Multiuser Detection for Overloaded CDMA Systems

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Abstract—**Multiuser detection for overloaded CDMA systems, in which the number of users is larger than the dimension of the signal space, is of particular interest when bandwidth is at a premium. In this paper, certain fundamental questions are answered regarding the asymptotic forms and performance of suboptimum multiuser detectors for cases where the desired and/or interfering signal subspaces are of reduced rank and/or have a non-trivial intersection. In the process, two new suboptimum detectors are proposed that are especially well suited to overloaded systems, namely the group pseudo-decorrelator and the group MMSE detector, and the former is seen to be the correct extension of the group decorrelator in the sense that it is the limiting form (in the low-noise regime) of the group MMSE detector. Pseudo-decorrelation is also used as a feedforward filter in a new decision feedback scheme. For the particular case of real-valued modulation, it is shown that the recent proposals of the so-called "improved" linear (aka "linear-conjugate" or "widely linear") detectors were more simply derived earlier using the idea of minimal sufficiency which we also apply to the new detectors of this paper.**

Index Terms—**asymptotics, bandwidth-efficient multiple-access, CDMA, decision feedback, multiuser detection**

I. Introduction

Suboptimum detection for multiuser CDMA communication systems has been studied extensively. There are however certain fundamental questions with regard to the asymptotic forms and performance of suboptimum multiuser group detectors with or without decision feedback that remain open. In this paper we classify the problems into two categories depending on the signal space geometry: one where the desired and/or interfering signal subspaces are of possibly reduced rank but are mutually linearly independent, and the other where they have a non-trivial intersection.

Overloaded CDMA systems, also known as low-rank or oversaturated systems, arise when the number of users is greater than the dimension of the signal space (if linear modulation is employed) and hence involve linearly dependent signature waveforms. They are of practical importance, for example, in bandwidth efficient multiuser communication [1]. Suboptimum multiuser detection for such systems has recently been studied in [2, 3], but with very specific constraints on the signal set. In this paper, we develop a general theory of suboptimum detection that accounts for linear dependence between and within the desired and interfering signal subspaces. We do so by focusing on group detection and sequential group detection schemes which subsume linear and decision feedback detection, respectively.

The linear decorrelator [4], the decorrelating decision feedback detector [5], and their group counterparts [6, 7] cancel out the multiple-access interference (MAI). These detectors, if naively applied to overloaded systems, can have an unacceptable performance when the desired signals are linearly dependent on the interfering signals. When the signals are linearly dependent, a "pseudo-inverse" linear detector (of group size one) was proposed in [4]. In this paper, we show that this approach has a compelling justification and generalizes to arbitrary group sizes. As for minimum mean-squared error (MMSE) detection, the linear MMSE detector is known to converge to the decorrelator in the low noise limit when the desired user's signal is linearly independent of the interfering users' signals [8]. Similarly, the group MMSE detector proposed in [9] was shown to converge to the group decorrelator for the full-rank signaling case. In either case, the asymptotic behavior and performance of linear and group MMSE detection is not known when the intersection of the desired and interfering signal subspaces has dimension greater than zero. Also, the MMSE decision feedback detector with full-rank signaling can be shown to converge to the decision feedback decorrelator of [5], but its asymptotic form and performance is not known for low-rank signaling.

We first consider the general problem of detecting a group of users such as would be relevant in multi-cell or multi-rate, multi-code CDMA systems [6, 10]. We specify a canonical description of group detection as a two stage process (a group filter followed by a decision rule that accounts for the residual MAI) and study its asymptotic performance without any assumption on the signal space geometry. Using this canonical description, we introduce two new group detectors, namely, the group pseudo-decorrelator and a group MMSE detector, with the former specializing to group decorrelation when the desired and interfering signal subspaces are linearly independent. We derive the asymptotic (low noise) form of our new group MMSE detector in the general case where the intersection of the desired and interfering signal subspaces has dimension greater than or equal to zero, and show that, irrespective of the signal space geometry, the group MMSE detector converges to the group pseudo-decorrelator in the limit of low noise. Moreover, these detectors can far outperform the previously proposed group MMSE detector of [9] which does not qualify, according to our definition, as a canonical group detector. The asymptotic form of the group MMSE detector allows us to derive the asymptotic effective energy (AEE) of the linear MMSE detector when the desired user's signal lies in the interfering signal subspace.

We also address the problem of how our results must be modified when some or all of the users employ real-valued modulation. This problem was first considered in the context of decorrelating (with and without decision feedback) group detection in [6], which also provided a succinct solution based on the idea of minimal sufficiency. We extend that idea to the more general case of the group pseudo-decorrelating and group MMSE de-

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tection for low-rank signaling. Unaware that real-valued modulation was specifically addressed in [6], several authors have recently proposed a less direct method for linear detection that yields the so-called widely linear or linear conjugate detectors in [11–14]. It is shown, in this paper, that the two approaches are equivalent for linear detection, and, in fact, that the minimal sufficiency approach of [6] is the simpler and the more direct of the two and it easily extends to the general case of group (not just linear) detection.

Our results are then extended to group decision feedback detection for overloaded systems. While this extension is applicable to arbitrary grouping of the users, we present only the particular case of decision feedback detection, where all groups have size one. We conclude that the MMSE decision feedback detector converges, in the low noise limit, to the decorrelating decision feedback detector of [5] if, and only if, the signals are linearly independent. In the linear dependent case, we derive its asymptotic form and performance, and use key results in [15] to find an ordering of the users that maximizes the exponential rate of decay of the joint error rate. The asymptotic form of the MMSE decision feedback detector is also used to define the new pseudo-decorrelating decision feedback detector, which can be used for all signal-to-noise ratios (SNRs). This new detector is the extension of the decorrelating decision feedback detector to overloaded systems. Moreover, unlike the MMSE decision feedback detector, it does not require the knowledge of the noise density but has the same performance for high SNRs.

The rest of this paper is organized as follows. We describe the synchronous CDMA model with additive white Gaussian noise in Section II. The canonical model and asymptotic performance of group detection are then presented in Section III, followed by a description and analysis of the new group pseudo-decorrelating and MMSE detectors in Sections IV and V, respectively. In Section VI, we extend our results to real-valued modulation and discuss widely linear detection. Finally, we present decision feedback detection in Section VII. The performance and asymptotic behavior of the various detectors discussed herein are illustrated via numerical examples in Section VIII. Section IX concludes this paper, and proofs of several results herein are relegated to the appendices.

II. CDMA System Model

We consider a multiuser system wherein K users employ M-ary quadrature amplitude modulation (M-QAM) to communicate simultaneously and synchronously over an additive white Gaussian noise (AWGN) channel using signature waveforms that satisfy the generalized Nyquist criterion [16, Sec. 6.8.3]. For such a system, detection on a symbol-by-symbol basis is optimal. The complex baseband equivalent representation of the received signal in a symbol interval is

$$r(t) = \sum_{k=1}^{K} \sqrt{E_k}\, e^{j\varphi_k}\, b_k s_k(t) + n(t), \tag{1}$$

where E_k, φ_k, b_k, and $s_k(t)$ represent, respectively, the received energy, phase, symbol, and the baseband equivalent representation of the signature waveform of user k. The energies and phases of all users are assumed to be known at the receiver, and their signature waveforms are normalized to have unit-energy. The users' data streams are independent, and each stream is comprised of i.i.d. symbols. Furthermore, all users employ the same M-QAM alphabet $\boldsymbol{\mathcal{A}} = \{\alpha_i\}_{i=1}^{M}$, which is also normalized to have unit average energy. Finally, $n(t)$ is a zero-mean, proper, complex, white Gaussian random process with power spectrum σ^2.

An equivalent discrete-time model can be obtained by passing $r(t)$ through a bank of matched-filters matched to a set of orthonormal basis functions that span the signal space of all users, and sampling the output appropriately. If N denotes the dimension of the signal space ($N \le K$), this operation yields the following N-dimensional sufficient statistic

$$\mathbf{r} = \mathbf{S}\mathbf{A}\mathbf{b} + \mathbf{n}, \tag{2}$$

where $\mathbf{S} \triangleq [\mathbf{s}_1, \dots, \mathbf{s}_K]$ denotes the $N \times K$ signal matrix, $\mathbf{A} \triangleq \mathrm{diag}\left(\sqrt{E_1}\, e^{j\varphi_1}, \dots, \sqrt{E_K}\, e^{j\varphi_K}\right)$ the diagonal matrix of complex received amplitudes, $\mathbf{b}^{\mathsf{T}} \triangleq [b_1, \dots, b_K]$ the symbols of all users (the superscript T denotes the transpose), and $\mathbf{n}$ is a zero-mean, proper, complex Gaussian random vector with covariance matrix $\sigma^2 \mathbf{I}_N$. This is denoted by $\mathbf{n} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2 \mathbf{I}_N\right)$. Throughout this paper, we denote the space spanned by any matrix, denoted by some bold upper-case letter (say $\mathbf{X}$), by the corresponding calligraphic letter ($\mathcal{X}$), and the projection orthogonal to this space by $\mathbf{P}_{\mathcal{X}^\perp}$. $\mathbf{I}_x$ denotes the $x \times x$ identity matrix for any positive integer x.

III. Group Detection

Group detection was introduced in [6] for systems wherein a subset $G \subseteq \{1, \dots K\}$ of users is to be detected. Let $|G|$ denote the cardinality of G, and $\bar{G}$ the complementary set. We refer to users whose indices are in $\bar{G}$ as the interfering users. In this section, we specify a canonical model for group detection as a two-stage process, and analyze its asymptotic group performance.

We introduce the subscripts G and $\bar{\mathrm{G}}$ to refer to parameters pertaining to the desired and interfering users, respectively, and rearrange users to partition the signal matrix as $\mathbf{S} = [\mathbf{S}_{\mathrm{G}}, \mathbf{S}_{\bar{\mathrm{G}}}]$. The model in (2) can then be written as $\mathbf{r} = \mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}} + \mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\mathbf{b}_{\bar{\mathrm{G}}} + \mathbf{n}$.

A. Canonical group detector

The first stage of a group detector is comprised of a bank of $|G|$ filters (one for each desired user), referred to as the group filter and denoted by the $N \times |G|$ matrix $\mathbf{F}_{\mathrm{G}}$. It is designed to combat the MAI from interfering users, and hence depends on G. Its output is

$$\mathbf{y}_{\mathrm{G}} = \mathbf{F}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}} + \mathbf{F}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\mathbf{b}_{\bar{\mathrm{G}}} + \boldsymbol{\eta}_{\mathrm{G}}, \tag{3}$$

where $\boldsymbol{\eta}_{\mathrm{G}} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\, \mathbf{F}_{\mathrm{G}}^{\mathsf{H}}\mathbf{F}_{\mathrm{G}}\right)$ (the superscript H denotes the conjugate transpose).

The second stage consists of a decision rule which, unlike the rule in linear detection, jointly decides which symbols the desired users have transmitted. Since, in general, the group filter does not completely remove the MAI from interfering users, the sufficient statistic in (3) is not MAI-free. Consequently, any joint decision rule that ignores the residual MAI is suboptimal. On the other hand, the maximum likelihood (ML) rule is too complex for practical implementation. To circumvent this difficulty, and motivated by [17] that proves that the distribution of the post-filtering MAI converges to being Gaussian for large systems (with fixed K/N as $N \to \infty$), we assume that the residual MAI has a Gaussian distribution. This assumption easily results in a suboptimal joint decision rule (with complexity exponential in $|G|$) and can, for good group filters, yield a reasonable compromise between simple implementation and good performance. The performance analysis (in Section III-B) is carried out however, without the Gaussian assumption.

The post-filtering statistic in (3) can be rewritten as $\mathbf{y}_{\mathrm{G}} = \mathbf{F}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}} + \boldsymbol{\gamma}_{\mathrm{G}}$, where $\boldsymbol{\gamma}_{\mathrm{G}}$ is the compound residual MAI $+$ noise term. It has mean zero and covariance matrix $\sigma^2\mathbf{K}_{\mathrm{G}}$, where $\mathbf{K}_{\mathrm{G}} \stackrel{\triangle}{=} \frac{1}{\sigma^2}\mathbf{F}_{\mathrm{G}}^{\mathsf{H}}\left(\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{E}_{\bar{\mathrm{G}}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}} + \sigma^2\mathbf{I}_N\right)\mathbf{F}_{\mathrm{G}}$ and $\mathbf{E} \stackrel{\triangle}{=} \mathrm{diag}(E_1, \ldots, E_K)$ is the matrix of received energies. Under the Gaussian residual MAI assumption, $\boldsymbol{\gamma}_{\mathrm{G}} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\mathbf{K}_{\mathrm{G}}\right)$, and, therefore, the ML decision rule is

$$\hat{\mathbf{b}}_{\mathrm{G}} \in \arg\max_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{A}}^{|G|}} \Big\{ 2\,\mathcal{R}e\Big(\boldsymbol{\alpha}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\mathbf{L}_{\mathrm{G}}\,\mathbf{r}\Big) - \boldsymbol{\alpha}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\mathbf{L}_{\mathrm{G}}\,\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\boldsymbol{\alpha}\Big\}, \tag{4}$$

where we have introduced the positive semi-definite matrix $\mathbf{L}_{\mathrm{G}} \stackrel{\triangle}{=} \mathbf{F}_{\mathrm{G}}\mathbf{K}_{\mathrm{G}}^{+}\mathbf{F}_{\mathrm{G}}^{\mathsf{H}}$ (the superscript $+$ denotes the Moore-Penrose pseudo-inverse [18]). Notice that, in general, $\mathbf{K}_{\mathrm{G}}$ and $\mathbf{L}_{\mathrm{G}}$ depend on the noise density σ^2.

A group detector is characterized as canonical if it accounts for the residual MAI as in (4).

B. Asymptotic group performance

Our analysis of the asymptotic group performance in this section is general in that it does not assume any specific criterion for the design of the group filter, nor any special geometry of the signal space. Moreover, while the joint decision rule is derived using the Gaussian assumption, the performance analysis accounts for the true distribution of the residual MAI.

The performance is measured by the group error rate (GER) that an erroneous decision is made for one or more users in G. A measure of the asymptotic behavior of the GER is the group symmetric energy (GSE), denoted by e_{G}, which is defined as the energy required by the matched-filter detector in a single-user channel to achieve the same GER in the high-SNR regime [6]. Using results from [15], it can be shown to be equal to the smallest AEE of the desired users (the AEE is a per-user measure similar to GSE but with the GER replaced by the error rate of the corresponding user). Consequently, $e_{\mathrm{G}} = \min_{k\in G} e_k$, where e_k denotes the AEE of user k and is given by the following proposition, which is proved in Appendix A.

Proposition 1: The AEE of user k (for $k \in G$) under group detection is given as in (5) at the bottom of the page, where Υ_k is the set of indecomposable error vectors of user k (as defined in [19, Sec. 4.3.2]), $\tilde{\mathbf{L}}_{\mathrm{G}}$ is limit of $\mathbf{L}_{\mathrm{G}}$ as σ goes to zero, $|\alpha_{\max}| \stackrel{\triangle}{=} \max_{\alpha\in\boldsymbol{\mathcal{A}}}|\alpha|$, $|\alpha_{\min}| \stackrel{\triangle}{=} \min_{i\neq j}|\alpha_j - \alpha_i|$, and $z_n = |z_n|e^{j\arg z_n}$ is the n^{th} component of the $|\bar{G}|$-dimensional vector $\mathbf{z} \stackrel{\triangle}{=} \mathbf{A}_{\bar{\mathrm{G}}}^{\mathsf{H}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}}\tilde{\mathbf{L}}_{\mathrm{G}}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\boldsymbol{\epsilon}_{\mathrm{G}}$.

Notice that the AEE of one or more users in the desired group can be equal to zero, in which case the GSE is zero and the GER does not decay exponentially as the noise density goes to zero. It is therefore of interest to characterize conditions under which $e_{\mathrm{G}} > 0$ (and hence the AEE of any desired user is non-zero). While such a characterization is not analytically tractable in general, we derive simple sufficient conditions in Appendix B.

For the special case of a group size one ($|G| = 1$), the analysis of this subsection simplifies to the derivation of the asymptotic efficiency of a linear multiuser detector and hence specializes to results in [4] when adapted for the model in (2).

C. Two examples of group detectors

The group decorrelator was derived in [6] in the context of full-rank CDMA systems by means of the generalized likelihood ratio test. Alternatively, it can be described as a canonical group detector that employs a decorrelating first stage, i. e., a group decorrelating filter that is designed to completely remove the MAI from interfering users. This group filter projects the received signal onto the subspace orthogonal to the interfering subspace $\mathcal{S}_{\bar{\mathrm{G}}}$, and match-filters the output to the resulting desired signals. Letting the compound subscript $\mathrm{G{-}D}$ refer to group decorrelation, and defining the matrix $\mathbf{M}_{\mathrm{G}} \stackrel{\triangle}{=} \mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$, the group decorrelating filter is then $\mathbf{F}_{\mathrm{G-D}} = \mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$, and its output is $\mathbf{y}_{\mathrm{G-D}} = \mathbf{M}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}} + \boldsymbol{\eta}_{\mathrm{G-D}}$, where $\boldsymbol{\eta}_{\mathrm{G-D}} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\,\mathbf{M}_{\mathrm{G}}\right)$. An equivalent statistic can be formed as $\mathbf{M}_{\mathrm{G}}^{+}\,\mathbf{y}_{\mathrm{G-D}}$, which suggests the alternative and more convenient expression

$$\mathbf{F}_{\mathrm{G-D}}^{\mathsf{H}} = \mathbf{M}_{\mathrm{G}}^{+}\,\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}} = \left(\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\right)^{+}. \tag{6}$$

Note that the group decorrelating filter is equivalent to selecting the $|G|$ corresponding columns of the joint decorrelating filter if, and only if, the desired signals are linearly independent.

Since the MAI from interfering users has been completely canceled, no Gaussian assumption is required and the canonical

$$e_k = \max{}^2\left\{0,\ \min_{\boldsymbol{\epsilon}_{\mathrm{G}}\in\Upsilon_k}\frac{\boldsymbol{\epsilon}_{\mathrm{G}}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\tilde{\mathbf{L}}_{\mathrm{G}}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\boldsymbol{\epsilon}_{\mathrm{G}} - \sqrt{2}\,|\alpha_{\max}|\displaystyle\sum_{n=1}^{|\bar{G}|}|z_n|\big(|\cos(\arg z_n)| + |\sin(\arg z_n)|\big)}{|\alpha_{\min}|\sqrt{\boldsymbol{\epsilon}_{\mathrm{G}}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\tilde{\mathbf{L}}_{\mathrm{G}}^{2}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\boldsymbol{\epsilon}_{\mathrm{G}}}}\right\} \tag{5}$$

group detector for the group filter $\mathbf{F}_{\mathrm{G-D}}$ is given as

$$\hat{\mathbf{b}}_{\mathrm{G-D}} \in \arg\max_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{A}}^{|G|}} \left\{ 2\mathcal{R}e\left(\boldsymbol{\alpha}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}}\mathbf{r}\right) - \boldsymbol{\alpha}^{\mathsf{H}}\mathbf{M}_{\mathrm{G}}\boldsymbol{\alpha}\right\}. \quad (7)$$

We refer to the group detector specified by (7) as the group decorrelator. Note that this form not only highlights that it is canonical, but also extends the form in [6] to allow for linear dependencies between desired signals and/or between interfering signals, i. e., $\mathbf{S}_{\mathrm{G}}$ and/or $\mathbf{S}_{\bar{\mathrm{G}}}$ can be column-rank deficient. Its asymptotic group performance can be obtained using the results of the previous subsection in general, and simplifies to results in [6] for full-rank signaling. While it possesses the attractive feature of requiring neither the interfering users' energy nor the noise power density, the group decorrelator completely cancels the MAI regardless of the signal space geometry. In particular, in overloaded systems (wherein the desired and interfering signals can be linearly dependent), the group decorrelating filter potentially reduces the desired signal subspace, i. e., $\mathrm{rank}(\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}}\mathbf{S}_{\mathrm{G}}) \leq \mathrm{rank}(\mathbf{S}_{\mathrm{G}})$. In the extreme case where a desired user's signal is a linear combination of interfering signals, the group decorrelating filter cancels the desired signal. Clearly, this leads to an unacceptable performance. The next section solves this problem by generalizing group decorrelation.

Non-canonical group MMSE detectors were proposed in [9] and [20] for the AWGN and fading channels, respectively. Let the subscript $\mathrm{G-ncM}$ refer to non-canonical group MMSE detection. In the case of an AWGN channel, it is comprised of the non-canonical group MMSE filter $\mathbf{F}_{\mathrm{G-ncM}} = \mathbf{H}^{-1}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$ that minimizes $\mathrm{E}\left[||\mathbf{F}_{\mathrm{G}}^{\mathsf{H}}\,\mathbf{r} - \mathbf{b}_{\mathrm{G}}||^2\right]$, the mean-squared error between the estimate and the desired symbols, among all $N \times |G|$ linear transformations $\mathbf{F}_{\mathrm{G}}$, and where $\mathbf{H} \triangleq \mathrm{E}\left[\mathbf{r}\mathbf{r}^{\mathsf{H}}\right]$ is the correlation matrix of the received signal. Even though this group filter does not cancel the MAI from interfering users, the subsequent joint decision rule of the non-canonical group MMSE detector ignores the residual MAI. Consequently, $\hat{\mathbf{b}}_{\mathrm{G-ncM}}$, the joint decision on the desired symbols, is given as in (4) with the matrices $\mathbf{K}_{\mathrm{G-ncM}} \triangleq \mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\mathbf{H}^{-2}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$ and $\mathbf{L}_{\mathrm{G-ncM}} \triangleq \mathbf{F}_{\mathrm{G-ncM}}\mathbf{K}_{\mathrm{G-ncM}}^{+}\mathbf{F}_{\mathrm{G-ncM}}^{\mathsf{H}}$. It was shown to converge (as the noise density goes to zero) to the group decorrelator for the case of full-rank signaling. In the low-rank signaling case, however, neither its asymptotic behavior nor its performance have been studied. We address these issues in Section V, where we propose a new canonical group MMSE detector and compare its performance with the non-canonical group MMSE detector.

IV. Group Pseudo-Decorrelation

Since decorrelation applied to overloaded systems can result in a reduction of the desired signal subspace and even in the cancellation of desired signals, we propose the new group pseudo-decorrelator. It is a canonical group detector with group filter that is obtained using the least-squares criterion. The group decorrelating and pseudo-decorrelating detectors are shown to be equivalent when the desired signals are linearly independent of the interfering signals. However, in the linear dependent case, the latter acts as a *partial decorrelator*. We use the subscripts PD and $\mathrm{G-PD}$ to refer to vectors and matrices pertaining to pseudo-decorrelation and group pseudo-decorrelation, respectively.

The first stage of the new group detector consists of an unconstrained least-squares estimate of the symbols $\mathbf{y} \in \arg\min_{\boldsymbol{\alpha}\in\mathbb{C}^K} ||\mathbf{r} - \mathbf{S}\mathbf{A}\boldsymbol{\alpha}||^2$, which yields $\mathbf{y} = (\mathbf{S}\mathbf{A})^{+}\mathbf{r}$. Notice that this is also the unconstrained ML estimate of the symbols $\mathbf{b}$, when these symbols are arbitrary complex numbers as opposed to being chosen in the discrete alphabet $\boldsymbol{\mathcal{A}}$. The estimate of the desired symbols then corresponds to the coordinates of $\mathbf{y}$ in G, and is denoted by $\mathbf{y}_{\mathrm{G}}$. We define the group pseudo-decorrelating filter to be the corresponding linear transformation and denote it by $\mathbf{F}_{\mathrm{G-PD}}$, so that $\mathbf{y}_{\mathrm{G}} = \mathbf{F}_{\mathrm{G-PD}}^{\mathsf{H}}\mathbf{r}$. It is given by $\mathbf{F}_{\mathrm{G-PD}}^{\mathsf{H}} = [\mathbf{I}_{|G|}, \mathbf{0}]\,(\mathbf{S}\mathbf{A})^{+}$, where $\mathbf{0}$ is a $|G| \times (K - |G|)$ matrix of zeros, i. e., by selecting the columns of the joint pseudo-decorrelating filter $\mathbf{F}_{\mathrm{PD}} = (\mathbf{S}\mathbf{A})^{+\mathsf{H}}$ whose indices are in $|G|$. Exactly specifying the group pseudo-decorrelating filter requires a partition of $(\mathbf{S}\mathbf{A})^{+}$, which, in turn, depends on the geometry of the signal subspace. Defining the matrices $\mathbf{K}_{\mathrm{G-PD}}$ and $\mathbf{L}_{\mathrm{G-PD}}$ for the group pseudo-decorrelator as suggested in Section III-A, the joint decision rule is given as in (4) by

$$\hat{\mathbf{b}}_{\mathrm{G-PD}} \in \arg\max_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{A}}^{|G|}} \Big\{ 2\mathcal{R}e\left(\boldsymbol{\alpha}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\mathbf{L}_{\mathrm{G-PD}}\,\mathbf{r}\right) - \boldsymbol{\alpha}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\mathbf{L}_{\mathrm{G-PD}}\,\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\boldsymbol{\alpha}\Big\}. \quad (8)$$

A. Linearly independent desired and interfering signals

Consider the case where the desired signals are linearly independent of the interfering signals, i. e., $\dim(\mathcal{S}_{\mathrm{G}} \cap \mathcal{S}_{\bar{\mathrm{G}}}) = 0$. We use the following lemma, proved in Appendix C, to derive the group pseudo-decorrelating filter.

Lemma 1: Let $\mathbf{Q}$ be an $m \times n$ matrix with $m < n$ and which is partitioned as $\mathbf{Q} = [\mathbf{A},\, \mathbf{B}]$, where $\mathbf{A}$ and $\mathbf{B}$ do not necessarily have full rank, but span linearly independent subspaces ($\mathcal{A}$ and $\mathcal{B}$, respectively), i. e., $\mathrm{rank}(\mathbf{Q}) = \mathrm{rank}(\mathbf{A}) + \mathrm{rank}(\mathbf{B})$. The Moore-Penrose pseudo-inverse of $\mathbf{Q}$ has the following partitioned form

$$\mathbf{Q}^{+} = \begin{bmatrix} \left(\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^{\perp}}\mathbf{A}\right)^{+}\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^{\perp}} \\ \left(\mathbf{B}^{\mathsf{H}}\mathbf{P}_{\mathcal{A}^{\perp}}\mathbf{B}\right)^{+}\mathbf{B}^{\mathsf{H}}\mathbf{P}_{\mathcal{A}^{\perp}} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{P}_{\mathcal{B}^{\perp}}\mathbf{A}\right)^{+} \\ \left(\mathbf{P}_{\mathcal{A}^{\perp}}\mathbf{B}\right)^{+} \end{bmatrix}. \quad (9)$$

Since $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{S}_{\bar{\mathrm{G}}}$ are linearly independent, Lemma 1 can be used to partition $(\mathbf{S}\mathbf{A})^{+}$ and yields $\mathbf{F}_{\mathrm{G-PD}}^{\mathsf{H}} = \left(\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\right)^{+}$. Therefore, the group pseudo-decorrelating filter is equivalent to the group decorrelating filter in (6). Furthermore, it can be shown that

$$\mathbf{K}_{\mathrm{G-PD}} = \mathbf{M}_{\mathrm{G}}^{+} \quad (10)$$

$$\mathbf{L}_{\mathrm{G-PD}} = \mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\,\mathbf{M}_{\mathrm{G}}^{+}\,\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}} \quad (11)$$

(notice they don't depend on the noise density) so that the decision rule of the group pseudo-decorrelator in (8) becomes

$$\hat{\mathbf{b}}_{\mathrm{G-PD}} \in \arg\max_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{A}}^{|G|}} \left\{ 2\mathcal{R}e\left(\boldsymbol{\alpha}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}}\mathbf{r}\right) - \boldsymbol{\alpha}^{\mathsf{H}}\mathbf{M}_{\mathrm{G}}\boldsymbol{\alpha}\right\}. \quad (12)$$

This decision rule is the same as the group decorrelator defined in (7). Consequently, the group pseudo-decorrelator requires only the knowledge of the received amplitudes of the desired users. Note again that we allow for linear dependencies within the set of desired signals and/or within the set of interfering signals, i. e., $\mathbf{S}_{\mathrm{G}}$ and/or $\mathbf{S}_{\bar{\mathrm{G}}}$ can be column-rank deficient.

B. Linearly dependent desired and interfering signals

When the desired signals are linearly dependent on the interfering signals, i. e., $\dim(\mathcal{S}_{\mathrm{G}} \cap \mathcal{S}_{\bar{\mathrm{G}}}) > 0$, there is no general closed form partition of $(\mathbf{SA})^{+}$ (except when $\mathcal{S}_{\mathrm{G}} \subseteq \mathcal{S}_{\bar{\mathrm{G}}}$, a case which we study in Section V-B). Instead, the behavior of the group pseudo-decorrelating filter can be explained by observing its output. This output depends on the product $(\mathbf{SA})^{+}\mathbf{SA}$, which, in turn, can be determined via the following lemma.

Lemma 2: If the k^{th} column of the matrix $\mathbf{Q}$ is linearly independent of the remaining columns, then $(\mathbf{Q}^{+}\mathbf{Q})_{kj} = \delta_{kj}$, otherwise $(\mathbf{Q}^{+}\mathbf{Q})_{kj} = \frac{1}{1+||\mathbf{x}||^2} \times \begin{cases} x_j & \text{if } j < k \\ ||\mathbf{x}||^2 & \text{if } j = k \\ x_{j-1} & \text{if } j > k \end{cases}$, where $\mathbf{x} \triangleq \bar{\mathbf{Q}}_k^{+}\mathbf{q}_k$, x_j is its j^{th} component, $\mathbf{q}_k$ is the k^{th} column of $\mathbf{Q}$, and $\bar{\mathbf{Q}}_k$ contains all other columns.

Proof: Without loss of generality, assume $k = 1$. Consider the partition $\mathbf{Q} = [\mathbf{q}_1,\ \bar{\mathbf{Q}}_1]$, denote the span of $\bar{\mathbf{Q}}_1$ by $\bar{\mathcal{Q}}_1$, and define the vector $\mathbf{x} \triangleq \bar{\mathbf{Q}}_1^{+}\mathbf{q}_1$.

If $\mathbf{q}_1$ is not a linear combination of the remaining columns, i. e., $\mathbf{q}_1 \notin \bar{\mathcal{Q}}_1$, then Lemma 1 can be used to derive a partitioned form of $\mathbf{Q}^{+}$, which yields $(\mathbf{Q}^{+}\mathbf{Q})_{1j} = \delta_{1j}$.

On the other hand, if $\mathbf{q}_1$ belongs to the subspace spanned by the remaining columns, i. e., $\mathbf{q}_1 \in \bar{\mathcal{Q}}_1$, then $\mathbf{x}$ represents the coordinates of $\mathbf{q}_1$ along the vectors in $\bar{\mathbf{Q}}_1$. Indeed, since $\bar{\mathbf{Q}}_1\bar{\mathbf{Q}}_1^{+}$ is the orthogonal projection onto $\bar{\mathcal{Q}}_1$, it follows that $\mathbf{q}_1 = \bar{\mathbf{Q}}_1\bar{\mathbf{Q}}_1^{+}\mathbf{q}_1 = \bar{\mathbf{Q}}_1\mathbf{x}$. In this case, the partitioned form of $\mathbf{Q}^{+}$ is obtained from Greville's formula for the update of a pseudo-inverse [18, Chap. 5] as

$$\mathbf{Q}^{+} = \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{Q}}_1^{+} \end{bmatrix} + \frac{1}{1+||\mathbf{x}||^2} \begin{bmatrix} \mathbf{x}^{\mathsf{H}} \\ -\mathbf{x}\mathbf{x}^{\mathsf{H}} \end{bmatrix} \bar{\mathbf{Q}}_1^{+}. \qquad (13)$$

It is then straightforward to verify that the first column of $\mathbf{Q}^{+}\mathbf{Q}$ is $\frac{1}{1+||\mathbf{x}||^2}\begin{bmatrix} ||\mathbf{x}||^2 \\ \mathbf{x} \end{bmatrix}$. ■

To understand how the group pseudo-decorrelating filter affects the output of a desired user, let $k \in G$, denote the corresponding output by $y_k = \left(\mathbf{F}_{\mathrm{G-PD}}^{\mathsf{H}}\,\mathbf{r}\right)_k$, and define $\mathbf{x}_k \triangleq \left(\bar{\mathbf{S}}_k\bar{\mathbf{A}}_k\right)^{+}\mathbf{s}_k$, where $\bar{\mathbf{S}}_k$ contains the remaining columns of $\mathbf{S}$ (it is comprised of signals of both interfering and other desired users), and $\bar{\mathbf{A}}_k$ is obtained from $\mathbf{A}$ by deleting its k^{th} row and column.

If $\mathbf{s}_k$ is not a linear combination of signals in $\bar{\mathbf{S}}_k$, then Lemma 2 yields that $y_k = b_k + \eta_k$. The group pseudo-decorrelating filter for user k has canceled the MAI from interfering users *and* from other desired users. As far as user k is concerned, it acts as a decorrelator.

On the other hand, if $\mathbf{s}_k$ is a linear combination of signals $\bar{\mathbf{S}}_k$, such as can arise in overloaded systems, Lemma 2 shows that $\mathbf{x}_k$ is the coordinate vector of $\mathbf{s}_k$, and that it has components equal to zero when they correspond to signals in $\bar{\mathbf{S}}_k$ of which $\mathbf{s}_k$ is linearly independent. These users, whether they are desired or interfering users, are therefore canceled from y_k. In particular, if $\mathbf{s}_k$ is a linear combination of other desired signals, i. e., $\mathbf{S}_{\mathrm{G}}\mathbf{a} = \mathbf{0}$ with $a_k \neq 0$, but is linearly independent of interfering signals, i. e., $\mathbf{s}_k \notin \mathcal{S}_{\bar{\mathrm{G}}}$, it follows that interfering users are canceled but not those desired signals on which $\mathbf{s}_k$ is linearly dependent. Consequently, as far as user k is concerned, the group pseudo-decorrelating filter acts like the group decorrelating filter in (6). They act differently, though, when $\mathbf{s}_k \in \mathcal{S}_{\bar{\mathrm{G}}}$, in which case the group pseudo-decorrelating filter for user k acts as a partial decorrelator.

Notice that, in general, the output of the group pseudo-decorrelating filter is not MAI-free, so that the matrices $\mathbf{K}_{\mathrm{G-PD}}$ and $\mathbf{L}_{\mathrm{G-PD}}$, and consequently the group pseudo-decorrelator, depend on the noise density. They also depend on the amplitudes of the interferers that are not canceled by the group pseudo-decorrelating filter. This contrasts with the linear independent case of the previous subsection. The special case $|G| = 1$ is equivalent to the pseudo-inverse detector of [4], which was proposed in place of the decorrelator when the signals are linearly dependent.

V. GROUP MMSE DETECTION

A new canonical group MMSE detector is proposed and its asymptotic form and performance studied. While this group detector can be derived using the framework of Section III-A by specifying the group MMSE filter (which yields the same group filter as the non-canonical group MMSE detector discussed in Section III-C), followed by an ML decision rule under the Gaussian residual MAI assumption, we propose another formulation that relies on a different Gaussian assumption and thereby offers another interpretation of MMSE detection.

Assume that $\mathbf{b}$ is a zero-mean, Gaussian random vector with covariance $\mathbf{I}_K$, and that it is jointly Gaussian with $\mathbf{r}$. The unconstrained MMSE estimate of $\mathbf{b}_{\mathrm{G}}$ given the observation $\mathbf{r}$ is then obtained from the Gauss-Markov theorem as $\mathbf{F}_{\mathrm{G-M}}^{\mathsf{H}}\mathbf{r}$, where we have introduced the group MMSE filter $\mathbf{F}_{\mathrm{G-M}} = \mathbf{H}^{-1}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$ (the subscripts M and $\mathrm{G-M}$ refer to MMSE and group MMSE detection, respectively). Note that the canonical and non-canonical group MMSE filters are identical (recall that it is the joint decision rule that determines whether a group detector is canonical or not), and that they require the knowledge of the signals and amplitudes of all users as well as the noise density.

The post-filtering statistic, conditioned on the transmitted symbol vector $\mathbf{b}_{\mathrm{G}}$, can be shown to be given by $\mathbf{y}_{\mathrm{G-M}} = \mathbf{N}_{\mathrm{G}}\mathbf{b}_{\mathrm{G}} + \boldsymbol{\gamma}_{\mathrm{G-M}}$, where $\mathbf{N}_{\mathrm{G}} \triangleq \mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\mathbf{H}^{-1}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$, and $\boldsymbol{\gamma}_{\mathrm{G-M}} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\mathbf{K}_{\mathrm{G-M}}\right)$ with $\sigma^2\mathbf{K}_{\mathrm{G-M}} = \mathbf{N}_{\mathrm{G}} - \mathbf{N}_{\mathrm{G}}^2$. The ML rule is then

$$\begin{aligned}\hat{\mathbf{b}}_{\mathrm{G-M}} \in \arg\max_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{A}}^{|G|}} \Big\{ & 2\mathcal{R}e\left(\boldsymbol{\alpha}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\mathbf{L}_{\mathrm{G-M}}\,\mathbf{r}\right) - \\ & \boldsymbol{\alpha}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\mathbf{L}_{\mathrm{G-M}}\,\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\boldsymbol{\alpha}\Big\}, \end{aligned} \qquad (14)$$

with $\mathbf{L}_{\mathrm{G-M}} = \mathbf{H}^{-1}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{K}_{\mathrm{G-M}}^{+}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\mathbf{H}^{-1}$. The resulting cascade of operations (group MMSE filtering and ML decision rule) defines the group MMSE detector. This is a canonical group detector. While it employs the same group MMSE filter as the non-canonical one in [9], they differ in their joint decision rules. For a group size one, however, they coincide and reduce to the linear MMSE detector of [8, 21].

A. Asymptotic form and performance of the canonical group MMSE detector

The asymptotic (low noise limit) form of the canonical group MMSE detector is derived by considering separately the limits of the group MMSE filter and of the subsequent decision rule. For the latter limit, we need to consider the linear independent and dependent cases separately.

Let $\mathbf{F}_{\mathrm{M}}$ denote the joint MMSE filter, and note that $\mathbf{F}_{\mathrm{G-M}}$ corresponds to the first $|G|$ columns of $\mathbf{F}_{\mathrm{M}}$. Therefore, the limit of the group MMSE filter is obtained from the limit of the joint MMSE filter. $\mathbf{F}_{\mathrm{M}}$ and its low noise limit are specified by the following lemma.

Lemma 3: The joint MMSE filter is $\mathbf{F}_{\mathrm{M}} = \mathbf{H}^{-1}\mathbf{SA}$, and its limit is $\lim_{\sigma\to 0}\mathbf{F}_{\mathrm{M}}^{\mathsf{H}} = (\mathbf{SA})^{+}$.

Proof: The joint MMSE filter is the solution to $\arg\min_{\mathbf{F}} \mathrm{E}\left[||\mathbf{F}^{\mathsf{H}}\mathbf{r}-\mathbf{b}||^2\right]$, and is easily shown to be given by $\mathbf{F}_{\mathrm{M}} = \mathbf{H}^{-1}\mathbf{SA}$. To derive its limit, consider the singular value decomposition $\mathbf{SA} = \mathbf{UDV}^{\mathsf{H}}$, where, because $\mathbf{S}$ has full row-rank equal to N, $\mathbf{U}$ is an $N\times N$ unitary matrix, $\mathbf{D}$ is the $N\times N$ diagonal matrix of singular values, and $\mathbf{V}$ is an $K\times N$ matrix such that $\mathbf{V}^{\mathsf{H}}\mathbf{V} = \mathbf{I}_N$. Consequently, $(\mathbf{SA})^{+} = \mathbf{VD}^{-1}\mathbf{U}^{\mathsf{H}}$, and the following sequence of equalities establishes the desired result.

$$\begin{aligned}\lim_{\sigma\to 0}\mathbf{F}_{\mathrm{M}}^{\mathsf{H}} &= \lim_{\sigma\to 0}(\mathbf{SA})^{\mathsf{H}}\left(\mathbf{SA}(\mathbf{SA})^{\mathsf{H}}+\sigma^2\mathbf{I}_N\right)^{-1}\\ &= \lim_{\sigma\to 0}\mathbf{VDU}^{\mathsf{H}}\left(\mathbf{UD}^2\mathbf{U}^{\mathsf{H}}+\sigma^2\mathbf{UU}^{\mathsf{H}}\right)^{-1}\\ &= \mathbf{VD}\lim_{\sigma\to 0}\left(\mathbf{D}^2+\sigma^2\mathbf{I}_N\right)^{-1}\mathbf{U}^{\mathsf{H}} = \mathbf{VD}^{-1}\mathbf{U}^{\mathsf{H}}.\end{aligned}$$

■

Lemma 3 yields that $\lim_{\sigma\to 0}\mathbf{F}_{\mathrm{G-M}}^{\mathsf{H}} = [\mathbf{I}_{|G|},\,\mathbf{0}]\,(\mathbf{SA})^{+}$. Hence, we conclude that the group MMSE filter is asymptotically equivalent to the group pseudo-decorrelating filter in general. From Section IV, it follows that it is asymptotically equivalent to the group decorrelating filter in the linear independent case only.

The asymptotic equivalence of the group MMSE and group pseudo-decorrelating decision rules is a consequence of the asymptotic equivalence of the matrices involved in these two rules. This is established by the following lemma, which we prove in Appendix D.

Lemma 4: In the linear independent case, i. e., when $\dim(\mathcal{S}_{\mathrm{G}}\cap\mathcal{S}_{\bar{\mathrm{G}}}) = 0$,

$$\begin{aligned}\lim_{\sigma\to 0}\mathbf{K}_{\mathrm{G-M}} &= \mathbf{K}_{\mathrm{G-PD}}\\ \lim_{\sigma\to 0}\mathbf{L}_{\mathrm{G-M}} &= \mathbf{L}_{\mathrm{G-PD}},\end{aligned}$$

with $\mathbf{K}_{\mathrm{G-PD}}$ and $\mathbf{L}_{\mathrm{G-PD}}$ given as in (10) and (11).

In the linear dependent case, i. e., when $\dim(\mathcal{S}_{\mathrm{G}}\cap\mathcal{S}_{\bar{\mathrm{G}}}) > 0$,

$$\begin{aligned}\lim_{\sigma\to 0}\sigma^2\mathbf{K}_{\mathrm{G-M}} &= \lim_{\sigma\to 0}\sigma^2\mathbf{K}_{\mathrm{G-PD}}\\ \lim_{\sigma\to 0}\frac{1}{\sigma^2}\mathbf{L}_{\mathrm{G-M}} &= \lim_{\sigma\to 0}\frac{1}{\sigma^2}\mathbf{L}_{\mathrm{G-PD}},\end{aligned}$$

and these limits are non-zero.

Combining Lemma 3 and 4, we obtain the result that the group MMSE detector is always asymptotically equivalent to the group pseudo-decorrelator. Consequently, if, and only if, the desired and interfering signal subspaces are mutually linearly independent, does it converge to the group decorrelator. Its asymptotic group and per-user performance can then be derived from the analysis in [6] for the full-rank signaling case, and from Section III-B for the case when $\mathbf{S}_{\mathrm{G}}$ is column-rank deficient. This result generalizes that of [8] which proves that the linear MMSE detector converges to the decorrelator when the desired user's signal is linearly independent of the interfering signals. Finally, note that the non-canonical group MMSE detectors proposed in [9] without the Gaussian assumption was shown to also converge to the group decorrelator, but in the full-rank signaling case only.

The second part of the result addresses the important problem where the desired and interfering subspaces have a non-trivial intersection, in which case the group MMSE detector asymptotically acts as a partial decorrelator. Its asymptotic performance can be derived using the results of Section III-B. The special case of group size one corresponds to linear MMSE detection in the linear dependent case and is of particular interest. It is summarized in the following proposition (where $1-\mathrm{M}$ denotes the MMSE filter for user 1) and is proved in Appendix E.

Proposition 2: Let user 1 be the desired user, and partition the signal matrix as $\mathbf{S} = [\mathbf{s}_1,\,\bar{\mathbf{S}}_1]$. When the desired user's signal lies in the interfering subspace, i. e., $\mathbf{s}_1\in\bar{\mathcal{S}}_1$, the MMSE filter is asymptotically equivalent to the pseudo-decorrelating filter and is proportional to

$$\lim_{\sigma\to 0}\mathbf{f}_{1-\mathrm{M}} \propto \left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-1}\mathbf{s}_1. \tag{15}$$

Consequently, the linear MMSE detector converges to the linear pseudo-decorrelator (which is equivalent to the pseudo-inverse detector of [4]) and its AEE is given in (16) at the bottom of the page, where $x_k = |x_k|\mathrm{e}^{j\arg x_k}$ is the k^{th} component of $\mathbf{x}\triangleq\left(\bar{\mathbf{S}}_1\bar{\mathbf{A}}_1\right)^{+}\mathbf{s}_1$, $\theta_k\triangleq\varphi_1+\arg x_k$, and $\alpha_{\boldsymbol{\mathcal{A}}}\triangleq\frac{\max_{\alpha\in\boldsymbol{\mathcal{A}}}|\alpha|}{\min_{i\neq j}|\alpha_j-\alpha_i|}$ (for a square M-QAM constellation, i. e., when $M=2^{2m}$, $\alpha_{\boldsymbol{\mathcal{A}}}=\frac{\sqrt{M}-1}{\sqrt{2}}$).

Note that if $\mathrm{rank}(\mathbf{S}_{\mathrm{G}}) = N$ (which implies that $|G|\geq N$), then $\mathbf{S}_{\mathrm{G}}\mathbf{S}_{\mathrm{G}}^{+} = \mathbf{I}_N$, and the group MMSE filter is equal, up to an invertible transformation, to the group matched-filter $[\mathbf{I}_{|G|},\,\mathbf{0}]^{\mathsf{T}}$. Consequently, the canonical group MMSE and matched-filter detectors are equivalent.

B. Performance comparison with non-canonical group MMSE detection

Next, we compare the asymptotic performance in terms of GSE of the canonical and non-canonical group MMSE detectors when their performances differ. This excludes the cases of a size-one group and of linearly independent desired and interfering signal subspaces. In the linear dependent case, since the group MMSE filter does not asymptotically cancel the MAI, it is reasonable to expect that the joint decision rule that accounts for the residual MAI outperforms the one that ignores it. However, comparing GSE for arbitrary signal space geometry does not appear to be analytically tractable. Instead, using the analysis in Appendix B, we derive sufficient conditions for non-zero GSE in the extreme case where the desired signals lie entirely in the interfering subspace, a case that yields simple and closed-form results. Our conclusion is that the non-canonical group MMSE detector is more likely than the canonical group MMSE detector to violate its sufficient condition.

The sufficient conditions for non-zero GSE derived in Appendix B require the low-noise form of the matrices $\mathbf{L}_{\mathrm{G-ncM}}$ and $\mathbf{L}_{\mathrm{G-M}}$ (which we denote by $\tilde{\mathbf{L}}_{\mathrm{G-ncM}}$ and $\tilde{\mathbf{L}}_{\mathrm{G-M}}$, respectively), and hence of the group MMSE filter. From the results in Section V-A, the latter is asymptotically equivalent to the group pseudo-decorrelating filter, which is given by $\mathbf{F}^{\mathsf{H}}_{\mathrm{G-PD}} = [\mathbf{I}_{|G|}, \mathbf{0}]\,(\mathbf{SA})^{+}$ in general. When $\mathcal{S}_{\mathrm{G}} \subseteq \mathcal{S}_{\bar{\mathrm{G}}}$, the following partition of $(\mathbf{SA})^{+}$ is obtained using a generalization of Greville's formula [18, Chap. 5] (i. e., a generalization of (13))

$$
\begin{aligned}
(\mathbf{SA})^{+} = &\begin{bmatrix} \mathbf{0} \\ (\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}})^{+} \end{bmatrix} \quad + \\
&\begin{bmatrix} \mathbf{I}_{|G|} \\ -\mathbf{X} \end{bmatrix} \left(\mathbf{I}_{|G|} + \mathbf{X}^{\mathsf{H}}\mathbf{X}\right)^{-1} \mathbf{X}^{\mathsf{H}}(\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}})^{\mathsf{H}},
\end{aligned}
$$

where $\mathbf{0}$ is a $|G| \times N$ matrix of zeros, and $\mathbf{X} \triangleq (\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}})^{+}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$ (the columns of $\mathbf{X}$ are the weighted coordinates of each desired signal along the interfering signals). This partition implies that, up to an invertible transformation, we have $\mathbf{F}_{\mathrm{G-PD}} = \left(\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{E}_{\bar{\mathrm{G}}}\mathbf{S}^{\mathsf{H}}_{\bar{\mathrm{G}}}\right)^{-1}\mathbf{S}_{\mathrm{G}}$, which is a generalization of the asymptotic form of the linear MMSE filter in Proposition 2 to arbitrary group sizes. Define the matrix $\mathbf{\Sigma}_{\bar{\mathrm{G}}} \triangleq \left(\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{E}_{\bar{\mathrm{G}}}\mathbf{S}^{\mathsf{H}}_{\bar{\mathrm{G}}}\right)^{-1}$ and note that, since $\mathcal{S}_{\mathrm{G}} \subseteq \mathcal{S}_{\bar{\mathrm{G}}}$, $\mathbf{S}_{\bar{\mathrm{G}}}$ has full row-rank (i. e., $N = |\bar{G}|$), $\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{E}_{\bar{\mathrm{G}}}\mathbf{S}^{\mathsf{H}}_{\bar{\mathrm{G}}}$ is positive-definite and hence $\mathbf{\Sigma}_{\bar{\mathrm{G}}}$ is well defined.

The desired low-noise forms can then be shown to be given by $\tilde{\mathbf{L}}_{\mathrm{G-ncM}} = \mathbf{F}_{\mathrm{G-PD}}\mathbf{F}^{+}_{\mathrm{G-PD}}$ and $\tilde{\mathbf{L}}_{\mathrm{G-M}} = \mathbf{\Sigma}_{\bar{\mathrm{G}}}\mathbf{S}_{\mathrm{G}}\left(\mathbf{S}^{\mathsf{H}}_{\mathrm{G}}\mathbf{\Sigma}_{\bar{\mathrm{G}}}\mathbf{S}_{\mathrm{G}}\right)^{+}\mathbf{S}^{\mathsf{H}}_{\mathrm{G}}\mathbf{\Sigma}_{\bar{\mathrm{G}}}$, respectively. Specializing conditions (SC 1) and (SC 2) of Appendix B to the non-canonical and canonical group MMSE detectors, respectively, we obtain

$$
\min_{i \neq j}\ \tilde{\mathbf{e}}^{\mathsf{H}}_{ij}\,\tilde{\mathbf{L}}_{\mathrm{G-ncM}}\,\tilde{\mathbf{e}}_{ij} \quad > \quad 4 \max_{\mathbf{b}_{\bar{\mathrm{G}}} \in \boldsymbol{\mathcal{A}}^{|\bar{G}|}} \tilde{\mathbf{b}}^{\mathsf{H}}_{\bar{\mathrm{G}}}\,\tilde{\mathbf{L}}_{\mathrm{G-ncM}}\,\tilde{\mathbf{b}}_{\bar{\mathrm{G}}}, \tag{17}
$$

$$
\min_{i \neq j}\ \tilde{\mathbf{e}}^{\mathsf{H}}_{ij}\,\mathbf{\Sigma}_{\bar{\mathrm{G}}}\,\tilde{\mathbf{e}}_{ij} \quad > \quad 4\,|\alpha_{\max}|^{2}\,|\bar{G}|, \tag{18}
$$

where $\tilde{\mathbf{e}}_{ij} \triangleq \mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{e}_{ij}$, $\mathbf{e}_{ij}$ is the difference between the desired symbol vectors that correspond to the joint hypothesis i and j ($i \neq j$), and $\tilde{\mathbf{b}}_{\bar{\mathrm{G}}} \triangleq \mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\mathbf{b}_{\bar{\mathrm{G}}}$. To derive (18), we have used that $\tilde{\mathbf{L}}_{\mathrm{G-M}}\mathbf{\Sigma}^{-1}_{\bar{\mathrm{G}}}\tilde{\mathbf{L}}_{\mathrm{G-M}} = \tilde{\mathbf{L}}_{\mathrm{G-M}}$ and $\tilde{\mathbf{e}}^{\mathsf{H}}_{ij}\,\tilde{\mathbf{L}}_{\mathrm{G-M}}\,\tilde{\mathbf{e}}_{ij} = \tilde{\mathbf{e}}^{\mathsf{H}}_{ij}\,\mathbf{\Sigma}_{\bar{\mathrm{G}}}\,\tilde{\mathbf{e}}_{ij}$, which follow from basic properties of pseudo-inverses.

These sufficient conditions are quadratic forms that depend, respectively, on the projection matrix $\tilde{\mathbf{L}}_{\mathrm{G-ncM}}$ and the positive-definite matrix $\mathbf{\Sigma}_{\bar{\mathrm{G}}}$. The left and right hand side of (17) involve a minimization and maximization, respectively, so that the sufficient condition (17) is more likely to be violated for some pair of hypotheses. Numerical examples in Section VIII-A further illustrate that the canonical group MMSE detector outperforms the non-canonical one for a variety of different scenarios and signal space geometry.

VI. Real-Valued Modulation

In this section, we adapt our results on group detection to the case where the modulation is real-valued, a case that arises for example when pulse amplitude modulation (PAM) is employed and that has recently received much attention.

A. The minimal sufficiency approach

When the alphabet $\boldsymbol{\mathcal{A}}$ is real-valued, and because the received phases are known, the complex vector $\mathbf{r}$ in (1) is not a minimal sufficient statistic (see e.g. [22, Chap. IV] for a definition of minimal sufficiency). A minimal sufficient statistic is in fact real-valued and was obtained in [6], where it was used to specify the PAM multiple access channel or PMAC, as opposed to the complex-valued QAM multiple access channel or QMAC of (1). It was suggested in [6] that suboptimum detectors can be obtained for real-valued modulation by simply specifying them for the PMAC model. It was also shown that PMAC linear and group decorrelators outperform their QMAC counterparts for real-modulation [6, Corollary 2] in the full-rank signaling case.

Similarly, using the minimal sufficiency approach, we can improve over the QMAC group pseudo-decorrelating and MMSE detectors of previous sections (when used for real modulation) by considering their PMAC counterparts. The PMAC group pseudo-decorrelator is thus the natural generalization of the PMAC group decorrelator to general signaling cases: it acts as a partial decorrelator in the linear dependent case, and they coincide in the linear independent case. In either case, it is the low noise limit of the PMAC group MMSE detector.

B. The "widely linear" approach

Recently, an alternative approach has been proposed for linear detection that relies rather on the complex sufficient statistic $\mathbf{r}$ to improve over the QMAC linear detectors. The so-called *widely linear* or *conjugate linear* multiuser detectors in [11–14] are essentially based on the results in [23] (derived for the MMSE criterion), which proved that the widely linear approach is optimal (in the sense of minimizing the MSE) when dealing with

$$
e_1 = \max{}^{2}\left\{0,\ \frac{\sqrt{E_1}\,\mathbf{s}^{\mathsf{H}}_1\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}^{\mathsf{H}}_1\right)^{-1}\mathbf{s}_1 - \alpha_{\boldsymbol{\mathcal{A}}}\sqrt{2}\displaystyle\sum_{k=1}^{K-1}|x_k|\Big(|\cos\theta_k| + |\sin\theta_k|\Big)}{\sqrt{\mathbf{s}^{\mathsf{H}}_1\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}^{\mathsf{H}}_1\right)^{-2}\mathbf{s}_1}}\right\}, \tag{16}
$$

complex, improper, random vectors, and thus has superior performance over the strictly linear approach.

In this approach, an estimate is formed by combining the observation $\mathbf{r}$ and its complex conjugate $\mathbf{r}^*$ as $\mathbf{f}^{\mathsf{H}}\mathbf{r} + \mathbf{g}^{\mathsf{H}}\mathbf{r}^*$, where $\mathbf{f}$ and $\mathbf{g}$ are suitably designed filters. In the context of multiuser detection, this approach exploits the fact that, when $\boldsymbol{\mathcal{A}}$ has real-valued elements, $\mathrm{E}[\mathbf{r}\mathbf{r}^{\mathsf{T}}] \neq \mathbf{0}$ and hence the observation is improper. Moreover, the filters $\mathbf{f}$ and $\mathbf{g}$ are complex-conjugates in this context, so that the widely linear structure simplifies to $2\mathcal{R}e\left(\mathbf{f}^{\mathsf{H}}\mathbf{r}\right)$. The decorrelating and MMSE widely linear detectors are derived for instance in [13].

The following proposition, which is proved in Appendix F, shows that the results of the widely linear approach are in fact subsumed by those obtained with the minimal sufficiency approach.

Proposition 3: The widely linear decorrelating and MMSE detectors derived from the QMAC are identical to their PMAC linear detectors derived using the minimal sufficiency approach.

Therefore, the widely linear or conjugate linear detectors recently proposed are not really new, but rather equivalent to linear detectors on the PMAC. Moreover, the idea of first obtaining a minimum sufficient statistic of [6] is more powerful in that it provides a simple solution to the problem of extending the group detectors (or any other suboptimum detector for that matter) to real-valued modulation.

Note also that the derivation of the widely linear detectors is more tedious and the performance comparison with the QMAC counterparts less sharp and general compared to the approach in [6]. For instance, it is claimed that the widely linear decorrelator has a higher asymptotic efficiency than the QMAC linear decorrelator in [13] but the proof was excluded because it was long and tedious (see also [24] on this topic). However, a sharper (and more general) result on the same comparison is to be found in [6, Corollary 2] which also answers the question, "by how much is the PMAC (or widely linear) decorrelator better than the QMAC decorrelator?".

VII. Decision Feedback Detection

In this section, we consider decision feedback detection, wherein users are detected sequentially, in the context of overloaded systems. The group decorrelating decision feedback detector (group D-DFD) was proposed in [6] for the full-rank case. However, because it relies on the group decorrelator discussed in Section III-C, it suffers from the same shortcomings in overloaded systems, and can result in a joint error rate (JER) that floors (i. e., a symmetric energy that is equal to zero). Instead, we propose the group pseudo-decorrelating and group MMSE decision feedback detectors (group PD-DFD and group MMSE-DFD, respectively), which are asymptotically equivalent, and which can yield non-zero symmetric energy in many overloaded cases. While these group decision feedback detectors can be specified for the general case where users are partitioned into distinct groups of arbitrary sizes, we present here the special case of groups of size one, and illustrate the performance of group DFDs via numerical examples in Section VIII-B. In particular, we focus on the PD-DFD in overloaded systems, thereby also characterizing the asymptotic form of the MMSE-DFD for such systems. Finally, we study the asymptotic performance of these two detectors and derive a greedy ordering rule that maximizes their symmetric energy.

A. The pseudo-decorrelating and MMSE decision feedback detectors

A general framework for (group size one) decision feedback detection is presented in [15] and is briefly summarized in this paragraph. A decision feedback detector is parameterized by a feedforward-feedback matrix pair $(\mathbf{F}, \mathbf{B})$. At stage k, it forms the statistics for user k as $y_k = \mathbf{f}_k^{\mathsf{H}}\mathbf{r} - \sum_{j=1}^{k-1} B_{kj}\hat{b}_j$, and feeds it to an appropriate decision rule (in our case a QAM slicer) to produce the decision $\hat{b}_k$. In the statistic y_k, $\{\hat{b}_j\}_{j=1}^{k-1}$ represent the estimated symbols of users that have already been detected, referred to as past users (users 1 to $k-1$), and $\mathbf{f}_k$ is the k^{th} column of the $N \times K$ feedforward matrix $\mathbf{F}$. Its purpose is to mitigate the MAI from future users (users $k+1$ to K). It is hence designed as the filter for user k under a given criterion (e.g. MMSE, decorrelating, pseudo-decorrelating, etc.) in a user-expurgated channel, i. e., a channel where only future users are active. The feedback matrix $\mathbf{B}$ is designed to completely cancel the MAI from past users when their decisions are correct. It is given as the strictly lower triangular part of the matrix $\mathbf{F}^{\mathsf{H}}\mathbf{S}$.

From results in Section IV-A, it follows that the PD-DFD is equivalent to the D-DFD of [5] if, and only if, the signature waveforms are linearly independent, i. e., $N = K$ (and hence $\mathbf{S}$ is non-singular). On the other hand, in overloaded cases, the PD-DFD for each user depends on the corresponding user-expurgated signal space geometry as is detailed next. Note that this also characterizes the asymptotic form of the MMSE-DFD.

Let $\mathbf{S}_k$ denote the user-expurgated signal matrix at stage k, i. e., $\mathbf{S}_k = [\mathbf{s}_k, \mathbf{S}_{k+1}]$. The pseudo-decorrelating filter for user k, denoted by $\mathbf{f}_{k-\mathrm{PD}}$, is then given, up to a scalar constant, by

$$\begin{array}{llll} \text{if} & \mathbf{s}_k \notin \mathcal{S}_{k+1}, & \mathbf{f}_{k-\mathrm{PD}} & = \left(\mathbf{s}_k^{\mathsf{H}}\mathbf{P}_{\bar{\mathcal{S}}_{k+1}^{\perp}}\mathbf{s}_k\right)^{-1}\mathbf{P}_{\bar{\mathcal{S}}_{k+1}^{\perp}}\mathbf{s}_k \\ \text{if} & \mathbf{s}_k \in \mathcal{S}_{k+1}, & \mathbf{f}_{k-\mathrm{PD}} & = \mu\left(\mathbf{S}_{k+1}\mathbf{E}_{k+1}\mathbf{S}_{k+1}\right)^{+}\mathbf{s}_k, \end{array}$$

where μ is chosen so that $\mathbf{f}_{k-\mathrm{PD}}^{\mathsf{H}}\mathbf{s}_k = 1$. In either case, the statistic for user k can be written as

$$y_k = A_k b_k + I_k + J_k + \eta_k, \tag{19}$$

where $I_k = \sum_{i=1}^{k-1} A_i\,\mathbf{f}_{k-\mathrm{PD}}^{\mathsf{H}}\mathbf{s}_i(b_i - \hat{b}_i)$ and $J_k = \mathbf{f}_{k-\mathrm{PD}}^{\mathsf{H}}\mathbf{S}_{k+1}\mathbf{A}_{k+1}\mathbf{b}_{k+1}$ represent the MAI from past and future users, respectively, and $\eta_k \sim \mathcal{CN}\left(0, \sigma^2||\mathbf{f}_{k-\mathrm{PD}}||^2\right)$. In the linear independent case, it follows that $J_k = 0$ (i. e., future users' MAI is canceled) so that, for user k, the PD-DFD reduces to the D-DFD of [5]. In the linear dependent case, the pseudo-decorrelating filter acts as a partial decorrelator and, as discussed in Section IV-B, rejects only those future users of which user k is linearly independent.

B. Asymptotic performance of the PD- and MMSE-DFD

In general, the asymptotic performance of each user under decision feedback detection is complicated, if not impossible, to derive. This is due to the difficulty of rigorously accounting for erroneous decisions from past users (such erroneous decisions yield $I_k \neq 0$), a problem that is known as error propagation.

We can, however, easily derive the AEE of the equivalent genie-aided decision feedback detector, where a "genie" ensures that past decisions are correct (and hence that $I_k = 0$, thereby avoiding error propagation). Let e_k and e_k^g denote the AEEs of user k with the MMSE-DFD and its genie-aided version, respectively. Note that e_k^g depends only on the user-expurgated signal space geometry. It is therefore given by the AEE of the decorrelator in the user-expurgated channel for the linear independent case, and thus $e_k^g = E_k \, \mathbf{s}_k^{\mathsf{H}} \mathbf{P}_{\mathcal{S}_{k+1}^{\perp}} \mathbf{s}_k$, and by the AEE of the pseudo-decorrelator in Proposition 2 for the linear dependent case.

If users are detected in the non-increasing order of their genie-aided AEEs, it is known that every user achieves genie-aided performance [15, Th. 3], i. e., that for all k, $e_k = e_k^g$. However, the existence of such an ordering is not always guaranteed, and, even if it does exist, it is not necessarily easy to determine. Instead, we propose a simple ordering that minimizes the asymptotic JER (or, equivalently, that maximizes the symmetric energy) of the MMSE-DFD among all orderings without assuming perfect feedback.

Proposition 4: Select the first user to be detected as the one that has highest AEE among all K users under linear MMSE detection. The k^{th} user to be detected is the one that has highest AEE among the remaining $K - k + 1$ users under linear MMSE detection in the user-expurgated channel. This ordering maximizes the symmetric energy of the MMSE-DFD among all orderings and hence minimizes its JER in the high-SNR regime.

Proof: The symmetric energy of a decision feedback detector is equal to the symmetric energy of its genie-aided version [15, Th. 2], which is in turn equal to the smallest genie-aided AEE. Therefore, maximizing the symmetric energy of the MMSE-DFD amounts to finding the ordering that maximizes the smallest genie-aided AEE.

Consider the argument in [25] that shows that the ordering rule that chooses users according to the highest signal-to-interference-plus-noise ratio (or SINR) (as opposed to AEE as proposed earlier in [6]) maximizes worst-case SINR under the simplifying assumption of perfect feedback. With genie-aided AEE instead of SINR (and hence without any assumption), the same argument shows that the minimum genie-aided AEE criterion is maximized by the ordering rule of the above proposition. ■

Surprisingly, however, the MMSE-DFD in the overloaded case cannot be guaranteed to user-wise outperform the linear MMSE detector in terms of AEE. This is in contrast with the performance for full-rank signaling, where [15, Th. 1] proves that the D-DFD (MMSE-DFD) user-wise outperforms the linear decorrelator (linear MMSE detector). Notice that if users are grouped so that each group signal matrix has rank N, the group MMSE-DFD reduces to the group matched-filter DFD. This generalizes to the group case the result that the MMSE-DFD reduces to successive cancellation (aka. onion peeling) when $N = 1$.

The result of Proposition 4 can be extended to optimally group and order users (in the sense of maximizing symmetric energy) under a constraint on the group sizes for the case of the group MMSE-DFD. Such an algorithm was proposed in [26] for the group D-DFD in the context of full-rank signaling and can be adapted to the group MMSE-DFD for general signaling using results in Section V-A.

In summary then, the PD-DFD generalizes the D-DFD to overloaded systems by properly accounting for the linear dependence between signature waveforms, and can yield good performance in cases where the D-DFD fails. Moreover, because it is asymptotically equivalent to the MMSE-DFD, it has the same medium/high SNR performance while not requiring the knowledge of σ^2. Finally, a simple greedy ordering rule (also independent of σ^2) ensures the highest rate of exponential decay of the JER for the MMSE-DFD (and PD-DFD) among all orders in which users can be detected.

VIII. Numerical examples

A. Group detection

We illustrate the performance and asymptotic behavior of group detectors for a CDMA system with $K = 8$ users each employing Quadrature Phase Shift Keying (QPSK). The performance is measured in terms of the simulated bit error rate (BER), obtained assuming Gray-encoded QPSK symbols, as a function of the SNR, measured by the average bit-energy (averaged over the desired users) to noise density ratio E_b/N_0. We assume that the users all have unit energies $\mathbf{E} = \mathbf{I}_K$ but that their phases are different. The set of desired users is $G = \{1, 2\}$ and we consider several signal space geometries. In each example, the signals and phases are randomly generated and fixed during the simulation. Hence, with probability one, the desired and interfering signal matrices have full column-rank. The linear dependence between the desired and interfering subspaces is characterized numerically by the difference in rank between $\mathbf{S}_{\mathrm{G}}$ and $\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}} \mathbf{S}_{\mathrm{G}}$.

Fig. 1 illustrates full-rank signaling, i. e., $N = 8$. Here, the group decorrelating and pseudo-decorrelating detectors are identical and their performance converges asymptotic to that of the group MMSE detector. Also, note that the non-canonical MMSE detector of [9, 20] has a worse performance than its canonical counterpart over the full range of SNRs considered. The asymptotic performances of these two detectors are, however, provably the same for full-rank signaling, but this could not be verified by simulations because the asymptotics don't take effect even at error rates of 10^{-6}.

Fig. 2 illustrates overloaded systems, i. e., $N < 8$. In Fig. 2 (a) and (b), the desired signal subspace intersects and is contained in the interfering signal subspace, respectively. For such scenarios, the group decorrelating and pseudo-decorrelating detectors are not equivalent. The latter is however asymptotically equivalent to the canonical group MMSE detector. This asymptotic equivalence is apparent for high SNRs in Fig. 2 (a) and even for low SNRs in Fig. 2 (b). Again, the canonical group MMSE detector far out-performs the non-canonical one, the gap increasing from Fig. 1 to Fig. 2 (a). In fact in Fig. 2 (b), the canonical detector yields an error rate that decays exponentially, whereas the non-canonical detector does not. Finally, the group decorrelator, which projects out part or all of the desired signal subspace, yields the worst performance in Fig. 2 (a), and floors in Fig. 2 (b). These results were found to be typical for several randomly generated signal sets of similar geometry.

B. Decision feedback detection

We illustrate the performance and asymptotic behavior of the (group) decision feedback detectors for two overloaded scenarios each with $K = 10$ users employing BPSK (hence we use the real-valued model discussed in Section VI). We consider two power distributions which correspond to, respectively, equal received energies with $E_k = 1$ as in a power-control-based system (such as in IS-95) and disparate received energies with $E_k = (K - k + 1)^2$ such as would arise, if no power control is implemented, from the natural geographical dispersion of users in a cell (such as in bandwidth-efficient multiple access [1]). The dimension of the signal space is fixed to $N = 8$ and $N = 5$ for the equal and disparate power distributions, respectively. For each power distribution and for the given parameters, the signal matrix is randomly generated and then fixed to numerically simulate the JER versus E_b/N_0. Fig. 3 compares the performance of all three DFDs, with users detected according to the optimum ordering specified by Proposition 4, and of the ML detector.

For the equal power distribution, Fig. 3 (a) illustrates that the D-DFD is useless. The asymptotic equivalence of the MMSE-DFD and PD-DFD occurs at E_b/N_0 greater than can be simulated practically, while for low/medium E_b/N_0, the performance gap can be quite large (as much as 10 dB at error rates of 10^{-2}). The gap between the MMSE-DFD and optimal detection is relatively small for low E_b/N_0 and increases to reach its maximum value, equal to the difference in symmetric energy of the two detectors, at high E_b/N_0.

The performance for the quadratic power distribution is illustrated in Fig. 3 (b). The optimum ordering corresponds to detecting users in the decreasing order of their received energies. As expected, the D-DFD is again useless. More interestingly, the MMSE-DFD and PD-DFD yield almost identical performance for all E_b/N_0, so that knowledge of the noise density does not improve performance here. Finally, note that their performance is near-optimal, up to very high E_b/N_0 (the symmetric energy of the ML detector and the MMSE-DFD we found to be 0 dB and -2.6 dB, respectively, for this example).

The ability of group decision feedback to bridge the gap with optimum detection is illustrated in Fig. 4 for the equal energy scenario of Fig. 3 (a). With the constraint that the maximum group size equals two, the optimal grouping and ordering is found, and the performance of the different group DFDs is simulated. The gap between the optimum and group MMSE-DFD detectors is only slightly reduced, but the reduction for the PD-DFD is much more important (from 11.5 dB in Fig. 3 (a) to 7.5 dB in Fig. 4 at $\text{JER} = 10^{-5}$). Moreover, unlike the case of group sizes one, the canonical and non-canonical group MMSE-DFDs are distinct and have different (though close) finite SNR and asymptotic performance. Finally, note that the group D-DFD is still useless, and the asymptotic equivalence of the group MMSE-DFD and group PD-DFD occurs at lower SNRs than in the case of group sizes one of Fig. 3 (a).

IX. Conclusion

This paper answers several fundamental but hitherto unaddressed questions in the theory of suboptimum multiuser detection in general, and those that pertain to overloaded CDMA systems in particular. In the process, new detectors are proposed and studied, namely the group pseudo-decorrelator, the group MMSE detector, and the pseudo-decorrelating decision feedback detector. These new detectors perform well in overloaded systems in which known detectors such as the (group) decorrelator or the (group) decorrelating decision feedback detector simply fail. New results are also obtained for known detectors. Specifically, the asymptotic forms and performance of MMSE linear, group, and decision feedback detectors are characterized for low-rank signaling, and their dependence on the signal space geometry is highlighted. For instance, pseudo-decorrelation is shown to be the correct extension of decorrelation to overloaded systems in the sense that it is asymptotically equivalent to MMSE detection. Similarly, the low-noise limit of the MMSE-DFD is shown to be the pseudo-decorrelating decision feedback detector. Finally, the idea of minimal sufficiency is used to extend these results to the case of real-valued modulation, and is shown to subsume the widely (or conjugate) linear approach.

Appendix A
Asymptotic Effective Energy of Group Detection

Let P_k denote the error probability of user k. It is given by

$$P_k = \frac{1}{M^{|\bar{G}|}} \sum_{\mathbf{b}_{\bar{G}} \in \boldsymbol{\mathcal{A}}^{|\bar{G}|}} P_k(\mathbf{b}_{\bar{G}}), \tag{20}$$

where $P_k(\mathbf{b}_{\bar{G}})$ denotes the conditional error probability conditioned on $\mathbf{b}_{\bar{G}}$, the vector of interfering users' symbols. Asymptotically tight upper and lower bound can be found using the approach of [19, Sec. 4.3] which yields

$$p\, Q\left(\frac{\varepsilon(\sigma, \mathbf{b}_{\bar{G}})}{\sigma}\right) \quad \leq \quad P_k(\mathbf{b}_{\bar{G}}) \quad \leq \quad \sum_{\boldsymbol{\epsilon}_G \in \Upsilon_k} M^{-w(\boldsymbol{\epsilon}_G)}\, Q\left(\frac{\varepsilon(\sigma, \boldsymbol{\epsilon}_G, \mathbf{b}_{\bar{G}})}{\sigma}\right), \tag{21}$$

where $p \in (0, 1]$, Υ_k denotes the set of indecomposable error vectors, $w(\boldsymbol{\epsilon}_G)$ the weight of the indecomposable error vector $\boldsymbol{\epsilon}_G$, and $\varepsilon(\sigma, \mathbf{b}_{\bar{G}}) = \min_{\boldsymbol{\epsilon}_G \in \Upsilon_k} \varepsilon(\sigma, \boldsymbol{\epsilon}_G, \mathbf{b}_{\bar{G}})$ with

$$\varepsilon(\sigma, \boldsymbol{\epsilon}_G, \mathbf{b}_{\bar{G}}) \triangleq \left(\boldsymbol{\epsilon}_G^{\mathsf{H}} \mathbf{A}_G^{\mathsf{H}} \mathbf{S}_G^{\mathsf{H}} \mathbf{L}_G^2 \mathbf{S}_G \mathbf{A}_G \boldsymbol{\epsilon}_G\right)^{-1/2} \left\{\boldsymbol{\epsilon}_G^{\mathsf{H}} \mathbf{A}_G^{\mathsf{H}} \mathbf{S}_G^{\mathsf{H}} \mathbf{L}_G \mathbf{S}_G \mathbf{A}_G \boldsymbol{\epsilon}_G - 2\mathcal{R}e\left(\boldsymbol{\epsilon}_G^{\mathsf{H}} \mathbf{A}_G^{\mathsf{H}} \mathbf{S}_G^{\mathsf{H}} \mathbf{L}_G \mathbf{S}_{\bar{G}} \mathbf{A}_{\bar{G}} \mathbf{b}_{\bar{G}}\right)\right\}. \tag{22}$$

The set Υ_k is a subset of E_k, the set of all error vectors for user k, defined as $E_k \triangleq \{\boldsymbol{\epsilon}_G = \boldsymbol{\alpha}_j - \boldsymbol{\alpha}_i, \text{with } \boldsymbol{\alpha}_j, \boldsymbol{\alpha}_i \in \boldsymbol{\mathcal{A}}^{|G|}, \text{ and such that } (\boldsymbol{\epsilon}_G)_k \neq 0\}$. The AEE of user k is then given by

$$\frac{1}{|\alpha_{\min}|} \max{}^2 \left\{0, \min_{\boldsymbol{\epsilon}_G \in \Upsilon_k} \min_{\mathbf{b}_{\bar{G}} \in \boldsymbol{\mathcal{A}}^{|\bar{G}|}} \lim_{\sigma \to 0} \varepsilon(\sigma, \boldsymbol{\epsilon}_G, \mathbf{b}_{\bar{G}})\right\},$$

where $|\alpha_{\min}| \triangleq \min_{i \neq j} |\alpha_j - \alpha_i|$. It requires the low noise limit of $\mathbf{L}_G$, which in turns requires the limits of $\mathbf{F}_G$ and $\mathbf{K}_G$ when these matrices depend on σ^2. Let us denote by $\tilde{\mathbf{L}}_G$ the limit of $\mathbf{L}_G$, and introduce $\mathbf{z}(\boldsymbol{\epsilon}_G) \triangleq \mathbf{A}_{\bar{G}}^{\mathsf{H}} \mathbf{S}_{\bar{G}}^{\mathsf{H}} \tilde{\mathbf{L}}_G \mathbf{S}_G \mathbf{A}_G \boldsymbol{\epsilon}_G$.

The limit $\lim_{\sigma\to 0}\varepsilon(\sigma,\boldsymbol{\epsilon}_{\mathrm{G}},\mathbf{b}_{\bar{\mathrm{G}}})$ is then obtained by substituting for $\mathbf{L}_{\mathrm{G}}$ its limit $\tilde{\mathbf{L}}_{\mathrm{G}}$ in (22). Therefore, in the limit of (22), the second term in the resulting difference can be written as $\mathcal{R}e\left(\mathbf{z}(\boldsymbol{\epsilon}_{\mathrm{G}})^{\mathsf{H}}\mathbf{b}_{\bar{\mathrm{G}}}\right)$, which is the sum of $|\bar{G}|$ terms of the form $|z_n|\left(\mathcal{R}e(b_{n+|G|})\cos\theta_n+\mathcal{I}m(b_{n+|G|})\sin\theta_n\right)$ for $n=1,\ldots,|\bar{G}|$, where $z_n=|z_n|e^{j\theta_n}$ is the n^{th} component of $\mathbf{z}(\boldsymbol{\epsilon}_{\mathrm{G}})$ and $b_{n+|G|}$ represents the symbol of interfering user $n+|G|$. For a square QAM constellation ($M=2^{2m}$), the worst case interference corresponds to the each interfering user employing a symbol that lies at one of the corner points of the QAM constellation, i. e., when

$$\forall n\in\{1,\ldots,|\bar{G}|\},\ b_n=\frac{|\alpha_{\max}|}{\sqrt{2}}\Big(\mathrm{sgn}(\cos\theta_n)+j\,\mathrm{sgn}(\sin\theta_n)\Big),$$

where $|\alpha_{\max}|\triangleq\max_{\alpha\in\boldsymbol{\mathcal{A}}}|\alpha|$. For this worst case interference, the second term of the difference in (22) becomes $\sqrt{2}\,|\alpha_{\max}|\sum_{n=1}^{|\bar{G}|}|y_n|\ (|\cos\theta_n|+|\sin\theta_n|)$. Hence the result.

APPENDIX B
SUFFICIENT CONDITIONS FOR NON-ZERO GROUP SYMMETRIC ENERGY

The conditional error exponent of the conditional pairwise error probability $\Pr\left(H_j\to H_i\,\middle|\mathbf{b}_{\bar{\mathrm{G}}}\right)$ that, given the interfering symbol vector $\mathbf{b}_{\bar{\mathrm{G}}}$, hypothesis H_j is more likely than the true hypothesis H_i (corresponding to the desired symbol vectors $\boldsymbol{\alpha}_j$ and $\boldsymbol{\alpha}_i$, respectively) can be shown to be given as in (23) at the bottom of the page, where we have introduced the error vector $\mathbf{e}_{ij}\triangleq\boldsymbol{\alpha}_j-\boldsymbol{\alpha}_i$.

In order to ensure that the joint error rate decays exponentially with decreasing noise density, it is necessary that the numerator in the error exponent (23) be positive for all possible interfering symbol vectors and pairs of hypothesis. Introducing the factorization $\tilde{\mathbf{L}}_{\mathrm{G}}=\mathbf{C}^{\mathsf{H}}\mathbf{C}$ (this is possible since $\tilde{\mathbf{L}}_{\mathrm{G}}$ is positive semi-definite), and the vectors $\tilde{\mathbf{e}}_{ij}\triangleq\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{e}_{ij}$ and $\tilde{\mathbf{b}}_{\bar{\mathrm{G}}}\triangleq\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\mathbf{b}_{\bar{\mathrm{G}}}$, this condition can be expressed by

$$\forall i\neq j,\ \forall\,\mathbf{b}_{\bar{\mathrm{G}}}\in\boldsymbol{\mathcal{A}}^{|\bar{G}|},\ ||\mathbf{C}\tilde{\mathbf{e}}_{ij}||^2>2\,\mathcal{R}e\left((\mathbf{C}\tilde{\mathbf{e}}_{ij})^{\mathsf{H}}\,\mathbf{C}\tilde{\mathbf{b}}_{\bar{\mathrm{G}}}\right).$$

To simplify these conditions, we use the Cauchy-Schwarz inequality to upper bound the right hand side (RHS) in the following two different ways

$$\begin{aligned}\mathrm{RHS}&\geq 2||\mathbf{C}\tilde{\mathbf{e}}_{ij}||\,||\mathbf{C}\tilde{\mathbf{b}}_{\bar{\mathrm{G}}}||\\ \mathrm{RHS}&\geq 2||\mathbf{b}_{\bar{\mathrm{G}}}||\,||\mathbf{A}_{\bar{\mathrm{G}}}^{\mathsf{H}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}}\tilde{\mathbf{L}}_{\mathrm{G}}\,\tilde{\mathbf{e}}_{ij}||.\end{aligned}$$

This yields the following two sufficient conditions for non-zero group symmetric energy

$$\begin{aligned}\min_{i\neq j}\quad&\tilde{\mathbf{e}}_{ij}^{\mathsf{H}}\,\tilde{\mathbf{L}}_{\mathrm{G}}\,\tilde{\mathbf{e}}_{ij}\ >\ 4\max_{\mathbf{b}_{\bar{\mathrm{G}}}\in\boldsymbol{\mathcal{A}}^{|\bar{G}|}}\tilde{\mathbf{b}}_{\bar{\mathrm{G}}}^{\mathsf{H}}\,\tilde{\mathbf{L}}_{\mathrm{G}}\,\tilde{\mathbf{b}}_{\bar{\mathrm{G}}} &&\text{(SC 1)}\\ \forall i\neq j,\quad&\tilde{\mathbf{e}}_{ij}^{\mathsf{H}}\,\tilde{\mathbf{L}}_{\mathrm{G}}\,\tilde{\mathbf{e}}_{ij}\ >\\ &2\,|\alpha_{\max}|\ \left(|\bar{G}|\,\tilde{\mathbf{e}}_{ij}^{\mathsf{H}}\tilde{\mathbf{L}}_{\mathrm{G}}\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{E}_{\bar{\mathrm{G}}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}}\tilde{\mathbf{L}}_{\mathrm{G}}\,\tilde{\mathbf{e}}_{ij}\right)^{1/2}. &&\text{(SC 2)}\end{aligned}$$

APPENDIX C
PSEUDO-INVERSE OF A PARTITIONED MATRIX

The pseudo-inverse of the matrix $\mathbf{Q}$ can always be written as $\mathbf{Q}^+=\left(\mathbf{Q}^{\mathsf{H}}\mathbf{Q}\right)^+\mathbf{Q}^{\mathsf{H}}$. Define the positive semi-definite matrix $\mathbf{R}\triangleq\mathbf{Q}^{\mathsf{H}}\mathbf{Q}$, which can be factored in two different ways by using a generalization of the normal LDU factorization (which requires invertibility of $\mathbf{A}^{\mathsf{H}}\mathbf{A}$ and $\mathbf{B}^{\mathsf{H}}\mathbf{B}$). Indeed, it is easily verified, using $\mathbf{P}_{\mathcal{A}^\perp}=\mathbf{I}-\mathbf{A}\mathbf{A}^+$, $\mathbf{P}_{\mathcal{B}^\perp}=\mathbf{I}-\mathbf{B}\mathbf{B}^+$ and the four properties of the pseudo-inverse, that

$$\begin{aligned}\mathbf{R}&=\begin{bmatrix}\mathbf{I}&\mathbf{A}^{\mathsf{H}}(\mathbf{B}^+)^{\mathsf{H}}\\ \mathbf{0}&\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}&\mathbf{0}\\ \mathbf{0}&\mathbf{B}^{\mathsf{H}}\mathbf{B}\end{bmatrix}\times\\ &\quad\begin{bmatrix}\mathbf{I}&\mathbf{0}\\ \mathbf{B}^+\mathbf{A}&\mathbf{I}\end{bmatrix}\triangleq\mathbf{X}_1\mathbf{D}_1\mathbf{X}_1^{\mathsf{H}}\\ &=\begin{bmatrix}\mathbf{I}&\mathbf{0}\\ \mathbf{B}^{\mathsf{H}}(\mathbf{A}^+)^{\mathsf{H}}&\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A}^{\mathsf{H}}\mathbf{A}&\mathbf{0}\\ \mathbf{0}&\mathbf{B}^{\mathsf{H}}\mathbf{P}_{\mathcal{A}^\perp}\mathbf{B}\end{bmatrix}\times\\ &\quad\begin{bmatrix}\mathbf{I}&\mathbf{A}^+\mathbf{B}\\ \mathbf{0}&\mathbf{I}\end{bmatrix}\triangleq\mathbf{X}_2\mathbf{D}_2\mathbf{X}_2^{\mathsf{H}},\end{aligned}$$

where the matrices $\mathbf{X}_i$ and $\mathbf{D}_i$ (for $i\in\{1,2\}$) are implicitly defined. The four properties that uniquely characterize the Moore-Penrose pseudo-inverse $\mathbf{X}$ of a matrix $\mathbf{C}$ are 1. $\mathbf{CXC}=\mathbf{C}$, 2. $\mathbf{XCX}=\mathbf{X}$, 3. $(\mathbf{XC})^{\mathsf{H}}=\mathbf{XC}$ and 4. $(\mathbf{CX})^{\mathsf{H}}=\mathbf{CX}$ [18]. Notice that the block triangular matrices $\mathbf{X}_i$ are non-singular but that the block diagonal matrices $\mathbf{D}_i$ can be rank deficient.

From these two factorizations, we can derive two different forms for the pseudo-inverse of $\mathbf{R}$. Note, however, that the “reverse-order” property of the pseudo-inverse of a product holds only for necessary and sufficient conditions stated in [18, p.181]. It is not easy to verify that these conditions are met for our case. Instead, we conjecture that $\mathbf{R}^+=\mathbf{X}_1^{-\mathsf{H}}\mathbf{D}_1^+\mathbf{X}_1^{-1}=\mathbf{X}_2^{-\mathsf{H}}\mathbf{D}_2^+\mathbf{X}_2^{-1}$ and verify that these two factorizations satisfy the four properties of the pseudo-inverse listed above.

Clearly, with either factorization, $\mathbf{R}\mathbf{R}^+\mathbf{R}=\mathbf{R}$ and $\mathbf{R}^+\mathbf{R}\mathbf{R}^+=\mathbf{R}^+$. To show that $\mathbf{R}^+\mathbf{R}$ and $\mathbf{R}\mathbf{R}^+$ are indeed hermitian symmetric requires a little more linear algebra. We detail the proof for $\mathbf{R}\mathbf{R}^+$ using the factorization $\mathbf{R}=\mathbf{X}_1\mathbf{D}_1\mathbf{X}_1^{\mathsf{H}}$ (the proof for $\mathbf{R}^+\mathbf{R}$ and/or using the second factorization is similar). Using our block notation, it can be shown that

$$\begin{aligned}\mathbf{R}\mathbf{R}^+&=\mathbf{X}_1\mathbf{D}_1\mathbf{D}_1^+\mathbf{X}_1^{-1}\\ &=\begin{bmatrix}\mathbf{P}_{\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}}&\mathbf{P}_{\left(\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}\right)^\perp}\mathbf{A}^{\mathsf{H}}\mathbf{B}^{\mathsf{H}+}\\ \mathbf{0}&\mathbf{P}_{\mathbf{B}^{\mathsf{H}}\mathbf{B}}\end{bmatrix},\end{aligned}\quad(24)$$

$$\varepsilon_{ij}(\mathbf{b}_{\bar{\mathrm{G}}})=\max{}^2\left\{0,\frac{\mathbf{e}_{ij}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\tilde{\mathbf{L}}_{\mathrm{G}}\,\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{e}_{ij}-2\,\mathcal{R}e\left(\mathbf{e}_{ij}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\tilde{\mathbf{L}}_{\mathrm{G}}\,\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\mathbf{b}_{\bar{\mathrm{G}}}\right)}{2\sigma\sqrt{\mathbf{e}_{ij}^{\mathsf{H}}\mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\tilde{\mathbf{L}}_{\mathrm{G}}^2\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\mathbf{e}_{ij}}}\right\},\quad(23)$$

where $\mathbf{P}_{\left(\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}\right)^\perp} = \mathbf{I} - \mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}(\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A})^+ = \mathbf{I} - \mathbf{P}_{\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}}$ is the orthogonal projection whose null space is the span of $\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}$. It remains to be shown that the north-east block in (24) is identically zero, a sufficient condition for which is that range($\mathbf{A}^{\mathsf{H}}$) $\subseteq$ range($\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}$). In fact we show that these two subspaces are equal. Indeed, we obviously have range($\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}$) $\subseteq$ range($\mathbf{A}^{\mathsf{H}}$) and, from our hypothesis that $\mathbf{A}$ and $\mathbf{B}$ span linearly independent subspaces, it follows that rank($\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}$) = rank($\mathbf{A}$). Consequently, range($\mathbf{A}^{\mathsf{H}}$) = range($\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A}$), the north-east block in (24) is zero and $\mathbf{R}\mathbf{R}^+$ is hermitian symmetric.

The two factorizations of $\mathbf{R}^+$ can be expressed in terms of our block notation as

$$\mathbf{R}^+ = \begin{bmatrix} (\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A})^+ & -(\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A})^+\mathbf{A}^{\mathsf{H}}(\mathbf{B}^+)^{\mathsf{H}} \\ -\mathbf{B}^+\mathbf{A}(\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A})^+ & (\mathbf{B}^{\mathsf{H}}\mathbf{B})^+ + \mathbf{B}^+\mathbf{A}(\mathbf{A}^{\mathsf{H}}\mathbf{P}_{\mathcal{B}^\perp}\mathbf{A})^+\mathbf{A}^{\mathsf{H}}(\mathbf{B}^+)^{\mathsf{H}} \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} (\mathbf{A}^{\mathsf{H}}\mathbf{A})^+ + \mathbf{A}^+\mathbf{B}(\mathbf{B}^{\mathsf{H}}\mathbf{P}_{\mathcal{A}^\perp}\mathbf{B})^+\mathbf{B}^{\mathsf{H}}(\mathbf{A}^+)^{\mathsf{H}} & -\mathbf{A}^+\mathbf{B}(\mathbf{B}^{\mathsf{H}}\mathbf{P}_{\mathcal{A}^\perp}\mathbf{B})^+ \\ -(\mathbf{B}^{\mathsf{H}}\mathbf{P}_{\mathcal{A}^\perp}\mathbf{B})^+\mathbf{B}^{\mathsf{H}}(\mathbf{A}^+)^{\mathsf{H}} & (\mathbf{B}^{\mathsf{H}}\mathbf{P}_{\mathcal{A}^\perp}\mathbf{B})^+ \end{bmatrix}, \quad (26)$$

and can be combined by considering only the north block-rows of (25) and the south block-rows of (26). We then multiply by $\mathbf{Q}^{\mathsf{H}}$ to obtain the two desired forms for the pseudo-inverse of $\mathbf{Q}$.

Appendix D
Asymptotic Equivalence of the Matrices Involved in the Group MMSE and Group Pseudo-decorrelating Decisions Rules

Define $\mathbf{W}(\sigma) \triangleq \mathbf{S}_{\bar{\mathrm{G}}}\mathbf{E}_{\bar{\mathrm{G}}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}} + \sigma^2\mathbf{I}_N$, and note that

$$\begin{aligned}\mathbf{F}_{\mathrm{G-M}}(\sigma) &= \left(\mathbf{W}(\sigma) + \mathbf{S}_{\mathrm{G}}\mathbf{E}_{\mathrm{G}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\right)^{-1}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}} \\ \mathbf{K}_{\mathrm{G-M}}(\sigma) &= \mathbf{F}_{\mathrm{G-M}}^{\mathsf{H}}(\sigma)\,\frac{1}{\sigma^2}\,\mathbf{W}(\sigma)\,\mathbf{F}_{\mathrm{G-M}}(\sigma),\end{aligned}$$

where the dependence on the noise density σ^2 has been made explicit.

In the linear independent case, recall that $\mathbf{K}_{\mathrm{G-PD}}$ and $\mathbf{L}_{\mathrm{G-PD}}$ are given by (10) and (11), respectively, and that they do not depend on the noise density. Therefore, we only need to show they are the low noise limits of $\mathbf{K}_{\mathrm{G-M}}(\sigma)$ and $\mathbf{L}_{\mathrm{G-M}}(\sigma)$, respectively. Using the Sherman-Morrison-Woodbury (SMW) identity [27, Sec. 2.1.3], it can be shown that the group MMSE filter and $\mathbf{K}_{\mathrm{G-M}}(\sigma)$ can be rewritten as

$$\begin{aligned}\mathbf{F}_{\mathrm{G-M}}(\sigma) &= \sigma^2\mathbf{W}^{-1}(\sigma)\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}\left(\sigma^2\mathbf{I}_{|G|} + \mathbf{Z}(\sigma)\right)^{-1} \\ \mathbf{K}_{\mathrm{G-M}}(\sigma) &= \left(\sigma^2\mathbf{I}_{|G|} + \mathbf{Z}(\sigma)\right)^{-1}\mathbf{Z}(\sigma)\left(\sigma^2\mathbf{I}_{|G|} + \mathbf{Z}(\sigma)\right)^{-1},\end{aligned}$$

where we have defined $\mathbf{Z}(\sigma) \triangleq \mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\,\sigma^2\mathbf{W}^{-1}(\sigma)\,\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$. Its low noise limit depends on the limit of $\sigma^2\mathbf{W}^{-1}(\sigma)$, which is given by

$$\begin{aligned}&\lim_{\sigma\to 0}\sigma^2\mathbf{W}^{-1}(\sigma) \\ &= \mathbf{I}_N - \mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\lim_{\sigma\to 0}\left(\sigma^2\mathbf{I}_{|\bar{G}|} + \mathbf{A}_{\bar{\mathrm{G}}}^{\mathsf{H}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}}\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\right)^{-1}\mathbf{A}_{\bar{\mathrm{G}}}^{\mathsf{H}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}} \\ &= \mathbf{I}_N - \mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\left(\mathbf{A}_{\bar{\mathrm{G}}}^{\mathsf{H}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}}\mathbf{S}_{\bar{\mathrm{G}}}\mathbf{A}_{\bar{\mathrm{G}}}\right)^{+}\mathbf{A}_{\bar{\mathrm{G}}}^{\mathsf{H}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}} = \mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^\perp},\end{aligned}$$

where the first equality results from the SMW identity. Hence, $\lim_{\sigma\to 0}\mathbf{Z}(\sigma) = \mathbf{A}_{\mathrm{G}}^{\mathsf{H}}\mathbf{S}_{\mathrm{G}}^{\mathsf{H}}\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^\perp}\mathbf{S}_{\mathrm{G}}\mathbf{A}_{\mathrm{G}}$.

Since $\mathbf{K}_{\mathrm{G-M}}(\sigma)$ is a product of matrices each with a finite limit, its limit is given by the product of the limits. It follows that $\lim_{\sigma\to 0}\mathbf{K}_{\mathrm{G-M}}(\sigma) = \mathbf{K}_{\mathrm{G-PD}}$ as given in (10) and (11). The limiting form of $\mathbf{L}_{\mathrm{G-M}}(\sigma)$ can be similarly computed as the product of limits and yields

$$\begin{aligned}\lim_{\sigma\to 0}\mathbf{L}_{\mathrm{G-M}}(\sigma) &= \lim_{\sigma\to 0}\mathbf{F}_{\mathrm{G-M}}(\sigma)\mathbf{K}_{\mathrm{G-M}}^+(\sigma)\mathbf{F}_{\mathrm{G-M}}(\sigma)^{\mathsf{H}} \\ &= \mathbf{F}_{\mathrm{G-PD}}\,\mathbf{K}_{\mathrm{G-PD}}^+\,\mathbf{F}_{\mathrm{G-PD}}^{\mathsf{H}} = \mathbf{L}_{\mathrm{G-PD}}.\end{aligned}$$

In the linear dependent case, the matrices $\mathbf{K}_{\mathrm{G-PD}}(\sigma)$ and $\mathbf{L}_{\mathrm{G-PD}}(\sigma)$ depend on σ^2 and do not have finite or non-zero limits. Nevertheless, using that $\lim_{\sigma\to 0}\mathbf{F}_{\mathrm{G-M}}(\sigma) = \mathbf{F}_{\mathrm{G-PD}}$ and $\lim_{\sigma\to 0}\mathbf{W}(\sigma) = \mathbf{S}_{\bar{\mathrm{G}}}\mathbf{E}_{\bar{\mathrm{G}}}\mathbf{S}_{\bar{\mathrm{G}}}^{\mathsf{H}}$, and that the limit of a product is equal to the product of the limits when the limits exist and are finite, it can be shown that $\sigma^2\mathbf{K}_{\mathrm{G-M}}(\sigma)\,(\mathbf{L}_{\mathrm{G-M}}(\sigma)/\sigma^2)$ has the same limit as $\sigma^2\mathbf{K}_{\mathrm{G-PD}}(\sigma)\,(\mathbf{L}_{\mathrm{G-PD}}(\sigma)/\sigma^2)$.

Appendix E
Asymptotic Effective Energy of the linear MMSE detector in the Linear dependent case

The pseudo-decorrelating filter in the linear dependent case, and hence the limiting form of the MMSE filter, are easily obtained by applying Greville's formula in (13) to $\bar{\mathbf{S}}_1\bar{\mathbf{A}}_1$.

For $G = \{1\}$ and MMSE detection, the AEE in (5) is simplified by noting that

$$\begin{aligned}\boldsymbol{\epsilon}_{\mathrm{G}} &= e_{ij} = \alpha_j - \alpha_i, \\ \tilde{\mathbf{L}}_{\mathrm{G}} &= \frac{1}{||\mathbf{x}||^2}\,\mathbf{f}_{1-\mathrm{PD}}\mathbf{f}_{1-\mathrm{PD}}^{\mathsf{H}}, \\ \mathbf{z}(e_{ij}) &= A_1 e_{ij}\mathbf{x}.\end{aligned}$$

Furthermore, the minimization in (5) is tractable. Indeed, the worst case error corresponds to hypothesis H_i and H_j being neighbors. In such a case, $\arg(e_{ij}) \in \{0, \pm\frac{\pi}{2}, \pi\}$ and since the function $|\cos\theta| + |\sin\theta|$ is $\frac{\pi}{2}$-periodic, the minimization yields

$$\frac{\sqrt{E_1}\,\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-1}\mathbf{s}_1|\alpha_{\min}| - \sqrt{2}\,|\alpha_{\max}|\sum_{k=1}^{K-1}|x_k|\,f(\theta_k)}{\sqrt{\mathbf{s}_1^{\mathsf{H}}\left(\bar{\mathbf{S}}_1\bar{\mathbf{E}}_1\bar{\mathbf{S}}_1^{\mathsf{H}}\right)^{-2}\mathbf{s}_1}},$$

where $\theta_k \triangleq \varphi_1 + \arg(x_k)$. Hence the result.

Appendix F
Equivalence of Widely Linear Detectors and the Minimal Sufficiency Approach

The PMAC is obtained by first considering the real-valued sufficient statistic $\hat{\mathbf{y}} = \mathcal{R}e\left(\boldsymbol{\Phi}^{\mathsf{H}}\mathbf{S}^{\mathsf{H}}\mathbf{r}\right) = \hat{\mathbf{R}}\mathbf{E}^{1/2}\mathbf{b} + \boldsymbol{\nu}$, where $\boldsymbol{\Phi} \triangleq \mathrm{diag}\left(e^{j\varphi_1}, \ldots, e^{j\varphi_K}\right)$, $\hat{\mathbf{R}} \triangleq \mathcal{R}e\left(\boldsymbol{\Phi}^{\mathsf{H}}\mathbf{S}^{\mathsf{H}}\mathbf{S}\boldsymbol{\Phi}\right)$, and $\boldsymbol{\nu} \sim$

$\mathcal{N}\left(\mathbf{0}, \frac{\sigma^2}{2}\hat{\mathbf{R}}\right)$, and then applying to $\hat{\mathbf{y}}$ the noise whitening transformation $\hat{\mathbf{S}}^{+\mathsf{T}}$, where $\hat{\mathbf{S}}$ represents the $\hat{N} \times K$ matrix of effective signals obtained from the factorization $\hat{\mathbf{R}} = \hat{\mathbf{S}}^{\mathsf{T}}\hat{\mathbf{S}}$ ($\hat{N}$ is the rank of $\hat{\mathbf{R}}$). The resulting statistic $\hat{\mathbf{r}} = \hat{\mathbf{S}}^{+\mathsf{T}}\hat{\mathbf{y}}$ can be shown to be a minimal sufficient statistic. It is given by $\hat{\mathbf{r}} = \hat{\mathbf{S}}\mathbf{E}^{1/2}\mathbf{b} + \hat{\mathbf{n}}$, where $\hat{\mathbf{n}} \sim \mathcal{N}\left(\mathbf{0}, \frac{\sigma^2}{2}\mathbf{I}_{\hat{N}}\right)$.

The proof of the equivalence is greatly simplified by considering augmented equivalent models for the PMAC and QMAC. These are obtained by considering, for a complex N-dimensional vector (or matrix) $\mathbf{x} = \mathbf{x}_R + j\mathbf{x}_I$, its real and complex augmented version defined by $\hat{\mathbf{x}}_a^{\mathsf{T}} = \left[\mathbf{x}_R^{\mathsf{T}}, \mathbf{x}_I^{\mathsf{T}}\right]$ and $\mathbf{x}_a^{\mathsf{T}} = \left[\mathbf{x}^{\mathsf{T}}, \mathbf{x}^{\mathsf{H}}\right]$, respectively.

For the augmented PMAC, we consider $\hat{\mathbf{T}}_a$, the real-valued augmented version of the matrix $\mathbf{T} \triangleq \mathbf{S}\mathbf{\Phi}$, and the alternative factorization of the matrix $\hat{\mathbf{R}}$ defined earlier as $\hat{\mathbf{R}} = \hat{\mathbf{T}}_a^{\mathsf{T}}\hat{\mathbf{T}}_a$. The model is then obtained by the transformation $\hat{\mathbf{r}}_a = \hat{\mathbf{T}}_a^{+\mathsf{T}}\hat{\mathbf{y}}$ so that the real-valued sufficient statistic (which is not minimal anymore) is $\hat{\mathbf{r}}_a = \hat{\mathbf{T}}_a\mathbf{E}^{1/2}\mathbf{b} + \hat{\mathbf{n}}_a$, where $\hat{\mathbf{n}}_a \sim \mathcal{N}\left(\mathbf{0}, \frac{\sigma^2}{2}\mathbf{I}_{2N}\right)$. Considering the partition $\hat{\mathbf{T}}_a = [\hat{\mathbf{t}}_{1a}, \hat{\mathbf{T}}_{1a}]$, the augmented PMAC decorrelating and MMSE filters are given, up to a scalar constant, by

$$\begin{aligned}
\hat{\mathbf{f}}_{\mathrm{D}a} &= \mathbf{P}_{\hat{\mathcal{T}}_{1a}^{\perp}}\hat{\mathbf{t}}_{1a}\left(\hat{\mathbf{t}}_{1a}^{\mathsf{T}}\mathbf{P}_{\hat{\mathcal{T}}_{1a}^{\perp}}\hat{\mathbf{t}}_{1a}\right)^{-1} \\
\hat{\mathbf{f}}_{\mathrm{M}a} &= \left(\hat{\mathbf{T}}_a\mathbf{E}\hat{\mathbf{T}}_a^{\mathsf{T}} + \frac{\sigma^2}{2}\mathbf{I}_{2N}\right)^{-1}\hat{\mathbf{t}}_{1a}.
\end{aligned}$$

Similarly, the augmented QMAC model is characterized by the complex-valued sufficient statistic $\mathbf{r}_a = \mathbf{T}_a\mathbf{E}^{1/2}\mathbf{b} + \mathbf{n}_a$, where $\mathbf{T}_a = [\mathbf{t}_{1a}, \mathbf{T}_{1a}]$ is the complex augmented version of $\mathbf{T}$, and $\mathbf{n}_a \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\mathbf{I}_{2N}\right)$. The augmented widely linear decorrelating and MMSE filters are then given, up to a scalar constant, by

$$\begin{aligned}
\mathbf{f}_{\mathrm{D}a} &= \mathbf{P}_{\mathcal{T}_{1a}^{\perp}}\mathbf{t}_{1a}\left(\mathbf{t}_{1a}^{\mathsf{H}}\mathbf{P}_{\mathcal{T}_{1a}^{\perp}}\mathbf{t}_{1a}\right)^{-1} \\
\mathbf{f}_{\mathrm{M}a} &= \left(\mathbf{T}_a\mathbf{E}\mathbf{T}_a^{\mathsf{H}} + \sigma^2\mathbf{I}_{2N}\right)^{-1}\mathbf{t}_{1a}.
\end{aligned}$$

Clearly the real and complex augmented vectors and matrices are related. Indeed, for complex vectors or matrices $\mathbf{x}$ and $\mathbf{z}$, it can easily be shown that $\mathbf{x}_a^{\mathsf{H}}\mathbf{z}_a = 2\,\hat{\mathbf{x}}_a^{\mathsf{T}}\hat{\mathbf{z}}_a$. Using this equality and recalling that $\mathbf{P}_{\mathcal{T}_{1a}^{\perp}} = \mathbf{I}_{2N} - \mathbf{T}_{1a}\left(\mathbf{T}_{1a}^{\mathsf{H}}\mathbf{T}_{1a}\right)\mathbf{T}_{1a}^{\mathsf{H}}$, we have $\mathbf{t}_{1a}^{\mathsf{H}}\mathbf{P}_{\mathcal{T}_{1a}^{\perp}}\mathbf{t}_{1a} = 2\hat{\mathbf{t}}_{1a}^{\mathsf{T}}\mathbf{P}_{\hat{\mathcal{T}}_{1a}^{\perp}}\hat{\mathbf{t}}_{1a}$ and $\mathbf{t}_{1a}^{\mathsf{H}}\mathbf{P}_{\mathcal{T}_{1a}^{\perp}}\mathbf{n}_a = 2\hat{\mathbf{t}}_{1a}^{\mathsf{T}}\mathbf{P}_{\hat{\mathcal{T}}_{1a}^{\perp}}\hat{\mathbf{n}}_a$. The equivalence of the widely linear detectors and the PMAC linear detectors then follows from noting that $\mathbf{f}_{\mathrm{D}a}^{\mathsf{H}}\mathbf{r}_a = \hat{\mathbf{f}}_{\mathrm{D}a}^{\mathsf{T}}\,\hat{\mathbf{r}}_a$ (a similar equality holds for the MMSE detectors).

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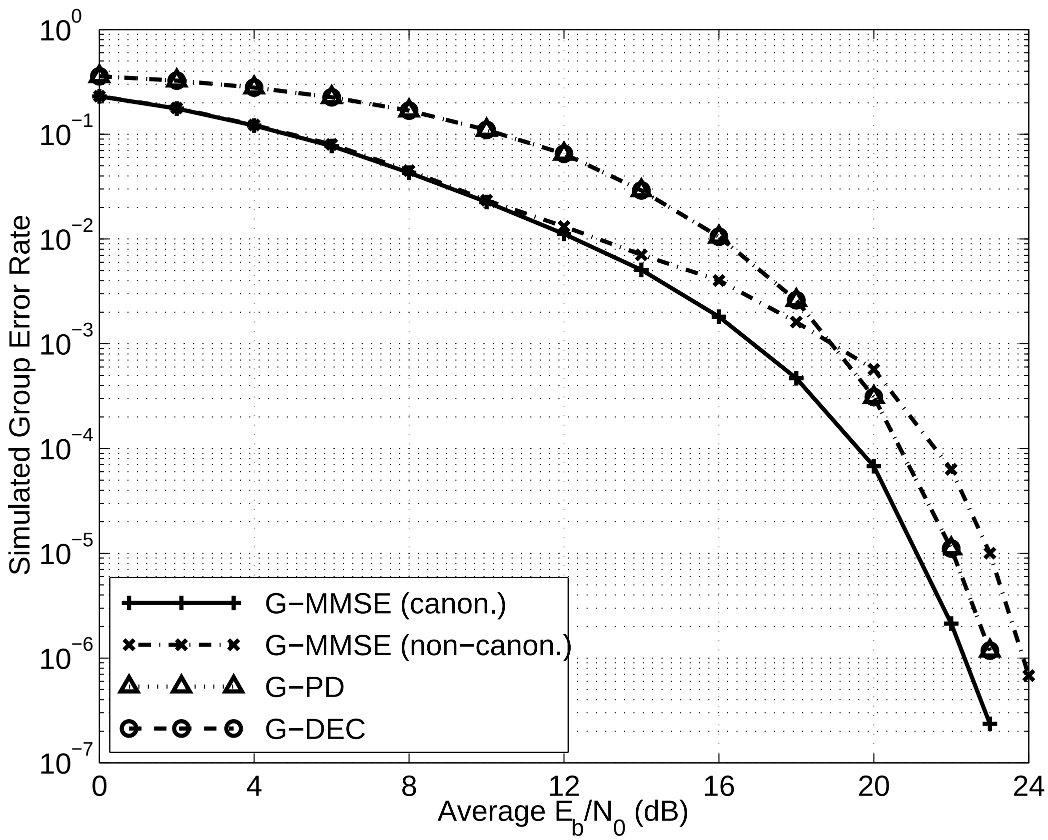
Fig. 1. Performance of group detectors for a fully-loaded CDMA system with $(N, K, |G|) = (8, 8, 2)$ and QPSK: rank$(\mathbf{S}_{\bar{\mathrm{G}}}) = 6$, rank$(\mathbf{S}_{\mathrm{G}}) = 2$ and rank$\left(\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}} \mathbf{S}_{\mathrm{G}}\right) = 2$, i. e., $\mathcal{S}_{\mathrm{G}} \cap \mathcal{S}_{\bar{\mathrm{G}}} = \{\mathbf{0}\}$.

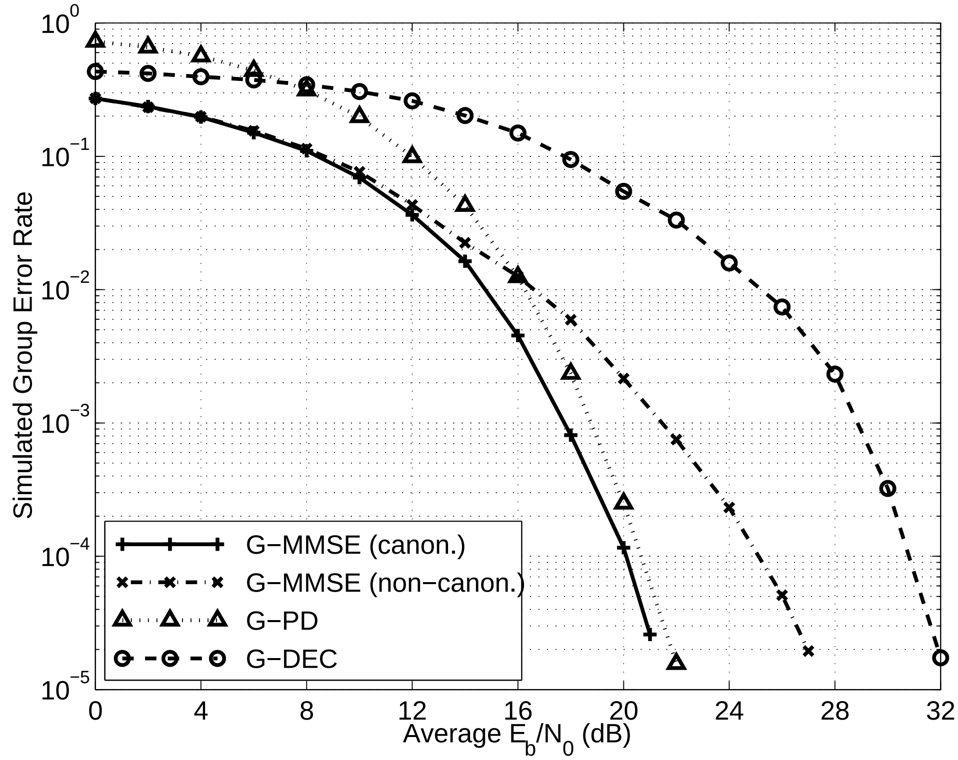
Fig. 2. Performance of group detectors for overloaded CDMA systems with QPSK modulation.

Fig. 3. Performance of decision feedback detectors for overloaded CDMA systems with BPSK modulation.

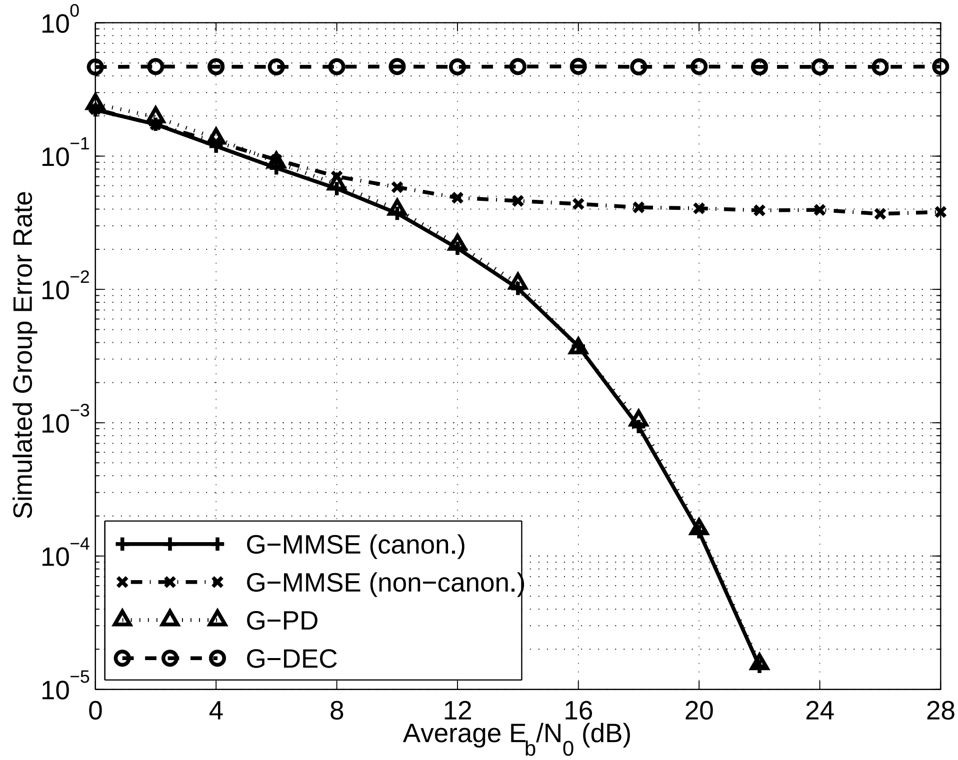
Fig. 4. Performance of group decision feedback detectors for the overloaded CDMA system of Fig. 3 (a).



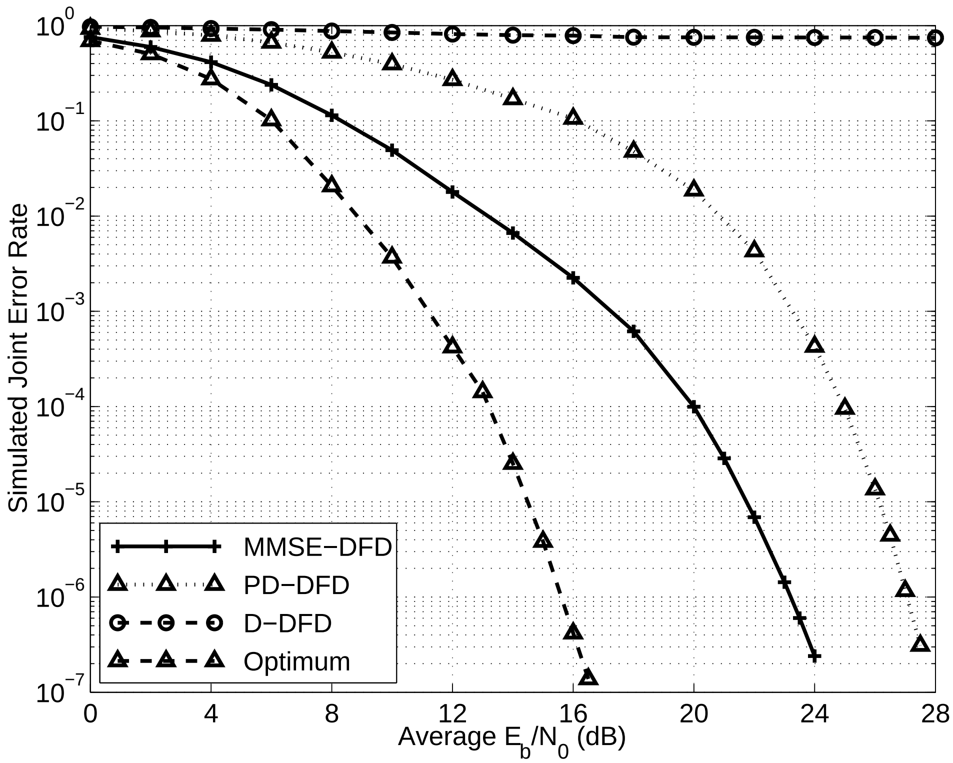
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
Simulated Group Error Rate
0
4
8
12
16
20
24
Average E_b/N_0 (dB)
G−MMSE (canon.)
G−MMSE (non−canon.)
G−PD
G−DEC



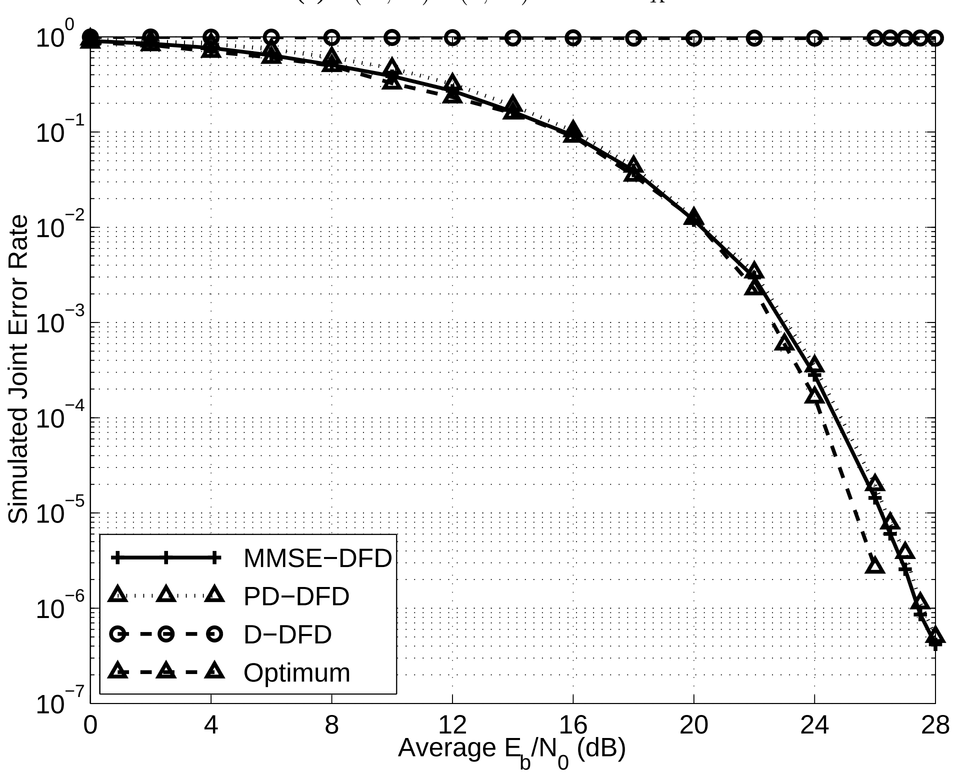
(a) $(N, K, |G|) = (7, 8, 2)$ and $\operatorname{rank}\left(\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}} \mathbf{S}_{\mathrm{G}}\right) = 1$



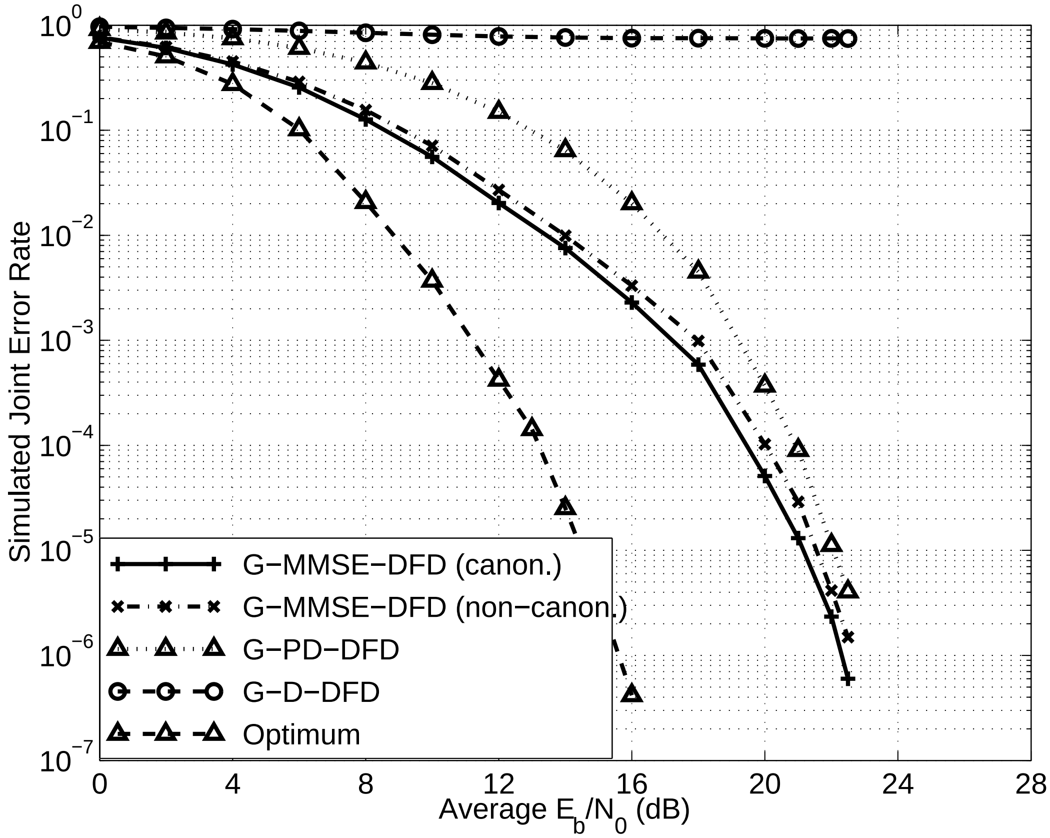
(b) $(N, K, |G|) = (6, 8, 2)$ and $\operatorname{rank}\left(\mathbf{P}_{\mathcal{S}_{\bar{\mathrm{G}}}^{\perp}} \mathbf{S}_{\mathrm{G}}\right) = 0$



(a) $(N, K) = (8, 10)$ and $\mathbf{E} = \mathbf{I}_K$



(b) $(N, K) = (5, 10)$ and $E_k = (K - k + 1)^2$



Simulated Joint Error Rate
Average E_b/N_0 (dB)
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
0
4
8
12
16
20
24
28
G−MMSE−DFD (canon.)
G−MMSE−DFD (non−canon.)
G−PD−DFD
G−D−DFD
Optimum