On the Limitation of Linear MMSE Detection and (Generalized) Welch Bound Equality Signals

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Abstract — **This paper addresses signal optimization for CDMA systems under linear minimum mean-squared error (MMSE) detection. In particular, generalized Welch-bound equality (WBE) signals, which have been found to be optimum for various measures/problems, are studied from a new perspective. Specifically, we consider the asymptotic effective energy (AEE) criterion which characterizes bit error rate (BER) faithfully in the high signal-to-noise ratio regime, and conservatively in low/medium regimes. Our analysis and numerical examples show that, with generalized WBE signals, at least one user has a BER that does not decay exponentially. In fact, in the equal-energy case, all users have an AEE that is provably equal to zero. Numerical simulations illustrate that, in general, the joint error rate floors at very high values. We also prove for sufficient overload and conjecture for other overloaded cases that no signal set yields a joint error rate that decays exponentially under linear MMSE detection, thereby highlighting a severe limitation of such detection. The results extend to the case where the received phases are unknown.**

I. Introduction

The problem of designing signals for efficient and reliable multiuser systems is receiving growing interest, an interest that draws on analytical results available about the performance of multiuser detectors. The so-called generalized Welch-bound equality (WBE) signals [1] have emerged as being particularly attractive with respect to this problem and have received considerable attention lately. These signals, which minimize total squared correlation in the equal-energy case [2], also maximize sum- and symmetric-capacity (the latter in the equal-energy case only) [1, 3]. Achieving this capacity requires nonlinear detection such as optimal multiuser coding/decoding, rate-splitting [4], or the single-user-code-based, vertex-achieving minimum mean-squared error (MMSE) decision feedback detector [5].

Recently, generalized WBE signals have been investigated in the context of *linear* detection, for which it was noted that linear MMSE detection collapses to matched-filter detection with such an allocation. In [6], they are shown to solve a quality-of-service- (QoS-) based joint signal design and power control problem, and practical issues regarding their implementation are considered in [7–10]. However, linear detection is understood to have limited performance in general, and in the context of overloaded systems in particular [11]. The analytical results to this effect rely mostly on the asymptotics of large systems (see e.g. [12]).

In this paper, we make the compelling case that, under linear MMSE (and matched-filtering) detection, generalized WBE signals are all but useless, and that linear MMSE detection is severely limited in the context of overloaded systems in terms of error performance. Specifically, we prove that generalized WBE signals yield a joint error rate (JER), defined as the probability that at least one user is in error, that always floors for such detectors. The result is even stronger in the equal-energy case, wherein every user has a bit error rate (BER) that floors. Moreover, we prove for the equal-energy case (and conjecture for the unequal-energy case) that any overloaded signal set yields a JER that floors. This is indicative of a severe limitation of linear MMSE detection, and further corroborates that nonlinear detection is necessary to achieve reliable and spectrally efficient multiple access communication [11]. Numerical examples not only confirm our results, but also illustrate that the JER of generalized WBE signals typically floors at a very high value. Our analytical approach relies on the fact that BER is faithfully characterized by the asymptotic effective energy (AEE) in the high signal-to-noise ratio (SNR) regime [13], and conservatively in low/medium regimes. We show that bounds on the AEE of any user can be maximized by the direction of least interference, and that, extending this approach to joint signal design yields generalized WBE signals. However, their symmetric energy, defined as the smallest AEE (over all users), is always equal to zero, and, consequently, at least one user has a BER that floors.

Our results also extend to the noncoherent case, where the received phases are unknown. We consider linear modulation, for which analytical results can be derived, and nonorthogonal multipulse modulation (NMM) [14], for which we have to rely on numerical examples. For either modulation, results and/or simulations suggest the same conclusion as in the coherent case.

The rest of this paper is organized as follows. The system model is described in Section II. Linear MMSE detection and its AEE are presented in Section III, and signal design for a power- and bandwidth-constrained user is considered in Section IV. This approach is extended to joint signal design in Section V, and to noncoherent detection in the appendix (Section VII). Section VI concludes this paper.

II. CDMA System Model

We consider a CDMA system wherein K users communicate independently and synchronously over an additive white Gaussian noise (AWGN) channel, by linearly modulating a signature waveform, in each signaling interval, with a symbol from an M-ary Quadrature or Pulse Amplitude Modulation (QAM or PAM). The signature waveforms satisfy the generalized Nyquist criterion [15] and are normalized to have unit-energy. The users data streams are independent, and each stream is an i.i.d. sequence of symbols drawn for a common M-ary alphabet denoted by $\boldsymbol{\mathcal{A}} = \{\alpha_1, \ldots, \alpha_M\}$, which is also normalized to have unit-energy. Coherent detection is assumed, except in the appendix.

A discrete-time model can be obtained by passing the baseband received signal through a bank of matched filters, matched to a set of orthonormal basis waveforms that span the signal space. The model is then given by the following complex N-dimensional vector

$$\mathbf{r} = \mathbf{S}\mathbf{A}\mathbf{b} + \mathbf{n}, \tag{1}$$

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where N is the dimension of the signal space, denoted by $\mathcal{S}$, and $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_K]$ is the $N \times K$ matrix of signals. The span of $\mathbf{S}$ is $\mathcal{S}$, it has full row-rank N, and its columns have unit-norm. $\mathbf{A} = \operatorname{diag}(A_1, \ldots, A_K)$, where $A_k = \sqrt{E_k}\, \mathrm{e}^{j\varphi_k}$ is the complex received amplitude of user k, $\mathbf{b}^{\mathsf{T}} = [b_1, \ldots, b_K] \in \boldsymbol{\mathcal{A}}^K$ represents the data of all users (T denotes the transpose), and $\mathbf{n}$ a zero-mean, proper, complex Gaussian random vector with known covariance $\sigma^2\mathbf{I}_N$. Note that when detection is coherent and the modulation real-valued, a real model can be derived that yields a minimal sufficient statistic [16].

The span of any matrix, denoted by a bold upper-case letter (say $\mathbf{X}$), is denoted by the corresponding calligraphic letter ($\mathcal{X}$), and the projection orthogonal to this span by $\mathbf{P}_{\mathcal{X}^\perp}$. Finally, we denote by $\mathbf{I}_x$ the $x \times x$ identity matrix for any positive integer x.

III. Linear MMSE Detection and its Asymptotic Performance

The (unbiased) linear MMSE detector for any user, say user 1, consists of the MMSE filter, denoted by $\mathbf{f}_{1-\mathrm{M}}$, followed by an appropriate decision rule. Specifically, the MMSE filter solves $\arg\min_{\mathbf{f}\in\mathbb{C}^N} \mathrm{E}\left[||\mathbf{f}^{\mathsf{H}}\mathbf{r} - b_1||_2^2\right]$ and is given by $\mathbf{f}_{1-\mathrm{M}} = A_1\mathbf{H}^{-1}\mathbf{s}_1$, where $\mathbf{H}$ is the correlation matrix of the received signal, i. e., $\mathbf{H} = \mathrm{E}[\mathbf{r}\mathbf{r}^{\mathsf{H}}]$ (H denotes hermitian transpose). The decision rule is the following minimum Euclidean distance rule, which is the maximum likelihood rule assuming that the residual MAI in the linear MMSE estimate has a Gaussian distribution,

$$\hat{b}_1 \in \arg\min_{\alpha\in\boldsymbol{\mathcal{A}}} \left|\mathbf{f}_{1-\mathrm{M}}^{\mathsf{H}}\left(\mathbf{r} - \alpha A_1\mathbf{s}_1\right)\right|^2. \tag{2}$$

Note that the decision rule is independent of any scaling of the MMSE filter. The Gaussian MAI assumption is useful to derive simple decision rules that perform well for group detection [17] and NMM [18].

Asymptotic Effective Energy

While BER is not analytically tractable in multiuser systems in general, its degradation with respect to a single-user system can be characterized in the low noise regime. This degradation is quantified by the AEE, defined, for say user 1, as the energy required by the matched-filter detector in a single-user channel to achieve, in high-SNR regimes, the same BER of the multiuser detector for user 1 in the multiuser channel [13]. Therefore, the AEE is a faithful measure of the BER performance at high SNR and, we argue, a conservative one for low/medium SNR. This is illustrated via numerical examples in the next section.

The AEE of the linear MMSE detector for user 1, denoted by e_1, depends on the signal space geometry [17]. Specifically, if $\bar{\mathbf{S}}_1$ denotes the matrix of interfering signals (and $\bar{\mathbf{E}}_1$ their corresponding energies), then e_1 is equal to the AEE of the decorrelator (defined as a detector that projects out the MAI) if, and only if, $\mathbf{s}_1 \notin \bar{\mathcal{S}}_1$ (for NMM, the condition is $\dim(\mathcal{S}_1 \cap \bar{\mathcal{S}}_1) = 0$). On the other hand, in the linear dependent case, i. e., when $\mathbf{s}_1 \in \bar{\mathcal{S}}_1$ (or $\dim(\mathcal{S}_1 \cap \bar{\mathcal{S}}_1) > 0$ for NMM), the linear MMSE detector converges, as $\sigma \to 0$, to the pseudo-decorrelator, which, as detailed in [17], only partially cancels the MAI.

For the linear dependent case, e_1 is given by

$$e_1 = \frac{E_1}{\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-2}\mathbf{s}_1} \times \Bigg[\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-1}\mathbf{s}_1 - \alpha_{\boldsymbol{\mathcal{A}}}\sqrt{\frac{2}{E_1}}\sum_{k=1}^{K-1}|x_k|\Big(|\cos\theta_k| + |\sin\theta_k|\Big)\Bigg]_+^2, \tag{3}$$

where $[a]_+ = \max\{0, a\}$, $\mathbf{x} = \left(\bar{\mathbf{S}}_1\bar{\mathbf{A}}_1\right)^+\mathbf{s}_1$ ($+$ denotes the pseudo-inverse), $x_k = |x_k|\mathrm{e}^{j\arg(x_k)}$ is its k^{th} component, $\theta_k = \varphi_1 + \arg(x_k)$, and $\alpha_{\boldsymbol{\mathcal{A}}} = \frac{\max_{\alpha\in\boldsymbol{\mathcal{A}}}|\alpha|}{\min_{i\neq j}|\alpha_j - \alpha_i|}$.[2] Note that in this case, $\bar{\mathbf{S}}_1$ has full row-rank and, hence, $\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}$ is positive definite. Simpler upper and lower bounds can be derived by first noting that $\forall\theta \in [0, 2\pi]$, $1 \le |\cos\theta| + |\sin\theta| \le \sqrt{2}$. It follows that $\tilde{\mathrm{L}}_1 \le e_1 \le \tilde{\mathrm{U}}_1$, where

$$\tilde{\mathrm{L}}_1 = \frac{E_1}{\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-2}\mathbf{s}_1}\left[\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-1}\mathbf{s}_1 - \frac{2\alpha_{\boldsymbol{\mathcal{A}}}}{\sqrt{E_1}}||\mathbf{x}||_1\right]_+^2, \tag{4}$$

and $\tilde{\mathrm{U}}_1$ is given by substituting $\sqrt{2}$ for 2 in the numerator of (4). Note that for real-valued modulation, i. e., $\boldsymbol{\mathcal{A}} \in \mathbb{R}^M$, the AEE is actually equal to the lower bound ($e_1 = \tilde{\mathrm{L}}_1$). Using the equivalence of the ℓ_1 and ℓ_2 norms, i. e., that $\forall\mathbf{x} \in \mathbb{C}^{K-1}$, $||\mathbf{x}||_2 \le ||\mathbf{x}||_1 \le \sqrt{K-1}||\mathbf{x}||_2$, we derive a lower bound L_1 on the AEE, where

$$\mathrm{L}_1 = E_1\left[\frac{\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-1}\mathbf{s}_1 - 2\beta_1\sqrt{\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-1}\mathbf{s}_1}}{\sqrt{\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-2}\mathbf{s}_1}}\right]_+^2, \tag{5}$$

where $\beta_1 = \sqrt{\frac{K-1}{E_1}}\,\alpha_{\boldsymbol{\mathcal{A}}}$. A similar upper bound U_1 is given by substituting $\alpha_{\boldsymbol{\mathcal{A}}}\sqrt{2/E_1}$ for $2\beta_1$ in the numerator of (5).

IV. Signal Optimization for a Desired User

If user 1 is power- and bandwidth-constrained, so that E_1 is fixed and $\mathbf{s}_1 \in \bar{\mathcal{S}}_1$, its AEE can be maximized, given the interference, by allocating it the signal that solves

$$\tilde{\mathbf{s}}_1 \in \arg\max_{\mathbf{s}_1\in\bar{\mathcal{S}}_1,\ ||\mathbf{s}_1||_2=1} e_1(\mathbf{s}_1), \tag{6}$$

where the dependence of the AEE on $\mathbf{s}_1$ has been made explicit. This problem does not appear tractable in general. The problem of maximizing the lower bound $\tilde{\mathrm{L}}_1$ does not appear to be tractable either, because of the ℓ_1-norm. Instead, we consider the following problem of maximizing the lower bound L_1

$$\hat{\mathbf{s}}_1 \in \arg\max_{\mathbf{s}_1\in\bar{\mathcal{S}}_1,\ ||\mathbf{s}_1||_2=1} \mathrm{L}(\mathbf{s}_1), \tag{7}$$

This constrained optimization problem is solved by the following proposition.

Proposition 1 *Consider the spectral decomposition* $\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathsf{H}}$, *where* $\boldsymbol{\Lambda}$ *is the* $N \times N$ *diagonal matrix of (positive) eigenvalues which we arrange in nondecreasing order, and* $\mathbf{U}$ *is the unitary matrix of corresponding eigenvectors.*

The upper and lower bounds U_1 *and* L_1 *are simultaneously maximized by the direction of least interference, i. e., by choosing* $\hat{\mathbf{s}}_1$ *proportional to* $\mathbf{u}_1$*, the eigenvector that corresponds to the minimum eigenvalue (denoted by* $\lambda_{\min}$*). The maximum lower bound is*

$$\mathrm{L}_1(\hat{\mathbf{s}}_1) = E_1\left[1 - 2\beta_1\sqrt{\lambda_{\min}}\right]_+^2. \tag{8}$$

Proposition 1 shows that one approach to optimizing the asymptotic BER for any user (say user 1) depends on the smallest singular value ($\sqrt{\lambda_{\min}}$) and on the corresponding left singular-vector ($\mathbf{u}_1$) of the weighted interference matrix ($\bar{\mathbf{S}}_1\bar{\mathbf{A}}_1$).

Note that $\hat{\mathbf{s}}_1$ also maximizes SIR under linear MMSE detection [9]. We have thus highlighted a previously unknown relation between the AEE and SIR measures. Moreover, with such a signal allocation (or with any other left singular vector), linear MMSE detection degenerates to matched-filter detection. Indeed, it can easily be shown that if $\mathbf{s}_1 = \mathbf{u}_j$, then $\mathbf{f}_{1-\mathrm{M}} \propto \mathbf{s}_1$.

[2] for square M-QAM constellations ($M = 2^{2m}$), $\alpha_{\boldsymbol{\mathcal{A}}} = \frac{\sqrt{M}-1}{\sqrt{2}}$.

Geometric Illustration

Consider a three-user example with unbalanced energies $\mathbf{E} = \mathrm{diag}(1, 8, 1)$, BPSK modulation (hence the model in Section II is real-valued and we adapt the results in Section IV accordingly), and a two-dimensional signal space. The interfering signals are chosen to have a correlation equal to $\mathbf{s}_2^{\mathsf{T}} \mathbf{s}_3 = 0.7$. The interfering signals and the optimum desired signal are represented in Figure 1, along with the unit circle. We have also included the region of $\mathbf{s}_1$ that ensures $e_1 > 0$. This region is found by specifying that the numerator of the ratio in e_1 be positive, and can be shown to be delimited by four elliptical arcs. The choices for the desired signal that result in a non-zero AEE are limited to that portion of the unit circle that lies outside this region.

From Figure 1, user 1 achieves a non-zero AEE for a wide choice of signals, out of which $\hat{\mathbf{s}}_1$ (or $-\hat{\mathbf{s}}_1$) maximizes its AEE. The optimum signal yields the correlations $\hat{\mathbf{s}}_1^{\mathsf{T}} \mathbf{s}_2 = -0.0623$ and $\hat{\mathbf{s}}_1^{\mathsf{T}} \mathbf{s}_3 = 0.67$. This allocation is intuitively clear: because user 2 is received with a much larger energy than user 3, it will be the main contributor to the total interference experienced by user 1, and hence the optimum signal should be almost orthogonal to $\mathbf{s}_2$. This example illustrates that even in an adverse environment with limited bandwidth and powerful interfering users, the AEE of the desired user can not only be non-zero, but can in fact be maximized by appropriate signal allocation, thereby minimizing its BER in the high SNR regime. Note, however, that in such overloaded scenarios (where $N < K$), the linear MMSE detector is not near-far resistant.

Finite SNR performance

Because it is optimized for an asymptotic performance measure, the signal specified by Proposition 1 yields a good BER performance in the high SNR regime, with the gap to single-user performance asymptotically equal to the asymptotic efficiency achieved. There is no such simple quantifiable measure of the performance for low/medium SNR. However, it is known that except for very low SNR, the AEE characterizes BER faithfully and, therefore, should be a good measure of performance at low/medium SNRs [19]. This is further confirmed by numerical examples in Figure 2, where η_1 denotes the asymptotic efficiency (i. e., $\eta_1 = e_1/E_1$), that illustrate the finite SNR performance of the optimum signal $\hat{\mathbf{s}}_1$ for several interfering signal matrices.

We consider an overloaded system wherein eight users communicate simultaneously in a seven-dimension space using QPSK modulation, i. e., $(K, N, M) = (8, 7, 4)$, with equal energies $\mathbf{E} = \mathbf{I}_K$ (as in a system with "perfect" power control, e.g. IS-95). The interfering signals are randomly generated for each example and fixed through the corresponding simulation. For each example, we find the signal designed in Proposition 1, compute the corresponding AEE, and simulate the BER assuming Gray encoding. Figure 2 illustrates that the gap to single-user performance is small for low SNR and increases to reach its limiting value, equal to the achieved AEE, at high SNR.

Limitation

Note that maximizing the lower bound does not guarantee that the resulting AEE is non-zero. From (8), a sufficient condition for $e_1 > 0$ is

$$\lambda_{\min} \leq \frac{E_1}{4\,\alpha_{\boldsymbol{\mathcal{A}}}^2\,(K-1)}. \tag{9}$$

If the transmit power of user 1 is adjustable, it is clear that we can always ensure that the AEE is non-zero. Condition (9) provides a (conservative) quantification of how much transmit power is required. Note also that the higher the modulation size is, the more likely the maximum AEE is zero. Indeed, $\alpha_{\boldsymbol{\mathcal{A}}}$ increases monotonically with M, and hence the most favorable case corresponds to binary modulation.

Of course, optimizing the signal for any user adversely affects the performance of the other users. To address this limitation, we consider, in the next section, joint signal optimization.

V. Joint Signal Optimization

In this section, we consider the problem of jointly optimizing signals for all users, given the total bandwidth and given the received energies. Specifically, K, N, and $\mathbf{E}$ are given, and we wish to minimize the joint error rate (JER). An asymptotic measure of the JER is the symmetric energy (see [13]), denoted by e, which is equal to the smallest AEE over all users, i. e., $e = \min_k e_k$.

The previous section has highlighted that the direction of least interference maximizes a lower bound on the AEE for a given user. The question is can we extend this approach to joint signal optimization? The answer is yes, if we can design a signal matrix such that each column is the left singular vector corresponding to the minimum singular value of the remaining columns.

Such a matrix exists and consists, interestingly, of the generalized WBE signals. Following the definition in [1], user k is said to be oversized if

$$E_k > \frac{\sum_{j=1}^{K} E_j\, 1_{\{E_k > E_j\}}}{N - \sum_{j=1}^{K} E_j\, 1_{\{E_j \geq E_k\}}}.$$

Let $\boldsymbol{\mathcal{O}}$ and $\boldsymbol{\mathcal{O}}^c$ denote the sets of oversized and nonoversized users, respectively. Generalized WBE signals are such that oversized users are assigned orthogonal signals, and nonoversized users are assigned WBE signals in the remaining signal subspace. Specifically, if $k \in \boldsymbol{\mathcal{O}}$, then the minimum singular value of $\bar{\mathbf{S}}_k \bar{\mathbf{A}}_k$ is equal to zero. Consequently, $\mathbf{s}_k \perp \bar{\mathcal{S}}_k$, the linear MMSE detector for user k is asymptotically equivalent to the decorrelator, and $e_k = E_k$. On the other hand, if $k \in \boldsymbol{\mathcal{O}}^c$, then the minimum singular value is non-zero, and $\mathbf{s}_k$ is the corresponding left singular vector. This corresponds to the linear dependent case discussed in Section III, for which e_k is given as in (3). The WBE signals assigned to the nonoversized users satisfy $\mathbf{S}_{\boldsymbol{\mathcal{O}}^c} \mathbf{E}_{\boldsymbol{\mathcal{O}}^c} \mathbf{S}_{\boldsymbol{\mathcal{O}}^c}^{\mathsf{H}} = E_{\boldsymbol{\mathcal{O}}^c} / (N - |\boldsymbol{\mathcal{O}}|)\, \mathbf{I}_{N-|\boldsymbol{\mathcal{O}}|}$, where $\mathbf{S}_{\boldsymbol{\mathcal{O}}^c}$ denotes their signal matrix, $\mathbf{E}_{\boldsymbol{\mathcal{O}}^c}$ their received energies, and $E_{\boldsymbol{\mathcal{O}}^c} = \sum_{k \in \boldsymbol{\mathcal{O}}^c} E_k$. Note that with such a signal allocation, the linear MMSE detector degenerates to the matched-filter detector for all users.

Performance of Generalized WBE Signals

As we have pointed out earlier, generalized WBE signals in conjunction with linear detection have been the focus of much attention lately. However, we prove here that when $N \leq K/2$, they never yield a JER that decays exponentially, and illustrate via numerical examples that this conclusion holds also for the unequal-energy case when $N > K/2$. In other words, for any K, N (with $N \leq K/2$), and $\mathbf{E}$, the symmetric energy is always zero, and, hence, at least one user has AEE equal to zero. In fact, a stronger result holds for the equal-energy case, wherein we prove that for any N (not just $N \leq K/2$) all users have an AEE that is equal to zero.

If the received energies are equal, and hence no user is oversized, we can assume, without loss of generality, that $\mathbf{E} = \mathbf{I}_K$. For WBE signals, the upper bound $\tilde{U}l_k, \forall k$, is easily shown to be given by

$$\tilde{\mathrm{U}}_k = \left[1 - \alpha_{\boldsymbol{\mathcal{A}}} \sqrt{2\,\lambda_{\min}^{(k)}}\ \left|\left|\mathbf{v}_1^{(k)}\right|\right|_1\right]_+^2,$$

where $\sqrt{\lambda_{\min}^{(k)}}$ and $\mathbf{v}_1^{(k)}$ denote the smallest singular value of $\bar{\mathbf{S}}_k \bar{\mathbf{A}}_k$ and the corresponding right singular-vector, respectively. Moreover, it can be shown that, for all $k = 1, \ldots, K$, $\lambda_{\min} = K/N - 1$ and that the ℓ_1-norm of $\mathbf{v}_1^{(k)}$ is greater than or equal to $(K/N - 1)^{-1/2}$. It follows that $\forall k$, $\tilde{\mathrm{U}}_k = 0$, and, consequently, that $e_k = 0$.

For the unequal-energy case, let us first assume that $\boldsymbol{\mathcal{O}} = \emptyset$, the empty set (i. e., there are no oversized users). In this case, the WBE signals, denoted by $\mathbf{S}_u$, satisfy $\mathbf{S}_u \mathbf{E} \mathbf{S}_u^{\mathsf{H}} = (E/N)\, \mathbf{I}_K$, where E is the sum of all energies. When $N \leq K/2$, it can be shown that their symmetric energy is smaller than that of WBE signals designed with the same total energy constraint, but with an equal-energy distribution. Let us denote the resulting WBE signal matrix by $\mathbf{S}_e$. These signals satisfy $\mathbf{S}_e \mathbf{S}_e^{\mathsf{H}} = (K/N)\, \mathbf{I}_N$, and, from the argument above, it follows that $e(\mathbf{S}_u) \leq e(\mathbf{S}_e) = 0$. Therefore, the result extends to the unequal-energy case provided that no user is oversized and that $N \leq K/2$. If, on the other hand, $\boldsymbol{\mathcal{O}} \neq \emptyset$, then the same argument yields that at least one nonoversized user has zero AEE if $N \leq K/2$, i. e., that $e(\mathbf{S}_{\boldsymbol{\mathcal{O}}^c}) = 0$. Hence, the symmetric energy is equal to zero. The following proposition summarizes these results.

Proposition 2 *For any number of users K, processing gain $N < K$, modulation size M, and when the received energies are equal, WBE signals yield an AEE that is equal to zero for all users. When the received energies are unequal and $N \leq K/2$, generalized WBE signals have symmetric energy always equal to zero. Specifically,*

- *if the energies are equal, all users have an AEE equal to zero:*
$$\mathbf{E} = E\, \mathbf{I}_K \quad \Rightarrow \quad \forall k = 1, \ldots, K, \quad e_k = 0.$$
- *if the energies are unequal, then oversized users have an AEE equal to their received energy, but at least one nonoversized user has an AEE equal to zero:*
$$\mathbf{E} \neq E\, \mathbf{I}_K \quad \Rightarrow \quad \begin{cases} \forall k \in \boldsymbol{\mathcal{O}}, & e_k = E_k, \\ \quad \textit{and} & e(\mathbf{S}_{\boldsymbol{\mathcal{O}}^c}) = 0. \end{cases}$$

Next, we consider numerical examples to illustrate the JER performance of generalized WBE signals under linear MMSE (and match-filtering) detection. This performance is a function only of the number of users, the processing gain, the energy distribution, and the modulation size. We consider numerical examples in which all users employ BPSK (therefore the model in (1) is real-valued), which is the most favorable modulation size. For fixed K and $\mathbf{E}$ (we choose $(K, \mathbf{E}) = (20, \mathbf{I}_K)$), Figure 3 illustrates the JER for different values of N. As N decreases, or equivalently as the load factor $\beta = K/N$ increases, the floor rises very quickly: for instance, when $N = 17$, the load $\beta = 1.1765$ is marginally better than full-rank signaling ($\beta = 1$), and the JER floors at 0.05 for $E_b/N_0 \geq 20$ dB. This suggests that performance is best when N is maximal, i. e., equal to $K - 1$, which corresponds to the least overload. In Figure 4, $N = K - 1$ and $\mathbf{E} = \mathbf{I}_K$ are fixed, and the performance is compared as K is varied. The figure illustrates that, as K increases, the floor has a lower value and is reached for higher E_b/N_0. The examples in Figures 3 and 4 correspond to scenarios where no user is oversized, and, hence, to WBE signals. Next, we fix $(K, N) = (200, 199)$ and vary the energy distribution. This yields generalized WBE signals with different numbers of oversized users $|\boldsymbol{\mathcal{O}}|$ for different distributions. Specifically, we consider that user k has energy E_k given as in the following table, which also includes the resulting $|\boldsymbol{\mathcal{O}}|$.

E_k	$\lvert\boldsymbol{\mathcal{O}}\rvert$
$(K-k+1)^2$	198
$K-k+1$	198
$\sqrt{K-k+1}$	196
$1+0.1(K-k+1)$	194

The simulated performance is shown in Figure 5, which illustrates that, as energy disparities decrease, the number of oversized users also decreases and the performance improves. Nevertheless, even mild energy disparities yield very poor performance. For instance, when $E_k = 1 + 0.1(K - k + 1)$, the JER floors at 0.06 for $E_b/N_0 \geq 35$ dB. Note that for the first two distributions, all users (except the two weakest ones) are allocated orthogonal signals, and, consequently, experience a single-user Gaussian channel. On the other hand, the two remaining users have the same signal allocation (only one dimension is left) and interfere in such a way that the AEE of the weakest user (user K) is equal to zero. Note that an alternative to reducing $|\boldsymbol{\mathcal{O}}|$ is, as suggested by the definition of oversized users, to choose a smaller N. However, the performance then degrades very rapidly. The examples in Figure 5 also suggest that the symmetric energy of generalized WBE signals is equal to zero when $N > K/2$, a result which has not been proved yet.

Improving the Performance

Next, we address the problem of improving over the performance of generalized WBE signals via alternative joint signal designs. Specifically, we would like to find signals that maximize symmetric energy. It turns out, however, that no other signal matrix yields a non-zero symmetric energy. Consequently, no matter how the signals are designed, at least one user will have an AEE equal to zero, and, hence, the JER will not decay exponentially as E_b/N_0 increases. The poor asymptotic BER performance is hence attributable not to generalized WBE signals but to linear MMSE detection. The following proposition states the result for the case when $N \leq K/2$ and no user is oversized (which encompasses the equal-energy and unequal-energy cases with no-oversized users).

Proposition 3 *Given K, $N \leq K/2$, and $\mathbf{E}$, such that $\boldsymbol{\mathcal{O}} = \emptyset$, any $N \times K$ signal matrix has symmetric energy equal to zero under linear MMSE (and match-filtering) detection, i. e.,*

$$\forall \mathbf{S} \in \mathbb{C}^{N \times K}, \quad e(\mathbf{S}) = 0. \tag{10}$$

While the proof is not available yet, we conjecture that the result holds when $\boldsymbol{\mathcal{O}} \neq \emptyset$ and for any N. Another interesting open problem is to consider our analysis for the optimal linear detector [20].

VI. Conclusion

We have shown that, under linear detection, generalized WBE signals yield JERs that never decay exponentially and typically floor at high levels. Therefore, these signals are all but useless when linear MMSE detection is employed. Because they actually minimize JER, this poor performance is not due to bad signal design, but rather to intrinsic limitations of linear MMSE detection. Therefore, reliable and bandwidth efficient multiple access requires nonlinear detection. Moreover, such nonlinear detectors are more suited to uplink transmission than linear, user-separating detectors. For SIR or capacity-related measures, optimum multiuser coding/decoding, rate-splitting multiple access [4], or the single-user-code-based, vertex-achieving MMSE decision feedback detector [5] should be considered. In [21], joint signal design under AEE constraints for the later detector is considered.

VII. Appendix
Extension to Noncoherent Detection

In this appendix, we extend our analysis and results to noncoherent detection, where the phases are unknown. We highlight, through analysis and numerical examples, that the same approach to signal design as in the coherent case is optimal, which suggests that the design depends only on the signal space geometry.

Noncoherent System Model

We assume that the phases $\{\varphi_k\}_{k=1}^K$ are i.i.d. random variables that are uniformly distributed in $[0, 2\pi]$. We consider both linear modulation, which consists of PAM with positive amplitudes ($\alpha_i > 0$), and nonorthogonal multipulse modulation (NMM). In the later case, each user communicates an M-ary symbol by transmitting one of M signature waveforms. Consequently, the signal matrix $\mathbf{S} = [\mathbf{S}_1, \ldots, \mathbf{S}_K]$ has dimension $N \times MK$, where $\mathbf{S}_k = [\mathbf{s}_{k1}, \ldots, \mathbf{s}_{kM}]$ is the $N \times M$ signal matrix for user k, and $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_K)$ is the $MK \times MK$ diagonal matrix of corresponding complex amplitudes. Defining the MK-dimensional data vector $\mathbf{b}^{\mathsf{T}} = [\mathbf{b}_1^{\mathsf{T}}, \ldots, \mathbf{b}_K^{\mathsf{T}}]$, where $\mathbf{b}_k$ represents the M-dimensional data vector for user k and is an equiprobable unit vector, it follows that NMM can be modeled as in (1) [14].

Noncoherent Linear MMSE detector and Asymptotic Performance

When the modulation is linear, the MMSE filter solves $\arg\min_{\mathbf{f}\in\mathbb{C}^N} \mathrm{E}\left[||\mathbf{f}^{\mathsf{H}}\mathbf{r} - e^{j\varphi_1} b_1||_2^2\right]$, which yields $\mathbf{f}_{1-\mathrm{M}} = \sqrt{E_1}\,\mathbf{H}^{-1}\mathbf{s}_1$. Under the same Gaussian MAI assumption in the coherent case, the subsequent decision rule is derived by averaging over the unknown phase φ_1 the conditional likelihood, conditioned on φ_1. However, the resulting decision rule involves the modified zeroth-order Bessel function and can be unwieldy to implement. Instead, using an asymptotic expansion of the Bessel function, we derive a simpler rule that has equivalent performance asymptotically. It is given by

$$\hat{b}_1 \in \arg\max_{\alpha\in\boldsymbol{\mathcal{A}}} \left\{ 2\,\alpha|\mathbf{f}_{1-\mathrm{M}}^{\mathsf{H}}\mathbf{r}| - E_1\,\mathbf{s}_1^{\mathsf{H}}\mathbf{H}^{-1}\mathbf{s}_1\,\alpha^2 \right\}. \tag{11}$$

The AEE of this rule is not tractable in the linear dependent case. However, a lower bound can be derived which, surprisingly, yields the same form as in (5)

$$\mathrm{L}_1 = E_1 \left[\frac{\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-1}\mathbf{s}_1 - \sqrt{2}\beta_1\sqrt{\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-1}\mathbf{s}_1}}{\sqrt{\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-2}\mathbf{s}_1}} \right]_+^2 .$$

A similar upper bound can also be derived.

The noncoherent NMM case is considered in [18], where three decision rules are proposed, namely the MMSE maximum magnitude (MM), generalized likelihood ratio test (G), and asymptotically optimal (AO) rules. When $\dim(\mathcal{S}_1 \cap \bar{\mathcal{S}}_1) > 0$, neither the AEE nor a useful lower bound appear to be tractable.

Signal Design

Proposition 1 extends to the noncoherent case with linear modulation, and maximizes the lower bound on the AEE. Therefore, the same approach to joint signal design as in the coherent case applies, but results in poor BER performance.

The NMM case is particularly interesting because each user has a signal subspace of dimension between 1 and M. For such cases, we illustrate, via numerical examples, that the same approach is optimal. We consider an overloaded system with $(K, M, N) = (10, 4, 16)$. The signals $\mathbf{S}$ are randomly generated so that rank$(\mathbf{S}_1) = 4$ and rank$(\bar{\mathbf{S}}_1) = 16$. Figure 6 illustrates the symbol error rate (SER) performance of the three noncoherent MMSE detectors (MM, G, and AO). They all yield SERs that floor. Next, for the given interfering signal and energy matrices ($\bar{\mathbf{S}}_1$ and $\bar{\mathbf{E}}_1$), we optimize the four signals of user 1 by choosing the four left singular vectors of the 16×16 matrix $\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1^{1/2}$ corresponding to its four smallest singular values. Consequently, the optimized $\hat{\mathbf{S}}_1$ consists of four orthogonal signals that lie in the interfering subspace. The resulting performance of the three MMSE detectors is simulated and shown in Figure 6, along with the single-user performance for 4-ary OMM. The signal design improves the performance dramatically. In fact, the SERs of the three MMSE detectors are close, and almost achieve single-user performance for the range of SNRs simulated.

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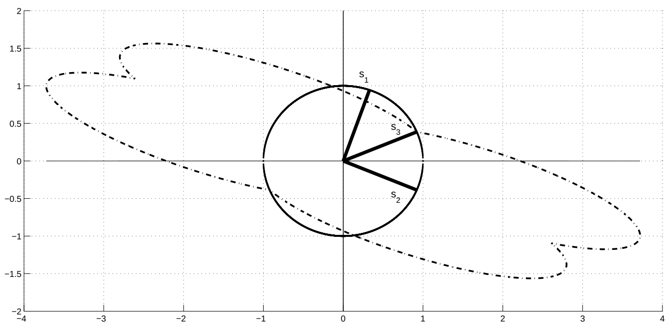

Figure 1: Signal space geometry and non-zero AEE region for $(K, N) = (3, 2)$, unbalanced received energies $\mathbf{E} = \mathrm{diag}(1, 8, 1)$, correlation $\mathbf{s}_2^T \mathbf{s}_3 = 0.7$, and BPSK modulation.

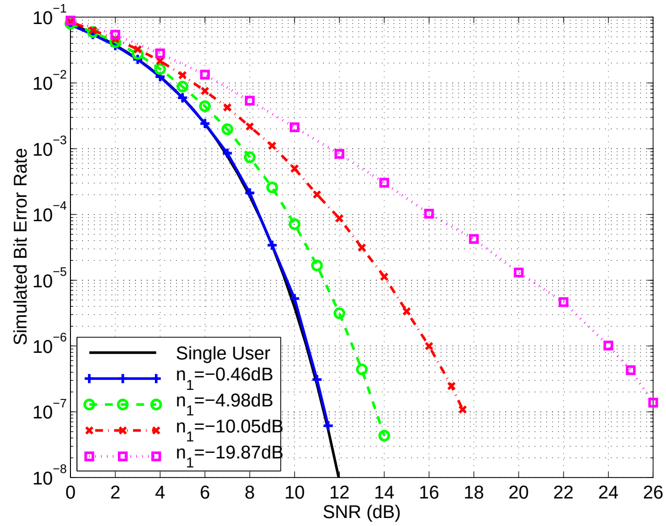

Figure 2: Finite SNR performance of the AEE-based signal design under linear MMSE detection in overloaded CDMA systems $(K, N, M) = (8, 7, 4)$ with different asymptotic efficiencies η_1 (in dB).

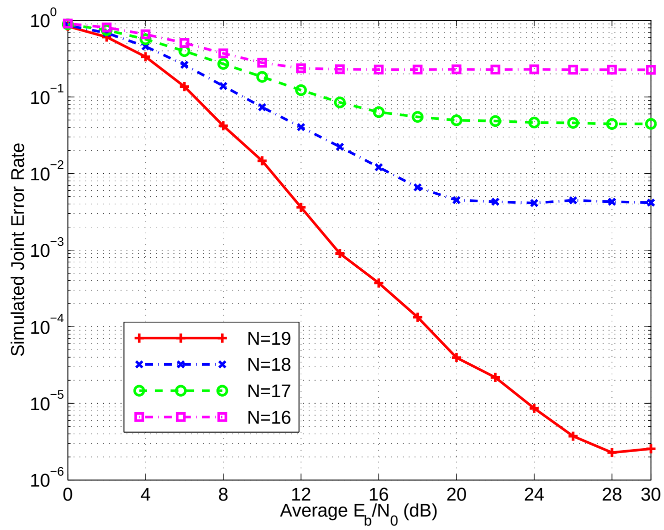

Figure 3: JER performance of WBE signals for $(K, \mathbf{E}) = (20, \mathbf{I}_K)$, BPSK modulation, and different values of N.

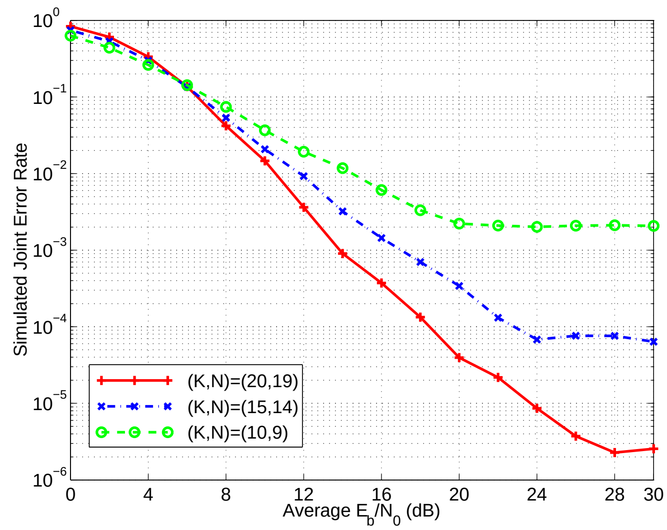

Figure 4: JER performance of WBE signals for $(K, N, \mathbf{E}) = (K, K-1, \mathbf{I}_K)$, BPSK modulation, and different values of K.

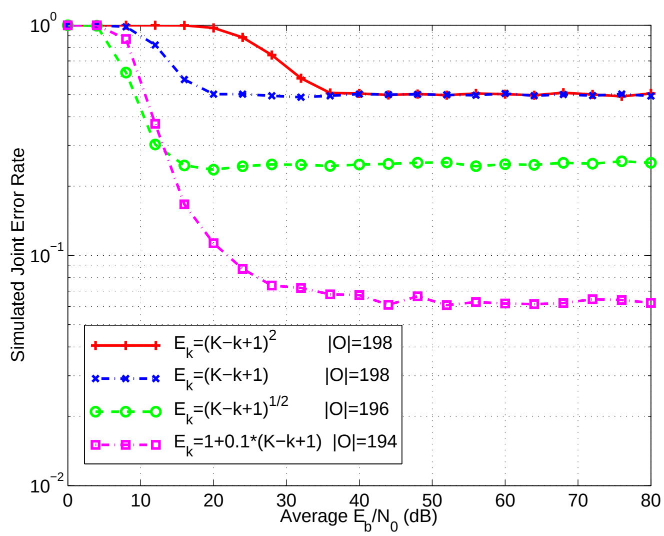

Figure 5: JER performance of generalized WBE signals for $(K, N) = (200, 199)$, BPSK modulation, and different energy distributions.

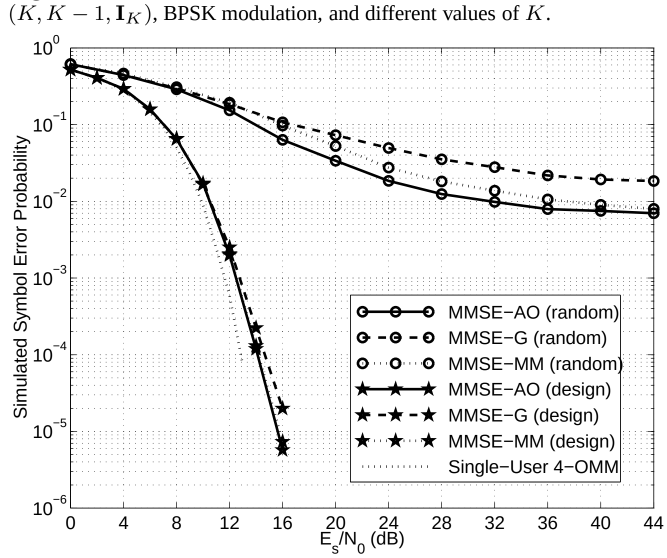

Figure 6: Noncoherent detectors for NMM in a large overloaded system: $(K, N, M) = (10, 16, 4)$ with user 1 employing linearly independent signals.