Signal Constellations for Noncoherent Space-Time Communications

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Abstract

We examine the problem of designing signal constellations for the multiple transmit-multiple receive antenna Rayleigh fading communication channel. By employing the asymptotic union bound on the probability of error, we find a new formulation of the problem of signal design for the noncoherent fading channel. Since unitary signals are optimal (in the limit of large signal to noise ratios), the design can be posed in terms of packings on the Grassmanian manifold. A key difference in our approach from that of other authors is that we use a notion of distance on this manifold which is suggested by the union bound. As a consequence of our use of this distance measure we obtain signals designs that are guaranteed to achieve the full diversity order of the channel, a claim which does not hold when the chordal distance is used. We introduce a new method of iteratively designing signals, termed successive updates, to approximately optimize this performance measure.

1 Introduction

In this paper we consider the problem of designing signal waveforms for use with multiple transmit and receive antennas over the wireless fading channel, also known as space-time communication. This problem has received considerable attention, due in large part to the information theoretical work of Telatar, Foschini, and Gans [1, 2], who showed that the use of transmit (and receive) antenna diversity could achieve considerable gains on the Rayleigh fading channel when the receiver has perfect side information about the channel state. This work was later extended to the noncoherent channel where the receiver is not allowed this side information in [3]. In that paper the capacity achieving signal designs were shown to have a unitary structure. The asymptotic structure of the noncoherent channel capacity was further explored in [4], where the problem was related to sphere packing on the Grassmanian manifold. The common theme in these papers is that diversity communications can yield significant improvements over single antenna communications.

In this paper we consider the *asymptotic union bound* (AUB) on the probability of error for a noncoherent space-time communication system operating over a Rayleigh fading channel. The AUB relates the error probability to a relatively simple algebraic function of the signals

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and hence can be used to generate a metric for designing space-time signal constellations. We introduce a method of iteratively designing signals, termed successive updates, to approximately optimize this performance measure. We also use the AUB to suggest modifications to the systematic designs of [5] and to the Grassmanian manifold based optimization of [6]. In particular, we are able to insure that the full potential order of diversity of the space-time channel is realized through our designs, a result which does not follow from the design metric employed in [5] and [6].

2 The Rayleigh Fading Channel

We adopt the usual block fading channel model for space-time communications wherein the discrete time received signal at the n^{th} antenna is of the form

$$\mathbf{y}_n = \sqrt{\overline{\gamma}}\mathbf{S}_i\mathbf{h}_n + \mathbf{n}_n. \tag{1}$$

Here $\mathbf{S}_i \in \mathbb{C}^{D\times M}$ is the i^{th} signal from a set of cardinality S, $\mathbf{h}_n \in \mathbb{C}^{M\times 1}$ is the vector of fading coefficients from each transmit antenna to sensor n, and $\mathbf{n}_n$ is additive noise.

Stacking the D observations per receiver antenna yields the DN sufficient statistics

$$\mathbf{y} = [\mathbf{y}_1^\mathsf{T}, \ldots,\ \mathbf{y}_N^\mathsf{T}]^\mathsf{T} = \sqrt{\overline{\gamma}}\boldsymbol{\mathcal{S}}_i\mathbf{h} + \mathbf{n}, \tag{2}$$

where $\boldsymbol{\mathcal{S}}_i = \mathbf{I}_N \otimes \mathbf{S}_i$ ($\otimes$ is the Kronecker product and $\mathbf{I}_N$ is the $N \times N$ identity matrix), $\mathbf{h} = [\mathbf{h}_1^\mathsf{T}, \ldots,\ \mathbf{h}_N^\mathsf{T}]^\mathsf{T}$, and $\mathbf{n} = [\mathbf{n}_1^\mathsf{T}, \ldots,\ \mathbf{n}_N^\mathsf{T}]^\mathsf{T}$. We assume that the additive noise is spatially white, but the fading need not be, i.e. $\mathbf{\Sigma} = E[\mathbf{h}\mathbf{h}^*]$ need not be diagonal. For the purposes of this paper we shall assume that "unitary" signals are used so that $\mathbf{S}_i^*\mathbf{S}_i = \mathbf{I}$. This assumption is justified by information theoretic arguments in [3] and by asymptotic error probability arguments in [7], each for the large SNR regime. Since our signal designs will be based on a distance measure valid for that regime, this assumption is not restrictive. In this model we normalize the noise so that $E[\mathbf{n}\mathbf{n}^*] = \mathbf{I}$, and the fading process so that $E[\mathbf{h}_n^*\mathbf{h}_n] = 1$. We denote the average SNR at each receiver antenna by the scalar $\overline{\gamma}$.

Conditioned on the transmission of signal $\boldsymbol{\mathcal{S}}_i$, the measurement has likelihood function

$$f_i(\mathbf{y}) = \frac{1}{\pi^{ND}\left|\mathbf{I}_{DN} + \overline{\gamma}\boldsymbol{\mathcal{S}}_i\mathbf{\Sigma}\boldsymbol{\mathcal{S}}_i^*\right|}\exp\left\{-\mathbf{y}^*\left(\mathbf{I}_{DN} + \overline{\gamma}\boldsymbol{\mathcal{S}}_i\mathbf{\Sigma}\boldsymbol{\mathcal{S}}_i^*\right)^{-1}\mathbf{y}\right\}. \tag{3}$$

The optimal detection strategy is the maximum likelihood rule [1]

$$\phi^{ML} : \widehat{i} = \arg\max_i f_i(\mathbf{y}) \quad = \quad \arg\max_i \mathbf{y}^*\boldsymbol{\mathcal{S}}_i\left(\mathbf{I}_{DN} + \overline{\gamma}^{-1}\mathbf{\Sigma}^{-1}\right)^{-1}\boldsymbol{\mathcal{S}}_i^*\mathbf{y} \tag{4}$$

where we have applied the Woodbury identity to the quadratic form in the likelihood function.

3 Performance of the Maximum Likelihood Detector

To evaluate the probability of error for the maximum likelihood test of (4) we employ a union upper bound on the probability of error:

$$P^e \leq \frac{1}{S}\sum_{i=1}^{S}\sum_{\substack{k=1\\ i\neq k}}^{S} P_{i,k}, \tag{5}$$

[1] We shall assume that the fading correlation is full rank. If $\mathbf{\Sigma}$ is singular with rank r, we may form the eigen-decomposition $\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^*$ with $\mathbf{U} \in \mathbb{C}^{NM\times r}$. We can then replace the signals by $\{\boldsymbol{\mathcal{S}}_i\mathbf{U}\}$ and treat the fading as full rank with correlation $\mathbf{\Lambda}$.

where P^e is the probability of a symbol error and $P_{i,k}$ is the probability that signal $\boldsymbol{\mathcal{S}}_k$ is preferred by the detector to signal $\boldsymbol{\mathcal{S}}_i$, when $\boldsymbol{\mathcal{S}}_i$ was transmitted.

Defining the vectors $\mathbf{v}_i = \left(\mathbf{I}_{DN} + \overline{\gamma}^{-1}\mathbf{\Sigma}^{-1}\right)^{-1/2} \boldsymbol{\mathcal{S}}_i^* \mathbf{y}$ and $\mathbf{v}_k = \left(\mathbf{I}_{DN} + \overline{\gamma}^{-1}\mathbf{\Sigma}^{-1}\right)^{-1/2} \boldsymbol{\mathcal{S}}_k^* \mathbf{y}$, where $(\mathbf{I}+\mathbf{\Sigma}^{-1})^{1/2}$ is the (unique) positive definite square root of the matrix $\mathbf{I}+\mathbf{\Sigma}^{-1}$, we make an error whenever $||\mathbf{v}_i||^2 < ||\mathbf{v}_k||^2$. If we define the matrix

$$\mathbf{J} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \tag{6}$$

and the vector $\mathbf{v} = [\mathbf{v}_i^{\mathsf{T}} \; \mathbf{v}_k^{\mathsf{T}}]$, the pairwise error probability is found to be

$$P_{i,k} = P\left[\beta = \mathbf{v}^* \mathbf{J} \mathbf{v} < 0\right]. \tag{7}$$

The characteristic function for β can be found using the results of [8, Appendix B] and is given by

$$G_\beta(s) = \frac{1}{|\mathbf{I} + s\mathbf{R}_{\mathbf{vv}}\mathbf{J}|}, \tag{8}$$

where

$$\mathbf{R}_{\mathbf{vv}} \;=\; E[\mathbf{v}\mathbf{v}^*] = \begin{bmatrix} \overline{\gamma}\mathbf{\Sigma} & \boldsymbol{\mathcal{F}}\left(\mathbf{I} + \overline{\gamma}\mathbf{\Sigma}\right)\boldsymbol{\mathcal{C}}_{i,k}\boldsymbol{\mathcal{F}} \\ \boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{C}}_{i,k}^*\left(\mathbf{I} + \overline{\gamma}\mathbf{\Sigma}\right)\boldsymbol{\mathcal{F}} & \boldsymbol{\mathcal{F}}\left(\mathbf{I} + \overline{\gamma}\boldsymbol{\mathcal{C}}_{i,k}^*\mathbf{\Sigma}\boldsymbol{\mathcal{C}}_{i,k}\right)\boldsymbol{\mathcal{F}} \end{bmatrix}$$

and $\mathbf{C}_{i,k} = \mathbf{S}_i^*\mathbf{S}_k$, $\boldsymbol{\mathcal{C}}_{i,k} = \boldsymbol{\mathcal{S}}_i^*\boldsymbol{\mathcal{S}}_k = \mathbf{I}_N \otimes \mathbf{C}_{i,k}$, and $\boldsymbol{\mathcal{F}} = \left(\mathbf{I} + \overline{\gamma}^{-1}\mathbf{\Sigma}^{-1}\right)^{-1/2}$. The performance of the detector depends on the non-zero eigenvalues, $\{\gamma_k\}_1^{2L}$, of the product $\mathbf{R}_{\mathbf{vv}}\mathbf{J}$ through

$$P_{i,k} = -\sum_{\gamma_j < 0} \text{Res}\left(\frac{1}{s\prod_{l=1}^{2L}\gamma_l\left(s + \frac{1}{\gamma_l}\right)}, s_j = -\frac{1}{\gamma_j}\right), \tag{9}$$

summing over the residues of the negative eigenvalues (see e.g. [7]). Notice that this bound holds whether or not full diversity order is achieved. When full diversity order is achieved we have $L = MN$.

3.1 The Asymptotic Union Bound

We consider the behavior of the pairwise error probabilities with arbitrary $\mathbf{\Sigma}$ and M as the SNR approaches infinity. Denote the asymptotic $L \leq MN$ positive eigenvalues of the matrix $\mathbf{R}_{\mathbf{vv}}\mathbf{J}$ by $\alpha_l^{i,k}$, i.e. $\alpha_l^{i,k}/\overline{\gamma} = \lim_{\overline{\gamma}\to\infty} \gamma_l^{i,k}/\overline{\gamma}$. We find that the pairwise error probabilities approach [7]

$$P_{ik} \to \binom{2L}{L} \frac{1}{\prod_{l=1}^{L}\alpha_l^{i,k}}. \tag{10}$$

When full diversity is achieved we have $L = MN$ and the corresponding error probabilities have the form

$$P_{ik} \to \binom{2MN}{MN} \frac{\overline{\gamma}^{-MN}}{|\mathbf{\Sigma}||\mathbf{I} - \mathbf{C}_{ik}\mathbf{C}_{ik}^*|^N}. \tag{11}$$

The corresponding union bound, denoted as the *asymptotic union bound* is given by

$$P^{AUB} = \frac{1}{S}\sum_{i=1}^{S}\sum_{k\neq i} P_{ik} = \frac{1}{S}\binom{2MN}{MN}\frac{\overline{\gamma}^{-MN}}{|\mathbf{\Sigma}|}\sum_{i=1}^{S}\sum_{k\neq i}\frac{1}{|\mathbf{I} - \mathbf{C}_{ik}\mathbf{C}_{ik}^*|^N}. \tag{12}$$

Notice that in the large SNR regime, the dependence of the bound on the signal geometry (through the determinant $|\mathbf{I} - \mathbf{C}_{ik}\mathbf{C}_{ik}^{*}|^N$) is decoupled from the fading correlation structure. That means that we can design signals without paying attention to the particular correlation structure if we use design techniques based on this large SNR performance measure.

4 Signal Design for the Rayleigh Fading Channel

To develop an optimality criterion for the signal design problem, we need a numerical function which characterizes (asymptotically) the performance of the detector as a function of the signal constellation. As argued in the preceding section, the probability of error for the Rayleigh fading channel is asymptotically determined by the function

$$d(\{\mathbf{S}_i\}) = \sum_{i=1}^{S} \sum_{k \neq i} \frac{1}{|\mathbf{I} - \mathbf{S}_i^{*}\mathbf{S}_k\mathbf{S}_k^{*}\mathbf{S}_i|^N}. \tag{13}$$

Notice that this metric differs from the worst-case chordal distance employed in [5] and [6], which is defined by

$$d_c(\{\mathbf{S}_i\}) = \min_{k \neq i} \operatorname{tr}\left\{\mathbf{I} - \mathbf{S}_i^{*}\mathbf{S}_k\mathbf{S}_k^{*}\mathbf{S}_i\right\}. \tag{14}$$

Optimizing signal constellations over this latter metric does *not* guarantee that full potential diversity order of the space-time channel is achieved, a point which is discussed further in Section 6.

With the measure of distance given by (13), we may state the signal design problem as follows: Given $M, N, D, S \in \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers) find the set $\{\mathbf{S}_i\}_{i=1}^{S}$ for which the value of $d(\{\mathbf{S}_i\})$ in (13) is minimized subject to the constraints 1) $\mathbf{S}_i{}^{*}\mathbf{S}_i = \mathbf{I}$ for all i, 2) $\mathbf{S}_i \in \mathbb{C}^{D \times M}$.

Since we constrain our signals to have a unitary structure, their properties are completely determined by the M-dimensional subspace, $< \mathbf{S}_i >$, which they span. This means that we may parameterize our optimization over the Grassmanian manifold $G(M, D)$, the space of all M-dimensional subspaces of $\mathbb{C}^D$. A parameterization of $G(M, D)$ which employs $2MD - M^2$ real parameters is explicitly detailed in [6] and was employed for our designs. The AUB provides a smooth differentiable function of these parameters and standard gradient based optimization may be performed to design the signals. Unfortunately, as the signal set cardinality grows large (corresponding to large spectral efficiencies), this optimization becomes infeasible due to the number of free parameters. For this reason we consider a suboptimal iterative procedure for approximately optimizing this function in the next section.

5 Successive Updates for Signal Design

In order to solve the signal design problem of the previous section, an optimization over $S(2DM - M^2)$ parameters describing the signals must be performed. In order to find a tractable procedure for designing large cardinality signal sets, we consider a suboptimal approach, termed *successive updates*.

Suppose that we have already chosen K signals and seek an additional signal, $\mathbf{S}_{K+1}$ to add to the set such that we increase $d(\{\mathbf{S}_k\})$ by as little as possible. The AUB metric may be expanded as

$$d(\{\mathbf{S}_k\}_{k=1}^{K+1}) = d(\{\mathbf{S}_k\}_{k=1}^{K}) + 2\sum_{k=1}^{K} \frac{1}{|\mathbf{I} - \mathbf{S}_{K+1}^{*}\mathbf{S}_k\mathbf{S}_k^{*}\mathbf{S}_{K+1}|^N}. \tag{15}$$

The strategy we employed was to minimize the second term in (15) under the constraint that the additional signal has a unitary structure, $\mathbf{S}_{K+1}^{*}\mathbf{S}_{K+1} = \mathbf{I}$. We initialized each optimization step with a random "unitary" signal. The optimization problem can be parameterized on the Grassmanian manifold as in Section 4.

It is easy to extend this algorithm to add multiple signals at each iteration. In the limit we have the full optimization problem described in Section 4. We can also use the signals designed with this algorithm with $S' < S$ updates per iteration to initialize the full, non-iterative, optimization.

5.1 Numerical results for M=1

We first consider the case of a single transmit antenna, using the successive updates algorithm with one signal per update. In Figure 1 we plot the probability of symbol error versus the SNR for 4 receive antennas and $D = 2$ dimensional constellations for several values of the spectral efficiency. We notice that we incur a $6dB$ loss in SNR for every increase by one in the spectral efficiency. This is the behavior predicted in [4] for the capacity of this channel, i.e. that the spectral efficiency should increase by $M^*(1 - M^*/D)$bps/Hz for every $3dB$ gain in SNR, where $M^* = min\{M, N, \lfloor D/2 \rfloor\}$. These results imply that our signal designs display the relationship between spectral efficiency and SNR expected of optimal constellations.

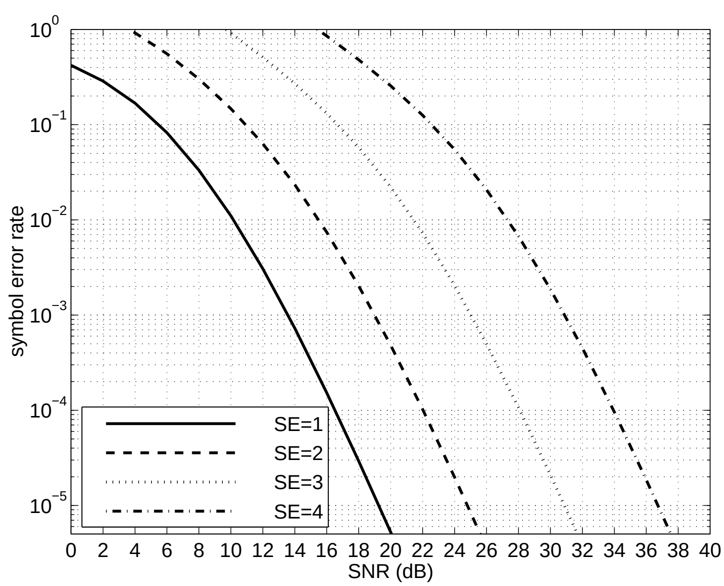

Figure 1: Probability of symbol error versus SNR for the $M = 1$ transmit, $N = 4$ receive antenna channel with $D = 2$ dimensional signal constellations for several values of the spectral efficiency.

5.2 Results for M=N=2

We now present results for multiple transmit and receive antennas. Using the successive updates algorithm with one signal added per update, a signal set with $S = 512$ signals in $D = 4$ dimensions for $M = N = 2$ transmit and receive antennas was designed. The spectral efficiency of the set is hence 2.25bps/Hz. To show the price in energy-efficiency for this relatively high spectral efficiency, the performance of the set using only the first 16, 64, 256 signals, respectively, is also shown in Figure 2. The figure shows simulated symbol error rates and the asymptotic union bounds on the symbol error probability. Note that the latter are a good indicator of the signals sets' performances. From the figure we see that the price paid for increasing the spectral efficiency from 1bps/Hz to 2bps/Hz is roughly 6dB, indicating a 3dB penalty when compared to the capacity results in [4]. This gap is most likely due to the poor fit between the linear approximation of [4] to the capacity curve at the spectral efficiencies considered.

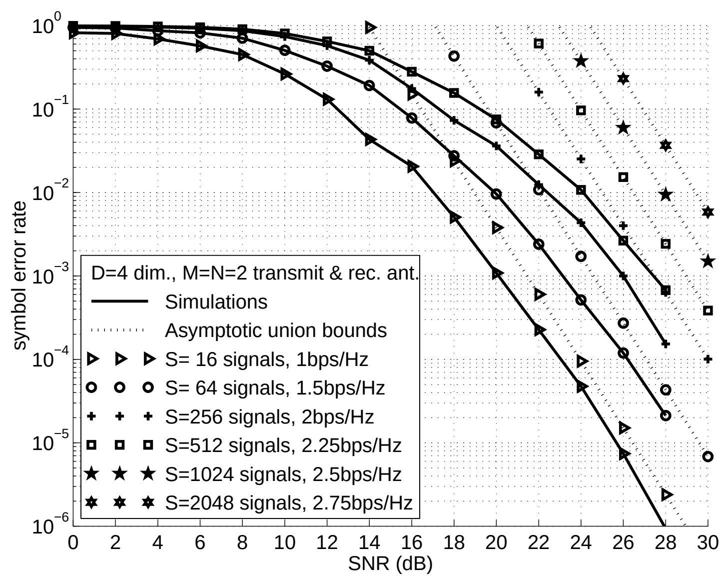

Figure 2: Symbol error rate comparison for increasing spectral efficiency for $M = N = 2$ transmit and receive antennas.

5.3 Results for M=2, N=2, with multiple signals per iteration

We end this section with an example demonstrating the utility of adding several signals per iteration. We consider $M = 2$ transmit antennas, $N = 2$ receive antennas and a $D = 4$ dimensional signal space. In Figure 3 we plot the AUB for $S = 128$ signals built with one and with 128 signals per iteration (this latter case corresponds to a joint optimization over all of the signals as in Section 4). We used the signals designed with one signal per iteration as an initial set for the joint optimization problem. We gained about 1dB in performance through the joint optimization relative to the signal by signal update for this example.

6 Modifications to Existing Techniques

The signal design techniques of [5] and [6] are based on the maximization of the worst-case chordal distance metric of (14). We can reformulate the algorithms presented in those refer-

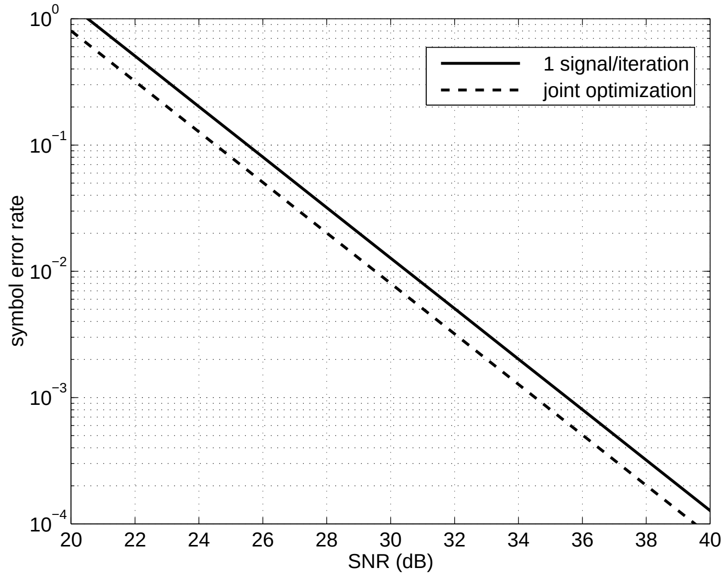

Figure 3: Symbol error rate comparison for signals designed with one signal update per iteration and with the joint optimization.

ences with the worst case chordal distance metric replaced by the AUB criterion of (13). The details of the resulting designs are presented in the subsequent sections.

6.1 Systematic Designs of Hochwald et. al

In [5] signal constellations are systematically generated by starting with an initial signal and successively rotating it. The method is shown to have an algebraic equivalence to block-codes in the ring of integers. Signal constellations are found by randomly searching the generating code matrices. The block-circulant structure in the signals simplifies the modulation and demodulation since PSK-type signals are employed. Furthermore, the design criterion of maximizing the minimum chordal distance does not necessarily yield good constellations. For example, the generating matrix given in [5] for $S = 2304$ signals, $M = 2$ transmit antennas, $N = 1$ receive antenna, and $D = 8$ dimensions does not yield a signal constellation that achieves full order of diversity, since the matrix $\mathbf{I} - \mathbf{C}^*_{i,k}\mathbf{C}_{i,k}$ is rank deficient for $\mathbf{S}_i$ generated from the (algebraic) input vector $\mathbf{x}_i = [0,\ 0]$ and $\mathbf{S}_k$ generated from $\mathbf{x}_k = [24,\ 24]$ (because of the cyclic structure of the code other input vectors yield similar low rank matrices). With a rather limited random search over the generating matrices (around 2000 matrices in a space of $48^{12} \approx 1.5 \cdot 10^{20}$ possible generating matrices were considered), we found a generating matrix[2] whose corresponding constellation achieves full order of diversity. Moreover, constellations with the same exponent in the union bound seem to be quite dense, meaning that several matrices were found which generated roughly equivalent signals. The asymptotic union bound on the probability of error of each design is shown in Figure 4. It is clear that at high SNRs the loss of diversity is significant, resulting in ever widening gap between the performance of the two designs. The minimum chordal distance of our constellation is 0.4062, while that of the constellation given in [5] is 0.4427, indicating the limited validity of the chordal distance as a figure of merit.

[2] $\begin{bmatrix} 3 & 26 & 14 & 10 & 2 & 46 \\ 1 & 30 & 39 & 47 & 38 & 16 \end{bmatrix}.$

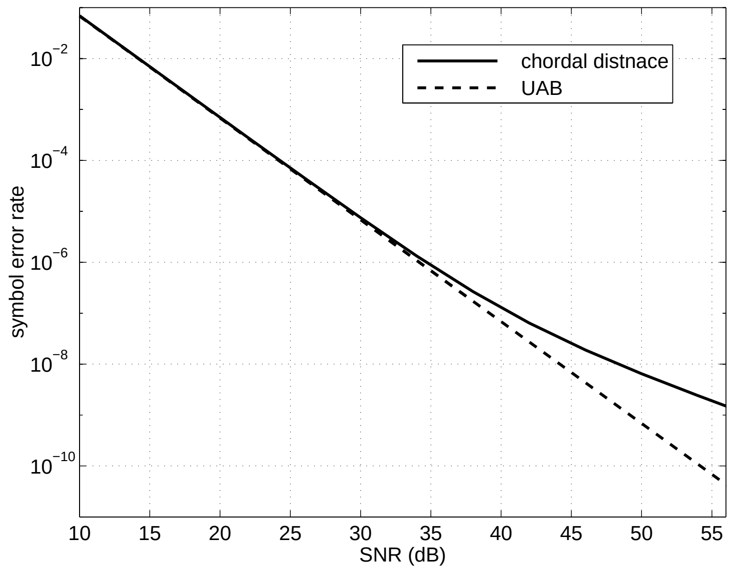

Figure 4: Probability of symbol error versus SNR for the $M = 2$ transmit, $N = 1$ receive, antenna channel for the $S = 2043$ constellation of [5], using the chordal distance on $G(M, D)$, and of this section, using the asymptotic union bound.

6.2 Sphere Packing on the Grassmanian Manifold

In [6], a sphere packing problem on $G(M, D)$, is described which is essentially the same as the joint optimization described in Section 4 with the worst case chordal distance of (14) employed in the place of (13). The authors of that paper used a relaxation technique to iteratively approximate the worst-case distance by smooth functions of the signal parameters.

A strong limitation to the sphere packing formulation of the signal design problem is the use of the minimum chordal distance as the objective function. This results in an l_∞ optimization problem which may not guarantee full diversity order. By using the asymptotic union bound of (12) we have an objective function which is a smooth differentiable function of the signal parameters and will guarantee that the full diversity order is achieved. We use a gradient search algorithm to solve the design problem, with gradients computed with respect to the Grassmanian parameters as in [6].

7 Conclusions

In this paper we have presented an approach to noncoherent signal design under dimensionality constraints for the multi-sensor Rayleigh fading channel based on the asymptotic union bound. By employing this distance measure, we can adapt the techniques of [5] and [6] to better reflect the error probability on the space time channel. We have also introduced a new technique, termed successive updates, which performs the signal design in an iterative fashion. The performance of our designs for two and three dimensions was shown to fit well with the theory of optimal designs presented in [4] for the case of single transmit antenna, while a $3dB$ gap was seen for the multiple transmit antenna case. This disparity is perhaps due to the failure of the linear approximation to the channel capacity at the spectral efficiencies considered and also appears in the designs of [5] and [6].

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