{2\mathbf{b}_{\mathrm{G}}^{\dagger}\mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\,\boldsymbol{\mathcal{K}}\,\mathbf{r} - \mathbf{b}_{\mathrm{G}}^{\dagger}\mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\,\boldsymbol{\mathcal{K}}\,\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}}\right\}, \tag{5}$$

where we have introduced $\boldsymbol{\mathcal{K}} \triangleq \mathbf{F}_{\mathrm{G}}\mathbf{K}^{+}\mathbf{F}_{\mathrm{G}}^{\dagger}$. Notice that in general, the matrices $\mathbf{K}$ and $\boldsymbol{\mathcal{K}}$ depend on the noise power σ^2.

3.1 Specific examples of Group detectors

The Group Decorrelator was derived in [1] by means of the generalized likelihood ratio test (GLRT). Alternatively, we can describe it as a decorrelating stage, with Group Decorrelating filter[2]

$$\mathbf{F}_{\mathrm{G-D}}^{\dagger} = \mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}}^{\perp}, \tag{6}$$

which completely removes the MAI. Therefore, no Gaussian assumption is required and the following ML rule is optimal

$$\hat{\mathbf{b}}_{\mathrm{G}} \in \arg\max_{\mathbf{b}_{\mathrm{G}}} \left\{2\mathbf{b}_{\mathrm{G}}^{\dagger}\mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}}^{\perp}\mathbf{r} - \mathbf{b}_{\mathrm{G}}^{\dagger}\mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}}^{\perp}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}}\right\}. \tag{7}$$

While it possesses the attractive feature of requiring neither the interfering users' energy nor the noise power, the Group Decorrelator completely cancels the MAI regardless of the signal space geometry. In low-rank (overloaded) systems where the desired and interfering signals

[2] Henceforth, $\mathcal{X}$ will denote the span (or range) of the matrix $\mathbf{X}$ and $\mathbf{P}_{\mathcal{X}}^{\perp}$ the projection orthogonal to $\mathcal{X}$.

can be linearly dependent, the Group Decorrelating filter will cancel the desired signals that lie in the interfering space. Clearly, this can lead to an unacceptable performance degradation.

Group MMSE detectors were proposed in [5] and [10] without, however, accounting for the residual MAI. They were shown to converge (as the noise power goes to zero) to the Group Decorrelator for the case of linearly independent signaling. In the low-rank case however, their asymptotic behavior was not studied nor was their performance analyzed. We will address such issues in Section 5 where we propose a Group MMSE detector based on the ML rule under the Gaussian MAI assumption.

4 Group Pseudo-Inverse Detection

The Group Decorrelation approach applied to low-rank signaling can result in the cancellation of the useful signals. Instead, we propose, in this Section, a Group Pseudo-Inverse Detector which generalizes the Group Decorrelator. While the two group detectors are equivalent when the desired signals are linearly independent from the interfering signals, the Group Pseudo-Inverse Detector acts as a *partial decorrelator* in the linearly dependent case.

The first stage of the new group detector consists of a least-squares estimation of the data $\mathbf{y} \in \arg\min_{\mathbf{b}} ||\mathbf{r} - \mathbf{SAb}||^2$ which yields the estimate $\mathbf{y} = (\mathbf{SA})^+\mathbf{r}$. Notice this is also the unconstrained ML estimate of $\mathbf{b}$, i. e., when $\mathbf{b} \in I\!R^K$ rather than $\boldsymbol{\mathcal{A}}^K$. The estimate $\mathbf{y}_{\rm G}$ of the desired data corresponds to the coordinates of $\mathbf{y}$ in G. The corresponding linear transformation $\mathbf{F}^\dagger_{\rm G-PI} = [\mathbf{I}_{\rm G},\, \mathbf{0}](\mathbf{SA})^+$ is the Group Pseudo-Inverse filter and its output is

$$\mathbf{y}_{\rm G} = [\mathbf{I}_{\rm G},\, \mathbf{0}](\mathbf{SA})^+\mathbf{SAb} + \mathbf{F}^\dagger_{\rm G-PI}\mathbf{n}. \tag{8}$$

The decision rule is then given as in (5) with the appropriate matrices $\mathbf{K}_{\rm PI}$, the compound residual MAI + noise covariance matrix, and $\boldsymbol{\mathcal{K}}_{\rm PI}$.

4.1 Linearly independent desired and interfering signals

Consider the case of desired signals being linearly independent of the interfering signals, i. e., $\dim(\mathcal{S}_{\rm G} \cap \mathcal{S}_{\bar{\rm G}}) = 0$. The following lemma (proofs of lemmas are omitted) is applied to the Group Pseudo-Inverse filter.

Lemma 1 *Let $\mathbf{Q}$ be an $m \times n$ matrix with $m < n$ which is partitioned as $\mathbf{Q} = [\mathbf{A},\, \mathbf{B}]$, where $\mathbf{A}$ and $\mathbf{B}$ are not necessarily full rank but such that the subspaces they span are linearly independent, i. e., rank$(\mathbf{Q})$ = rank$(\mathbf{A})$ + rank$(\mathbf{B})$. The Moore-Penrose pseudo-inverse of $\mathbf{Q}$ has the following partitioned form*

$$\mathbf{Q}^+ = \begin{bmatrix} \left(\mathbf{A}^\dagger\mathbf{P}^\perp_{\mathcal{B}}\mathbf{A}\right)^+ \mathbf{A}^\dagger\mathbf{P}^\perp_{\mathcal{B}} \\ \left(\mathbf{B}^\dagger\mathbf{P}^\perp_{\mathcal{A}}\mathbf{B}\right)^+ \mathbf{B}^\dagger\mathbf{P}^\perp_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{P}^\perp_{\mathcal{B}}\mathbf{A}\right)^+ \\ \left(\mathbf{P}^\perp_{\mathcal{A}}\mathbf{B}\right)^+ \end{bmatrix}. \tag{9}$$

Using Lemma 1 we find that $\mathbf{F}^\dagger_{\rm G-PI} = \left(\mathbf{P}^\perp_{\mathcal{S}_{\bar{\rm G}}}\mathbf{S}_{\rm G}\mathbf{A}_{\rm G}\right)^+$, i. e., the Group Pseudo-Inverse filter is equivalent to the Group Decorrelating filter in (6). Furthermore, it can be shown that

$$\mathbf{K}_{\rm PI} = \left(\mathbf{A}^\dagger_{\rm G}\mathbf{S}^\dagger_{\rm G}\mathbf{P}^\perp_{\mathbf{S}_{\bar{\rm G}}}\mathbf{S}_{\rm G}\mathbf{A}_{\rm G}\right)^+ \text{ and } \boldsymbol{\mathcal{K}}_{\rm PI} = \mathbf{P}^\perp_{\mathbf{S}_{\bar{\rm G}}}\mathbf{S}_{\rm G}\mathbf{A}_{\rm G}\,\mathbf{K}\,\mathbf{A}^\dagger_{\rm G}\mathbf{S}^\dagger_{\rm G}\mathbf{P}^\perp_{\mathbf{S}_{\bar{\rm G}}} \tag{10}$$

(notice they don't depend on the noise power) so that the decision rule of the Group Pseudo-Inverse detector is

$$\hat{\mathbf{b}}_{\rm G} \in \arg\max_{\mathbf{b}_{\rm G}} \left\{2\mathbf{b}^\dagger_{\rm G}\mathbf{A}^\dagger_{\rm G}\mathbf{S}^\dagger_{\rm G}\,\mathbf{P}^\perp_{\mathbf{S}_{\bar{\rm G}}}\mathbf{r} - \mathbf{b}^\dagger_{\rm G}\mathbf{A}^\dagger_{\rm G}\mathbf{S}^\dagger_{\rm G}\,\mathbf{P}^\perp_{\mathbf{S}_{\bar{\rm G}}}\,\mathbf{S}_{\rm G}\mathbf{A}_{\rm G}\mathbf{b}_{\rm G}\right\}. \tag{11}$$

Consequently, the Group Pseudo-Inverse Detector simplifies to the Group Decorrelator. Note that our derivation is more general than in [1] because we allow for linear dependencies within the set of desired signals or of interfering signals, i. e., $\mathbf{S}_{\mathrm{G}}$ and/or $\mathbf{S}_{\bar{\mathrm{G}}}$ can be column-rank deficient.

4.2 Linearly dependent desired and interfering signals

When $\dim(\mathcal{S}_{\mathrm{G}} \cap \mathcal{S}_{\bar{\mathrm{G}}}) > 0$, i. e., the desired signals are linearly dependent on the interfering signals, we use the following result in linear algebra which can be proved by using Greville's formula for the update of a pseudo-inverse [11].

Lemma 2 *If the k^{th} column of the matrix $\mathbf{Q}$, denoted $\mathbf{q}_k$, is linearly independent from the remaining columns, then $(\mathbf{Q}^+\mathbf{Q})_{kj} = \delta_{kj}$, otherwise $(\mathbf{Q}^+\mathbf{Q})_{kj} = \frac{1}{1+||\mathbf{x}_k||^2} \times \begin{cases} ||\mathbf{x}_k||^2 & \text{if } j = k \\ (\mathbf{x}_k)_j & \text{if } j \neq k \end{cases}$, where $\mathbf{x}_k \triangleq \bar{\mathbf{Q}}_k^+\mathbf{q}_k$ is the "coordinate" vector and $(\mathbf{x}_k)_j$ is its j^{th} component.*[3]

Using Lemma 2 to determine the output of the Group Pseudo-Inverse filter, we find that $\mathbf{F}_{\mathrm{G-PI}}$ acts as a partial decorrelator. Indeed, for each desired signal $\mathbf{s}_k$, only those signals in $\bar{\mathbf{S}}_k$ from which it is linearly independent have a corresponding component in $\mathbf{x}_k$ equal to zero. These users are therefore canceled.

The Group filter output is not MAI-free so that the matrices $\mathbf{K}_{\mathrm{PI}}$ and $\boldsymbol{\mathcal{K}}_{\mathrm{PI}}$, and consequently the Group Pseudo-Inverse detector, depend on the noise power and on the energies of the interferers that are not canceled by $\mathbf{F}_{\mathrm{G-PI}}$.

5 Group MMSE Detection

In this section, we propose a new Group MMSE Detector and study its asymptotic form. Let us assume that $\mathbf{b}$ is a zero-mean Gaussian random vector with covariance $\mathbf{I}_K$. Under this assumption, $\mathbf{r}$ and $\mathbf{b}_{\mathrm{G}}$ are jointly Gaussian. Therefore, the (unconstrained) MMSE estimate of $\mathbf{b}_{\mathrm{G}}$ given the observation $\mathbf{r}$ is easily obtained from the Gauss-Markov theorem as the linear transformation $\mathbf{F}_{\mathrm{G-M}}^{\dagger}\mathbf{r}$, where we have introduced the Group MMSE filter

$$\mathbf{F}_{\mathrm{G-M}} = \mathbf{H}^{-1}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}} \tag{12}$$

and $\mathbf{H} \triangleq E\left[\mathbf{r}\mathbf{r}^{\dagger}\right] = \mathbf{S}\mathbf{E}\mathbf{S}^{\dagger} + \sigma^2\mathbf{I}_N$. This filter requires the knowledge of the signals and amplitudes of all users as well as the noise power. Under the Gaussian residual MAI assumption, the post-filtering statistic is $\mathbf{y}_{\mathrm{G}} = \mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\mathbf{H}^{-1}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}} + \boldsymbol{\gamma}_{\mathrm{G}}$, where $\boldsymbol{\gamma}_{\mathrm{G}}$ has covariance $\sigma^2\mathbf{K}_{\mathrm{M}} \triangleq \mathbf{G} - \mathbf{G}^2$ and $\mathbf{G} \triangleq \mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\mathbf{H}^{-1}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$. The ML rule is

$$\hat{\mathbf{b}}_{\mathrm{G}} \in \arg\max_{\mathbf{b}_{\mathrm{G}}} \left\{2\mathbf{b}_{\mathrm{G}}^{\dagger}\mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\,\boldsymbol{\mathcal{K}}_{\mathrm{M}}\,\mathbf{r} - \mathbf{b}_{\mathrm{G}}^{\dagger}\mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\,\boldsymbol{\mathcal{K}}_{\mathrm{M}}\,\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}}\right\}, \tag{13}$$

with $\boldsymbol{\mathcal{K}}_{\mathrm{M}} = \mathbf{H}^{-1}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{K}_{\mathrm{M}}^{+}\mathbf{A}_{\mathrm{G}}^{\dagger}\mathbf{S}_{\mathrm{G}}^{\dagger}\mathbf{H}^{-1}$. The resulting cascade of operations (Group MMSE filtering and ML decision rule) is termed the Group MMSE Detector. It differs from the one in [5] and [10] in that it accounts for the residual MAI. However, for a group size one, they coincide and reduce to the classical MMSE solution of [4].

[3] $\bar{\mathbf{Q}}_k$ contains all columns of $\mathbf{Q}$ except $\mathbf{q}_k$

5.1 Asymptotic form of the Group MMSE Detector

We find the asymptotic (low noise power) form of the Group MMSE Detector by considering the limits of the Group MMSE filter and of the subsequent decision rule. For the latter limit, we need to consider the linear independent and dependent cases separately.

First, we write the Group MMSE filter in terms of $\mathbf{F}_{\mathrm{M}}$, the joint MMSE filter for all users, as $\mathbf{F}_{\mathrm{G-M}}^{\dagger} = [\mathbf{I}_{\mathrm{G}},\, \mathbf{0}]^{\dagger}\mathbf{F}_{\mathrm{M}}$.

Lemma 3 *The limit of* $\mathbf{F}_{\mathrm{G-M}}$ *can be found from that of* $\mathbf{F}_{\mathrm{M}}$ *which is* $\lim_{\sigma^2\to 0} \mathbf{F}_{\mathrm{M}}^{\dagger} = (\mathbf{SA})^{+}$.

Therefore, the Group MMSE filter is asymptotically equivalent to the Group Pseudo-Inverse filter.

Lemma 4 *In the linear independent case, when* $\dim(\mathcal{S}_{\mathrm{G}} \cap \mathcal{S}_{\bar{\mathrm{G}}}) = 0$,

$$\lim_{\sigma^2\to 0} \mathbf{K}_{\mathrm{M}} = \mathbf{K}_{\mathrm{PI}} \quad \text{and} \quad \lim_{\sigma^2\to 0} \boldsymbol{\mathcal{K}}_{\mathrm{M}} = \boldsymbol{\mathcal{K}}_{\mathrm{PI}}$$

with $\mathbf{K}_{\mathrm{PI}}$ *and* $\boldsymbol{\mathcal{K}}_{\mathrm{PI}}$ *given as in (10).*

In the linear dependent case, when $\dim(\mathcal{S}_{\mathrm{G}} \cap \mathcal{S}_{\bar{\mathrm{G}}}) > 0$,

$$\lim_{\sigma^2\to 0} \sigma^2\mathbf{K}_{\mathrm{M}} = \lim_{\sigma^2\to 0} \sigma^2\mathbf{K}_{\mathrm{PI}} \quad \text{and} \quad \lim_{\sigma^2\to 0} \sigma^2\boldsymbol{\mathcal{K}}_{\mathrm{M}} = \lim_{\sigma^2\to 0} \sigma^2\boldsymbol{\mathcal{K}}_{\mathrm{PI}}$$

and these limits are non-zero.

Therefore, when the desired and interfering signal spaces are linearly independent of each other, the Group MMSE Detector converges to the Group Pseudo-Inverse Detector (which in turn reduces to the Group Decorrelator). Notice we have made no assumptions on the ranks of the desired and interfering signal matrices. This result generalizes that of [4] which proves that the linear MMSE detector converges to the decorrelator when the desired user's signal is linearly independent from the interfering signals. The Group MMSE detectors proposed in [5,10] without the Gaussian assumption were also shown to converge to the Group Decorrelator in the case of linearly independent signaling.

The second part of the lemma states that even when the desired and interfering spaces have a non-trivial intersection, the asymptotic form of the Group MMSE Detector is the Group Pseudo-Inverse Detector. The special case of group size one is as follows.

Fact 1 *The linear MMSE detector converges to the pseudo-inverse detector of [3] when the desired signal lies in the interference subspace.*

5.2 Numerical examples

We illustrate the performance and asymptotic behavior of the group detectors we have discussed for a CDMA system with $K = 8$ users employing 4-QAM. We assume they all have unit energies $\mathbf{E} = \mathbf{I}_K$ but that their phases are different. The set of desired users is $G = \{1, 2\}$ and we consider several signal space geometries. In all examples, the signals and phases are randomly generated.

In Figure 1, the desired signals are linearly independent from the interfering signals. As expected the Group Decorrelating and Pseudo-Inverse Detectors are equivalent. The Group MMSE Detector accounts for the MAI so that is outperforms the one proposed in [5, 10].

Figures 2 and 3 illustrate overloaded systems. In Figure 2 the desired signals are linearly dependent on the interfering signals and in Figure 3 they lie entirely in the interfering space. In

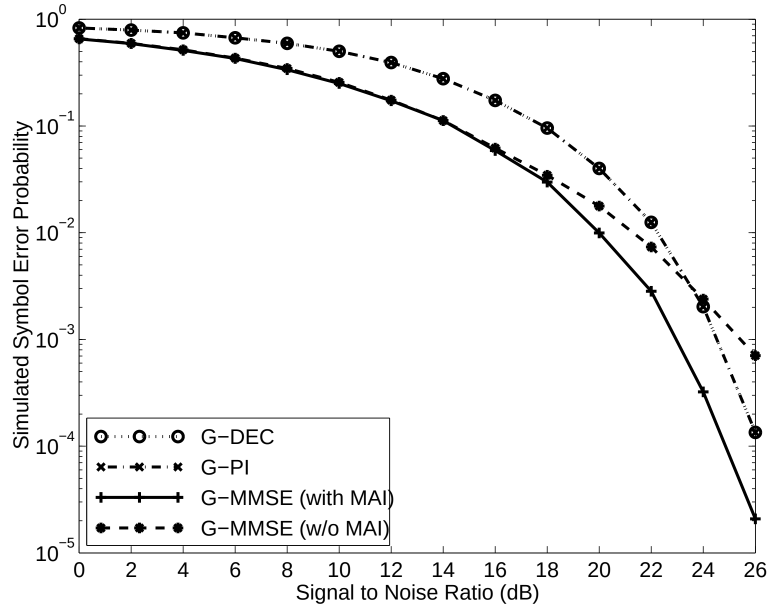

Figure 1: Fully loaded system with $(N, K, |G|) = (8, 8, 2)$ and 4-QAM: $\mathrm{rank}(\mathbf{S}_{\bar{\mathrm{G}}}) = 6$, $\mathrm{rank}(\mathbf{S}_{\mathrm{G}} = 2)$ and $\mathrm{rank}\left(\mathbf{P}^{\perp}_{\mathcal{S}_{\bar{\mathrm{G}}}} \mathbf{S}_{\mathrm{G}}\right) = 2$, i. e., the desired and interfering spaces are linearly independent.

both cases, the Group Pseudo-Inverse and MMSE detectors, which are asymptotically equivalent, have almost identical performance even at low SNRs. The Group Decorrelator cancels some or all the desired signals so that it is useless while the Group MMSE detector of [5, 10] is interference limited and its error probability floors.

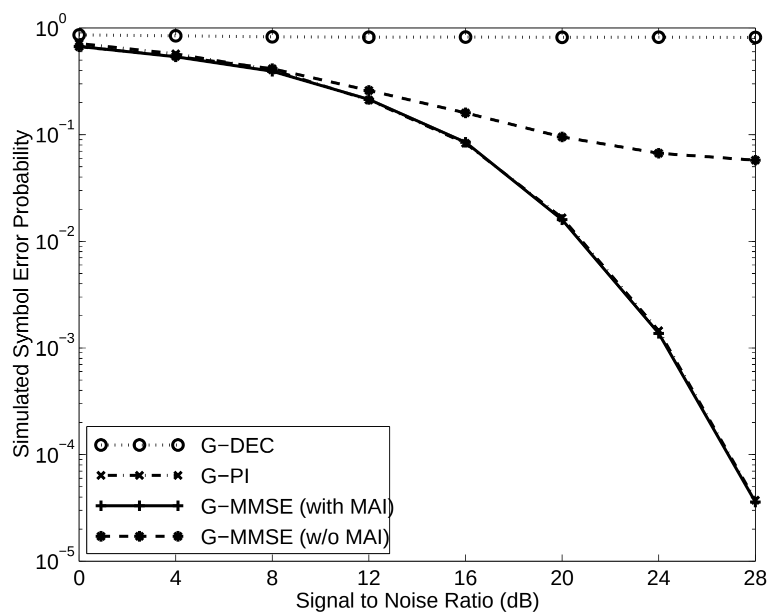

Figure 2: Overloaded system with $(N, K, |G|) = (7, 8, 2)$ and 4-QAM: $\mathrm{rank}(\mathbf{S}_{\bar{\mathrm{G}}}) = 7$, $\mathrm{rank}(\mathbf{S}_{\mathrm{G}} = 2)$ and $\mathrm{rank}\left(\mathbf{P}^{\perp}_{\mathcal{S}_{\bar{\mathrm{G}}}} \mathbf{S}_{\mathrm{G}}\right) = 1$.

6 Signal Design for Linear Detection in Low-Rank Systems

In this section, we assume that the system has $K-1$ active users. Their signal matrix, complex amplitudes and data are denoted by $\bar{\mathbf{S}}_1$, $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{b}}_1$, respectively. The problem we pose is to

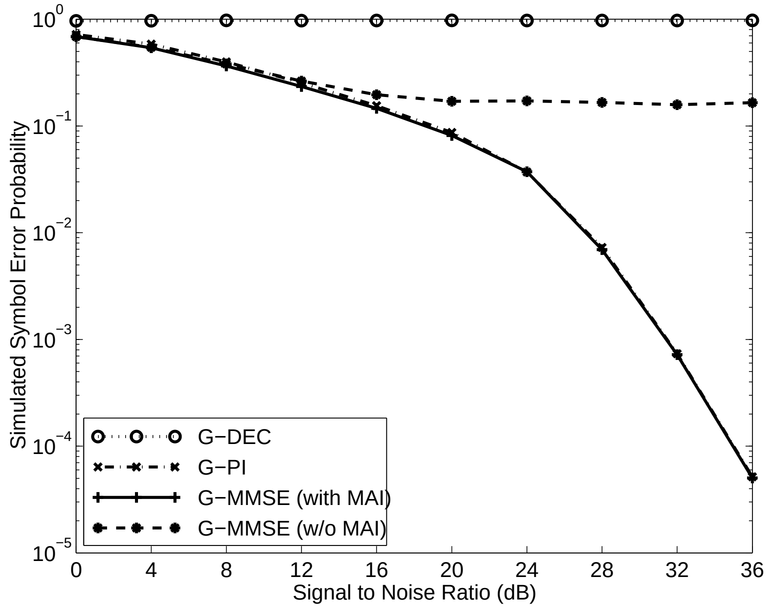

Figure 3: Overloaded system with $(N, K, |G|) = (6, 8, 2)$ and 4-QAM: rank$(\mathbf{S}_{\bar{\mathrm{G}}}) = 8$, rank$(\mathbf{S}_{\mathrm{G}} = 2)$ and rank$\left(\mathbf{P}^{\perp}_{\mathcal{S}_{\bar{\mathrm{G}}}} \mathbf{S}_{\mathrm{G}}\right) = 0$, i. e., the desired signal space lies entirely in the interference.

assign a signal to a new user so as to maximize its asymptotic efficiency with MMSE detection *without* increasing the signal space dimension. Since MMSE and pseudo-inverse detection are asymptotically equivalent in this case, we can equivalently maximize the asymptotic efficiency using the pseudo-inverse detector.

When the desired signal $\mathbf{s}_1$ lies in the space spanned by the interfering signals denoted by $\bar{\mathbf{S}}_1$, we define the coordinate vector $\mathbf{x} \triangleq \left(\bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1\right)^+ \mathbf{s}_1$, where $\bar{\mathbf{A}}_1$ is the diagonal matrix of the interferers complex amplitudes.[4] The pseudo-inverse filter $\mathbf{f}_{\mathrm{G-PI}}$ is then obtained from Greville's formula for the column update of a pseudo-inverse [11] and is given, up to a scalar constant, by

$$\mathbf{f}^{\dagger}_{\mathrm{G-PI}} = \mathbf{x}^{\dagger} \left(\bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1\right)^+ . \tag{14}$$

Since $\mathbf{x}$ is in the range of $(\bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1)^+$ and $(\bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1)^+ \bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1$ represents the projection orthogonal to the range of $(\bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1)^+$, it follows that $(\bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1)^+ \bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1 \mathbf{x} = \mathbf{x}$, or equivalently $\mathbf{x}^{\dagger} \left(\bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1\right)^+ \bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1 = \mathbf{x}^{\dagger}$. Consequently, the post-filtering statistic $y_1 = \mathbf{f}^{\dagger}_{\mathrm{G-PI}} \mathbf{r}$ becomes

$$y_1 = A_1 ||\mathbf{x}||^2 b_1 + \mathbf{x}^{\dagger} \bar{\mathbf{b}}_1 + \eta_1, \tag{15}$$

where η_1 is a zero-mean Gaussian random variable with variance $\sigma^2 \mathbf{x}^{\dagger} \left(\bar{\mathbf{A}}_1^{\dagger} \bar{\mathbf{S}}_1^{\dagger} \bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1\right)^+ \mathbf{x}$ and $\bar{\mathbf{b}}_1$ represents the data of the interfering users.[5] The decision rule reduces to a QAM slicer.

Our signal design is based on an exact performance analysis where we make no assumptions about the residual MAI. The conditional error exponent $\varepsilon_{ij}(\bar{\mathbf{b}}_1)$ of the conditional pairwise error probability that hypothesis H_j is more likely than the true hypothesis H_i (corresponding to hypothesis α_j and α_i, respectively), can be shown to be given by

$$\varepsilon_{ij}(\bar{\mathbf{b}}_1) = \max{}^2 \left\{ 0,\ \frac{\sqrt{E_1}\, ||\mathbf{x}||^2 |e_{ij}| - 2\mathcal{R}e\left(e^{j(\phi_1 - \arg(e_{ij}))}\, \mathbf{x}^{\dagger} \bar{\mathbf{b}}_1\right)}{2\sigma \sqrt{\mathbf{x}^{\dagger} \left(\bar{\mathbf{A}}_1^{\dagger} \bar{\mathbf{S}}_1^{\dagger} \bar{\mathbf{S}}_1 \bar{\mathbf{A}}_1\right)^+ \mathbf{x}}} \right\}, \tag{16}$$

[4]Notice that in this case, the $\bar{\mathbf{S}}_1$ has have full row rank, so that $\bar{\mathbf{S}}_1 \bar{\mathbf{S}}_1^+ = \mathbf{I}_N$.

[5]To compute the noise variance, we have used the result that for any matrix $\mathbf{X}$, $\mathbf{X}^+ \mathbf{X}^{+\dagger} = \left(\mathbf{X}^{\dagger} \mathbf{X}\right)^+$.

where we introduce the error event $e_{ij} \stackrel{\triangle}{=} \alpha_j - \alpha_i = |e_{ij}|e^{j\arg(e_{ij})}$. Solving for the unit energy signal $\mathbf{s}_1$ which will maximize the worst case conditional error exponent (minimized over all interfering data vectors and pairs of hypothesis) is a problem that is not analytically tractable.

Instead, we use the Cauchy-Schwarz inequality to lower bound the worst case conditional error exponent and optimize this lower bound. Recalling that $\mathbf{x} = (\bar{\mathbf{S}}_1\bar{\mathbf{A}}_1)^+\mathbf{s}_1$, the lower bound is obtained as

$$\min_{i\neq j}\min_{\bar{\mathbf{b}}_1}\varepsilon_{ij}(\bar{\mathbf{b}}_1) \geq \tag{17}$$

$$\max{}^2\left\{0,\ \frac{\sqrt{E_1}\,\mathbf{s}_1^\dagger\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^\dagger\right)^+\mathbf{s}_1|\alpha_{\min}| - 2\sqrt{\mathbf{s}_1^\dagger\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^\dagger\right)^+\mathbf{s}_1}\sqrt{K-1}|\alpha_{\max}|}{2\sigma\sqrt{\mathbf{s}_1^\dagger\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^\dagger\right)^{2+}\mathbf{s}_1}}\right\} \tag{18}$$

where $|\alpha_{\max}| \stackrel{\triangle}{=} \max_{\alpha_k\in\boldsymbol{\mathcal{A}}}|\alpha_k|$ and $|\alpha_{\min}| \stackrel{\triangle}{=} \min_{i\neq j}|\alpha_j - \alpha_i|$.

Proposition 1 *The lower bound in (18) is maximized by choosing* $\mathbf{s}_1$ *as the eigenvector of* $\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^\dagger$ *corresponding to its minimum non-zero eigenvalue which we denote as* $\lambda_{\min}$. *With such a choice, the lower bound (18) becomes*

$$\max{}^2\left\{0,\ \frac{\sqrt{E_1}}{2\sigma}\left(|\alpha_{\min}| - 2\sqrt{\lambda_{\min}\frac{K-1}{E_1}}|\alpha_{\max}|\right)\right\}. \tag{19}$$

We illustrate our signal design in Figure 4 by considering the same system as in Section 5.2 with $(N, K, M) = (7, 8, 4)$. The optimum signal which maximizes the worst case conditional error exponent is found numerically. Its simulated performance is virtually identical to that of the signal which maximizes the lower bound and far outperforms that of randomly generated signals for the new user.

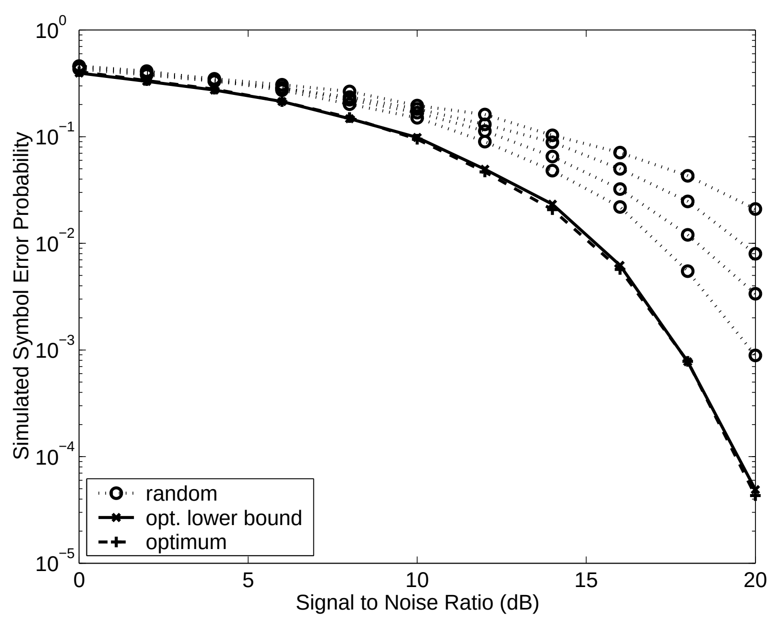

Figure 4: Signal design for linear MMSE or psuedo-inverse detection in an overloaded system: $(N, K, M) = (7, 8, 4)$.

7 Conclusion

Suboptimum linear and group detection was studied for general CDMA systems with special attention to over-loaded CDMA systems. Two new group detectors, namely, the Group Pseudo-Inverse and a new Group MMSE Detector were introduced. While it reduces to the Group Decorrelator when the interfering and desired signal spaces are linearly independent, the Group Pseudo-Inverse generalizes the notion of (group) decorrelation to the general CDMA model that allows for linear dependencies between the desired and interfering signal spaces. It also finds rigorous justification in the fact that it is the asymptotic (low noise power) form of the Group MMSE Detector independently of signal space geometry. Our asymptotic analysis for over-loaded systems is also used to solve a signal design problem of finding a signal for a new user that must be added to a fully- or over-loaded system so as to maximize that user's performance in terms of asymptotic efficiency with linear MMSE and pseudo-inverse detection without increasing the system bandwidth.

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